Control, Optimization, and Diffusion Limits for Queuing Systems

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PhD Thesis Seminar, July 17

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Department of Electrical Engineering
Electronics
Computers
Communications

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Israel Institute of Technology

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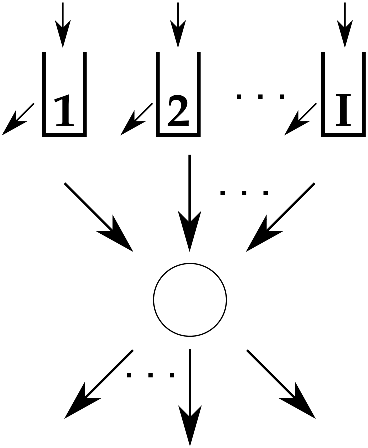
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The Model

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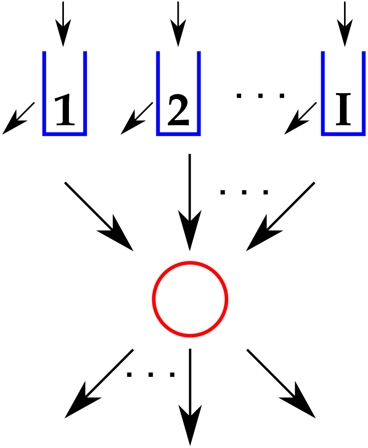

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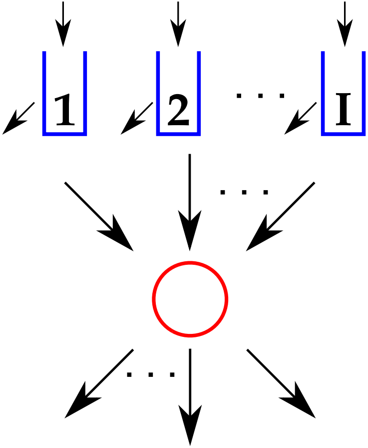

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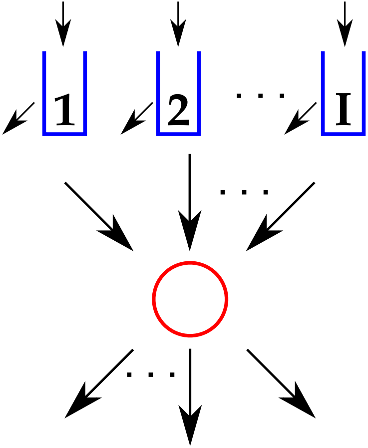

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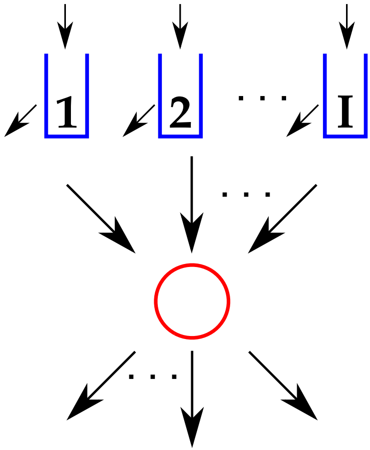

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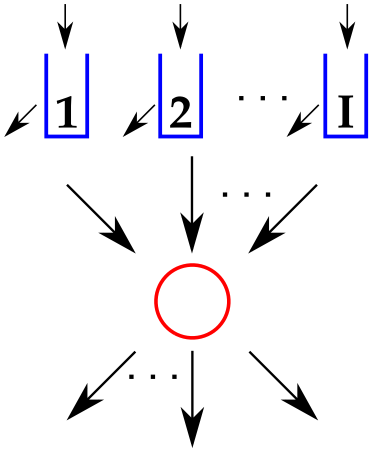

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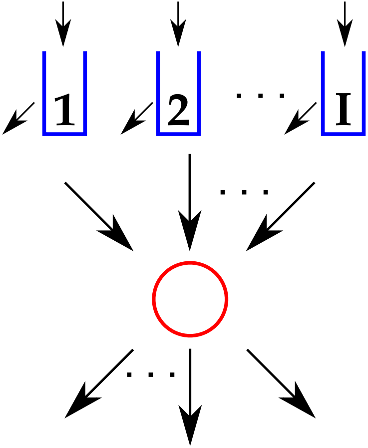

Problem: Choose who to serve next to minimize the cost

- $X_i(t)$ = the number of customers in the i-th class at time t

$$= \underbrace{X_i(0)}_{\text{initial state}} + \underbrace{A_i(t)}_{\text{Arrivals}} - \underbrace{S_i(\int_0^t B_i(s)ds)}_{\text{Departures (service)}} - \underbrace{R_i(t)}_{\text{Reneging}}$$

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$$J(B) = \mathbb{E}\Big(\int_0^\infty e^{-\alpha t} b' dR(t)\Big) = \alpha \mathbb{E}\Big(\int_0^\infty e^{-\alpha t} c' X(t) dt\Big)$$

where $\alpha > 0$, b_i reneging cost for class i, c_i holding cost for class i ($c_i = \theta_i b_i$, where θ_i is abandonment rate).

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- **Approach:** Use diffusion approximation.

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These results DO NOT apply in Diffusion scale.

More Related Work

1. Samim Ghamami and Amy R. Ward. Dynamic scheduling of a two-server parallel server system with complete resource pooling and reneging in heavy traffic: asymptotic optimality of a two-threshold policy.
   Math. Oper. Res., 38(4):761–824, 2013
2. Jeunghyun Kim and Amy R. Ward. Dynamic scheduling of a $GI/GI/1 + GI$ queue with multiple customer classes.
   Queueing Syst., 75(2-4):339–384, 2013
3. Barış Ata and Mustafa H. Tongarlak. On scheduling a multiclass queue with abandonments under general delay costs.
   Queueing Syst., 74(1):65–104, 2013
4. Melanie Rubino and Baris Ata. Dynamic control of a make-to-order, parallel-server system with cancellations.
   Operations Research, 57(1):94–108, 2009

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Find Asymptotically Optimal Control - Solution Steps

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   (The Bellman equation characterize the value function)

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3. **AO**: prove this control is Asymptotically Optimal.

Step 1 - Define BCP

- $X_t = x + \underbrace{W_t}_{\text{BM}} - \int_0^t \underbrace{\Theta}_{\mathrm{diag}(\theta)} X_s ds + \underbrace{Y_t}_{\text{Control}} \in \mathbb{R}_+^I$

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- **Heavy traffic condition:** $\sum_{i=1}^I \lambda_i/\mu_i = 1$
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- **Goal:** To minimize the cost function

$$J(x, Y) = \mathbb{E}\Big(\int_0^\infty e^{-\alpha t} \underbrace{c'}_{\text{holding cost}} X_t dt \Big).$$

for Y being admissible

Definition

admissible control system with initial condition $x \in \mathbb{R}^I_+$ is a tuple $(\bar{\Sigma}, W_t, Y_t, X_t)$, where $\bar{\Sigma} = (\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}, \bar{P})$ is a filtered probability space, $\{W_t\}$ is an I-dimensional $\bar{\mathcal{F}}_t$-adapted (y, σ)-BM, the processes $\{X_t\}$ and $\{Y_t\}$ have sample paths in $\mathbb{D}_{\mathbb{R}^I}(\mathbb{R}_+)$ and are $\bar{\mathcal{F}}_t$-adapted, and the following hold:

i. For all $t, s \geq 0$, $W_{t+s} - W_t$ is independent of $\bar{\mathcal{F}}_t$ under $\bar{P}$,
ii. X_t satisfies $X_t \in \mathbb{R}^I_+$ for all t $\bar{P}$-a.s.,
iii. The process $m'Y$, where $m = (1/\mu_1, ..., 1/\mu_I)$, is non-negative and non-decreasing.

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- $\mathcal{A}(x)$ the set of all admissible control systems with initial condition $x \in \mathbb{R}_+^I$.
- **The value function**

$$V(x) = \inf_{\mathcal{A}(x)} J(x, Y), \qquad x \in \mathbb{R}_+^I.$$

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The two problems are equivalent

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- **Cost:**

$$\tilde{J}(\tilde{x}, (U, \tilde{Y})) = \mathbb{E}\Big(\int_0^\infty e^{-\alpha t} q' U_t \tilde{X}_t dt \Big).$$

Definition

An *admissible control system* with initial condition $\tilde{x} \in \mathbb{R}_+$ is a tuple $(\tilde{\Sigma}, \tilde{W}, U, \tilde{Y}, \tilde{X})$, where $\tilde{\Sigma} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{P})$ is a filtered probability space, $\{\tilde{W}_t\}$ is a 1-dimensional $\tilde{\mathcal{F}}_t$-adapted $(\tilde{y}, \tilde{\sigma})$-BM, and the processes $\{U_t\}$, $\{\tilde{Y}_t\}$, $\{\tilde{X}_t\}$ have sample paths in $\mathbb{D}_{\mathcal{S}_1}(\mathbb{R}_+)$, $\mathbb{D}_{\mathbb{R}^I}(\mathbb{R}_+)$ and $\mathbb{D}_{\mathbb{R}}(\mathbb{R}_+)$, resp., are $\tilde{\mathcal{F}}_t$-adapted and the following hold:

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- $\tilde{\mathcal{A}}(\tilde{x})$ the set of all the admissible controls for the initial condition $\tilde{x}$.
- **Value function:**

$$\tilde{V}(\tilde{x}) = \inf_{(U,\tilde{Y}) \in \tilde{\mathcal{A}}(\tilde{x})} \tilde{J}(\tilde{x}, (U, \tilde{Y})), \qquad \tilde{x} \in \mathbb{R}_+.$$

$$\begin{cases} -\frac{\tilde{\sigma}^2}{2}\frac{d^2 v}{dx^2} - \tilde{y}\frac{dv}{dx} + xF\left(\frac{dv}{dx}\right) + \alpha v = 0, & 0 < x < \infty, \\ \frac{dv}{dx}(0) = 0, \qquad |v(x)| \leq C(1+x)^C, & x \in [0, \infty). \end{cases}$$

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- The Bellman equation admits a unique classical solution - the value function $\tilde{V}$.
- The optimal control is deduced from finding the maximum in $F(y) = \max_u g(u,y)$.
- Can be solved numerically (equation in one variable).

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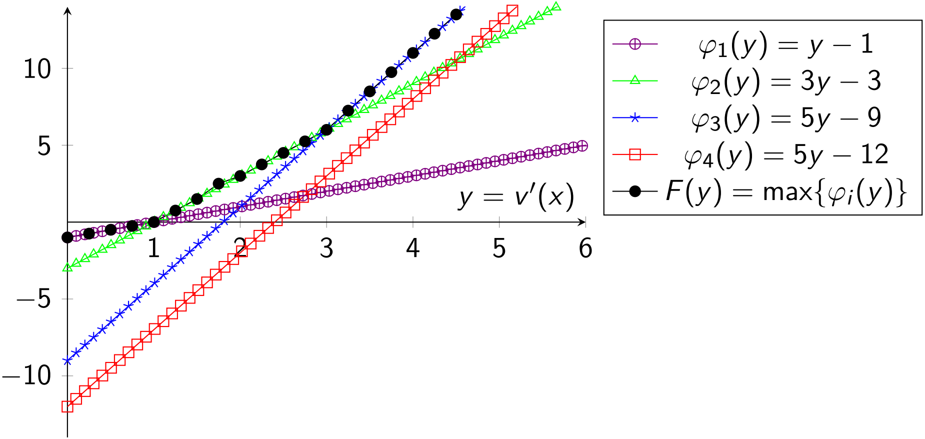
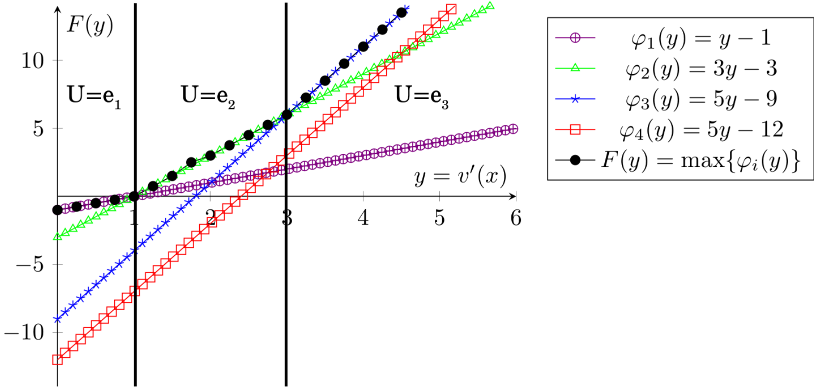
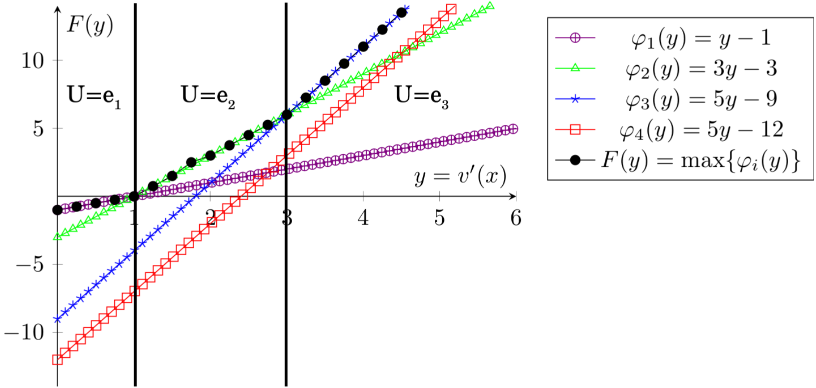
$$\Rightarrow F(y) = \max_u g(u, y) = \max_i \varphi_i(y), \qquad \varphi_i(y) = \theta_i y - c_i \mu_i, \quad y \in \mathbb{R}_+$$

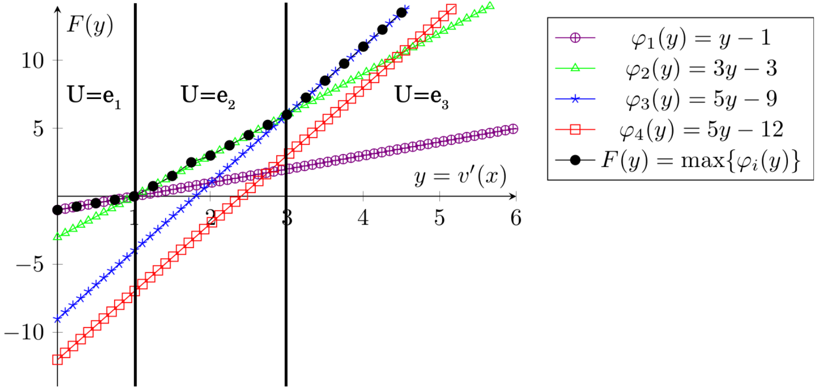


10
5
−5
−10
1
2
3
4
5
6
$y = v'(x)$
$\varphi_1(y) = y - 1$
$\varphi_2(y) = 3y - 3$
$\varphi_3(y) = 5y - 9$
$\varphi_4(y) = 5y - 12$
$F(y) = \max\{\varphi_i(y)\}$


F(y)
10
5
−5
−10
U=e1
U=e2
U=e3
y = v′(x)
1
2
3
4
5
6
φ1(y) = y − 1
φ2(y) = 3y − 3
φ3(y) = 5y − 9
φ4(y) = 5y − 12
F(y) = max{φi(y)}



- 3 different intervals for v': $[0, 1), [1, 3), [3, \infty)$



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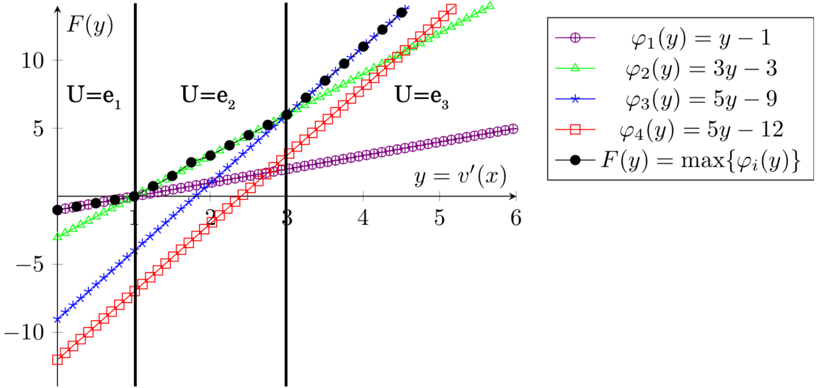
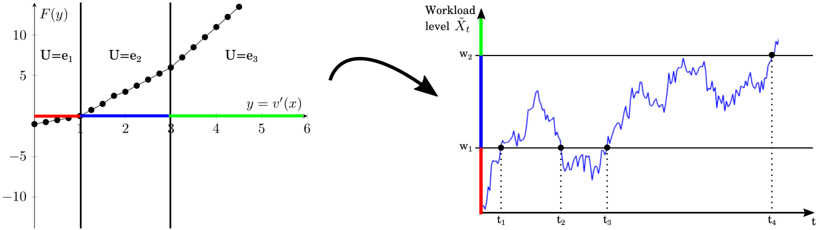
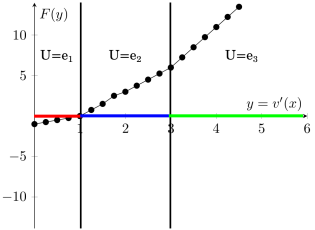
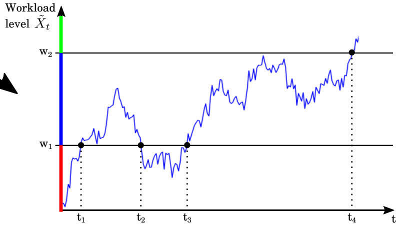


- 3 different intervals for v': $[0,1), [1,3), [3,\infty)$
- On each interval the control process satisfy $U = e_i$
  $\rightarrow$ all the workload remains in the i-th class.

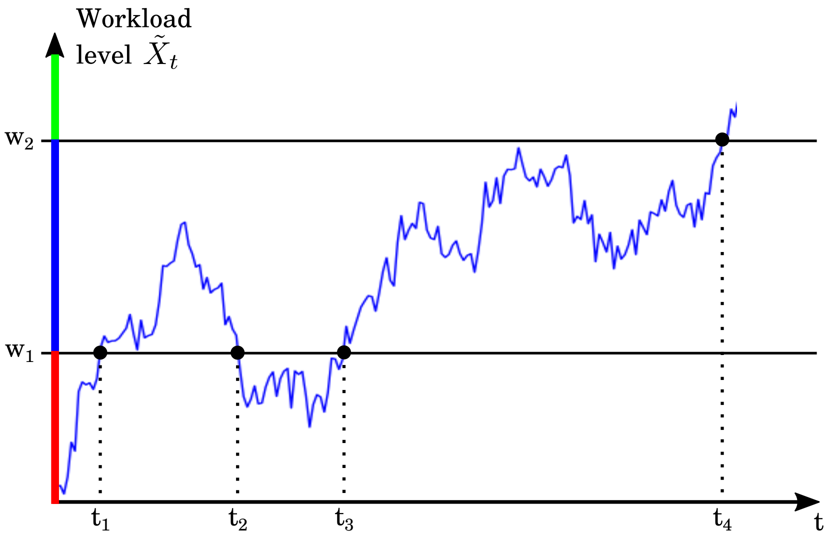


- v is nondecreasing and convex





- v is nondecreasing and convex
- The intervals for v' correspond to interval for $\tilde{X}$ - $[0, w_1), [w_1, w_2), [w_2, \infty)$



Ex. Below w_1 ($[0, t_1)$, $[t_2, t_3)$,...): $U = e_1 \rightarrow$ least priority to class 1.

$\mathcal{K}$ is a minimal subset of $\mathcal{I}$ such that $\max_{k\in\mathcal{K}} \varphi_k(y) = \max_{i\in\mathcal{I}} \varphi_i(y)$

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1. Non-interruptible, non-idling policy that allows no processor sharing.
2. At time t, the server becomes available and some of the buffers are non-empty, the current value of the scaled workload $\tilde{X}^n_t$ is computed and the **least priority class** k is set to be the unique i for which $\tilde{X}^n_t \in L_i$.

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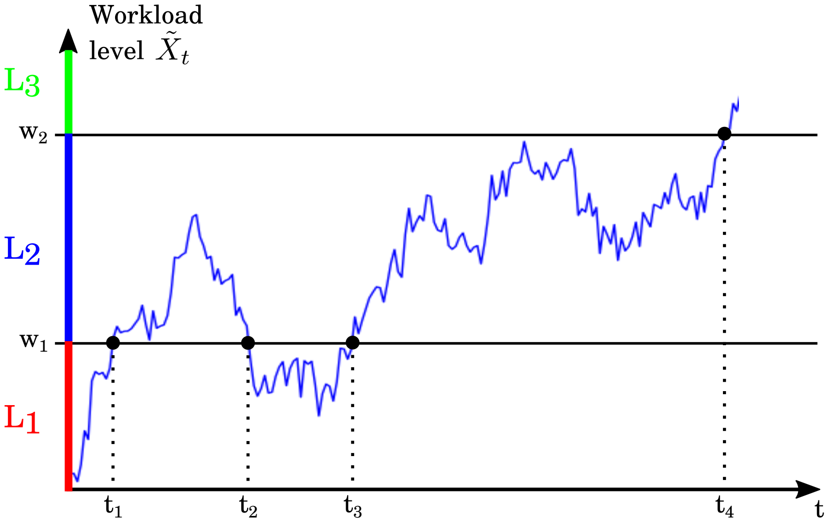
3. The customer to be served is then picked according to the ordering

$$(k+1, k+2, \dots, I, 1, 2, \dots, k).$$



$\mathcal{I} = \{1, 2, 3, 4\}$, $\mathcal{K} = \{1, 2, 3\}$, $c_1\mu_1 \leq c_2\mu_2 \leq c_3\mu_3$

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Theorem

Let v denote the solution of the Bellman equation. Then
i. The limit value is determined by the function v. In particular,

$$\lim_{n\to\infty} \hat{V}^n = v(m' x_0).$$

ii. The control process $B^{n,*}$ described above is AO, that is,

$$\lim_{n\to\infty} \hat{J}^n(B^{n,*}) = v(m' x_0).$$

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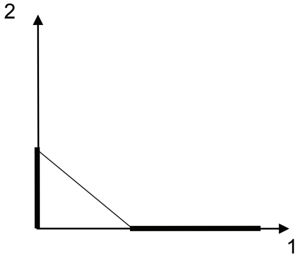
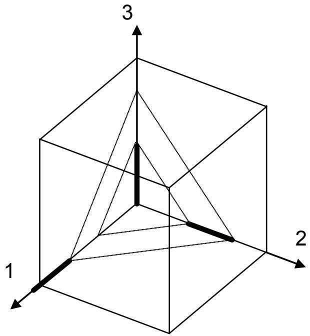
   Under the control process $B^{n,*}$ described earlier
   $$\limsup_{n\to\infty} \hat{J}^n(B^{n,*}) \leq v(m'x_0).$$


2
1


3
2
1

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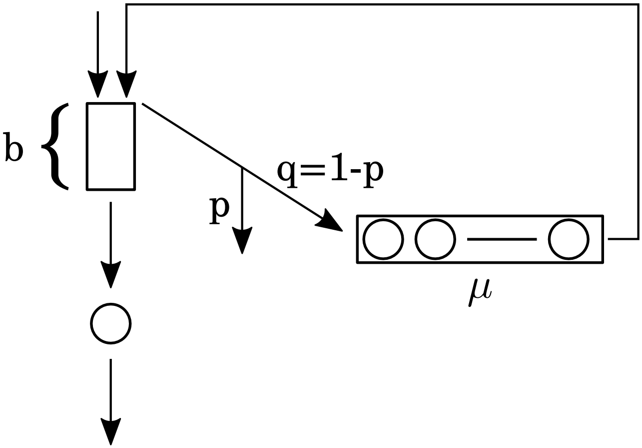
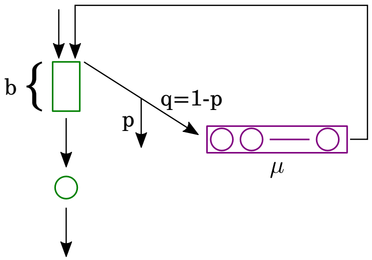
  2. Show that the time spent around the discontinuity points w_i is small $\rightarrow$ can be neglected.


b
q=1-p
p
μ

2 **stations:** main and retrial.



2 **stations:** main and retrial.

- Main station:
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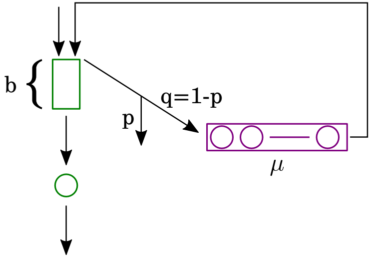

The Model

2 **stations:** main and retrial.

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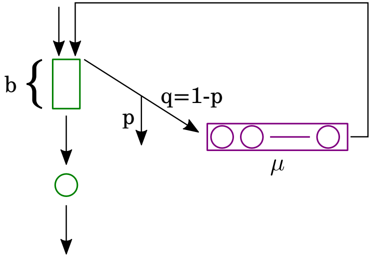
  - Reject incoming customers when buffer full.



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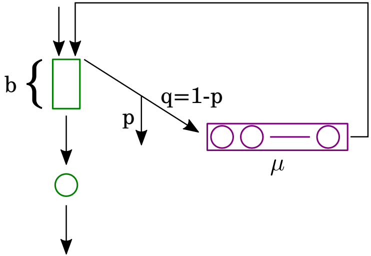

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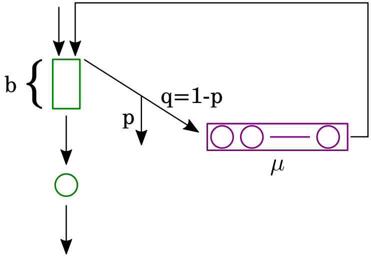

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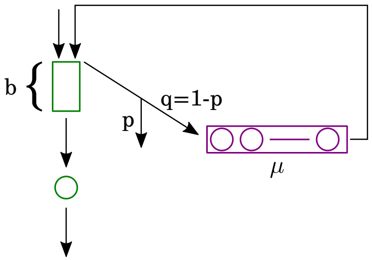

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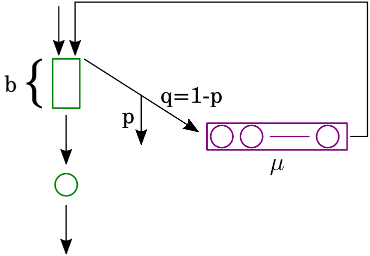

- **First Problem:** Diffusion limits.
- **Second Problem:** Optimizing b with cost

$$\mathbb{E}\int_0^\infty e^{-\alpha t}(c_1 \underbrace{X(t)}_{\text{system state}} dt + c_2 d \underbrace{C(t)}_{\text{rejections count}})$$

Related Work

1. Avi Mandelbaum, William A Massey, and Martin I Reiman. Strong approximations for markovian service networks.
Queueing Systems, 30(1):149–201, 1998
2. GI Falin and JR Artalejo. Approximations for multiserver queues with balking/retrial discipline.
OR Spectrum, 17(4):239–244, 1995

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- The diffusion model is characterize using the Skorokhod Map on $[0, b]$.

Definition

The Skorokhod Problem on $[0, b]$ (Tanaka '79)

Given $\psi \in \mathcal{D}[0, \infty)$, find $(\phi, \eta_l, \eta_u) \in (\mathcal{D}[0, \infty))^3$, such that

1. $\phi(t) = \psi(t) + \eta_l(t) - \eta_u(t), \qquad \phi(t) \in [0, b] \qquad \text{for all } t.$
2. η_l, η_u are non negative and non decreasing and one has

$$\int_{[0,\infty)} \mathbb{I}_{\{\phi(t)>0\}} d\eta_l(t) = 0, \qquad \int_{[0,\infty)} \mathbb{I}_{\{\phi(t)<b\}} d\eta_u(t) = 0.$$

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- Thus $(\phi, \eta_l, \eta_u) = \Gamma_{0,b}(\psi)$.

$$\begin{cases} (X, L, C) = \Gamma_{0,b}(Z) \\ Z(t) = x + \underbrace{W(t)}_{\text{BM}} + \int_0^t \mu R(s)ds \\ R(t) = r + qC(t) - \int_0^t \mu R(s)ds. \end{cases}$$

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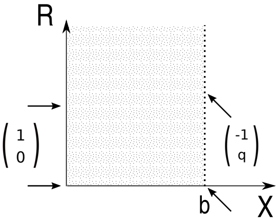
Reflection vector field:



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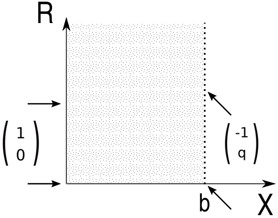

Two dimensional reflected diffusion process which is degenerate in the second diffusion coefficient

Theorem

As $n \to \infty$,

$$(\hat{X}^n, \hat{R}^n, \hat{L}^n, \hat{C}^n) \Rightarrow (X, R, L, C).$$

- **Cost:**

$$\mathbb{E}\Big(\int_0^\infty e^{-\alpha t}(c_1 X(t)dt + c_2 dC(t))\Big)$$

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- **Value:**

$$V(x) = \inf_b \mathbb{E}\Big(\int_0^\infty e^{-\alpha t}(c_1 X(t)dt + c_2 dC(t))\Big)$$

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- For general $\mu \in (0, \infty)$ - simulation.

- Harrison Taksar '83.

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- Controls the buffer size to minimize the same cost.

$$\begin{cases} (\tilde{X}, \tilde{L}, \tilde{C}) = \Gamma_{0,b}(x + W), & \text{where } W \text{ is a BM} \\ J^b_{\mathrm{HT}}[x; c_1, c_2] = \mathbb{E} \int_0^\infty e^{-\alpha t}[c_1 \tilde{X}_t + c_2 \tilde{C}_t] dt, \end{cases}$$

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- Has a unique optimal buffer size: $b_{\mathrm{HT}}[c_1, c_2]$.
- The value function

$$v[x; c_1, c_2] = \inf_{b \in (0,\infty)} J^{b}_{\mathrm{HT}}[x; c_1, c_2].$$

is the unique solution to a 1-dimensional Bellman equation:

$$\begin{cases} \frac{1}{2}\sigma^2 f'' + \hat{y} f' - \alpha f + c_1 = 0, \quad \text{in } (0, b_0), \\ \\ f'(0) = 0, \qquad f'(b_0) = \frac{c_2}{\alpha}. \end{cases}$$

(unknowns: f and b_0)

Theorem

One has

$$\liminf_{\mu\to 0}\liminf_{n\to\infty} V^{n,\mu} = \limsup_{\mu\to 0}\limsup_{n\to\infty} V^{n,\mu} = v[x; c_1, c_2],$$

and $b_{\mathrm{HT}}[c_1, c_2]$ *is an AO scaled buffer size, namely, with* $b = b_{\mathrm{HT}}[c_1, c_2]$,

$$\liminf_{\mu\to 0}\liminf_{n\to\infty} J^{n,\mu,b} = \limsup_{\mu\to 0}\limsup_{n\to\infty} J^{n,\mu,b} = v[x; c_1, c_2].$$

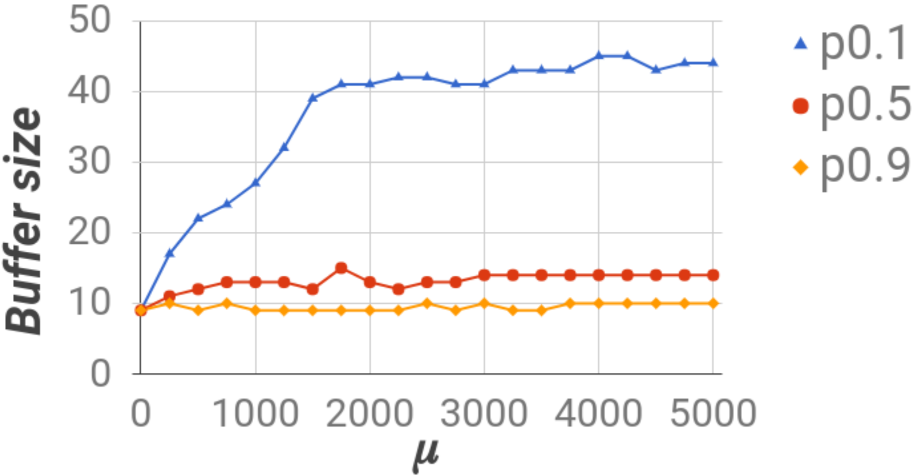
Theorem

One has

$$\liminf_{\mu\to\infty} \liminf_{n\to\infty} V^{n,\mu} = \limsup_{\mu\to\infty} \limsup_{n\to\infty} V^{n,\mu} = v[x; c_1, \frac{c_2}{p}].$$

Moreover, $b_{\mathrm{HT}}[c_1, \frac{c_2}{p}]$ *is an AO scaled buffer size, in the sense that, with* $b = b_{\mathrm{HT}}[c_1, \frac{c_2}{p}]$,

$$\liminf_{\mu\to\infty} \liminf_{n\to\infty} J^{n,\mu,b} = \limsup_{\mu\to\infty} \limsup_{n\to\infty} J^{n,\mu,b} = v[x; c_1, \frac{c_2}{p}].$$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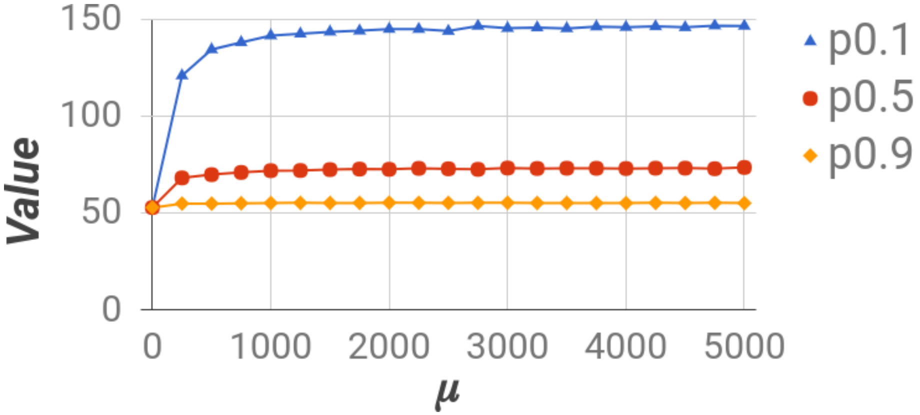
p0.9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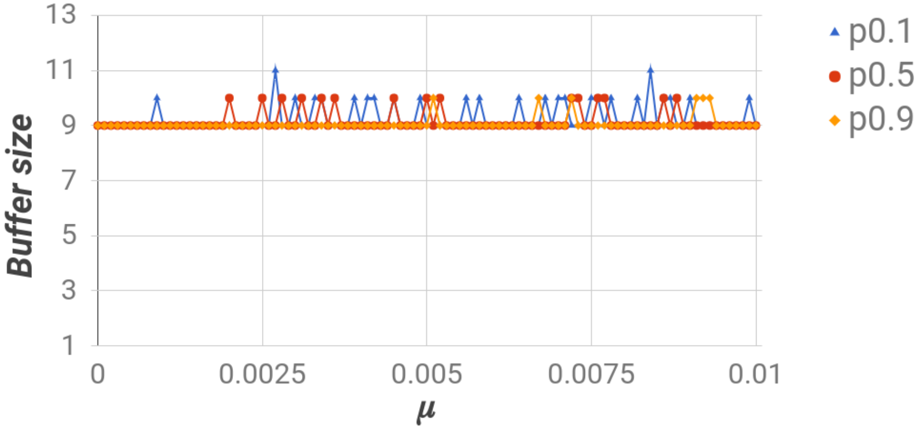

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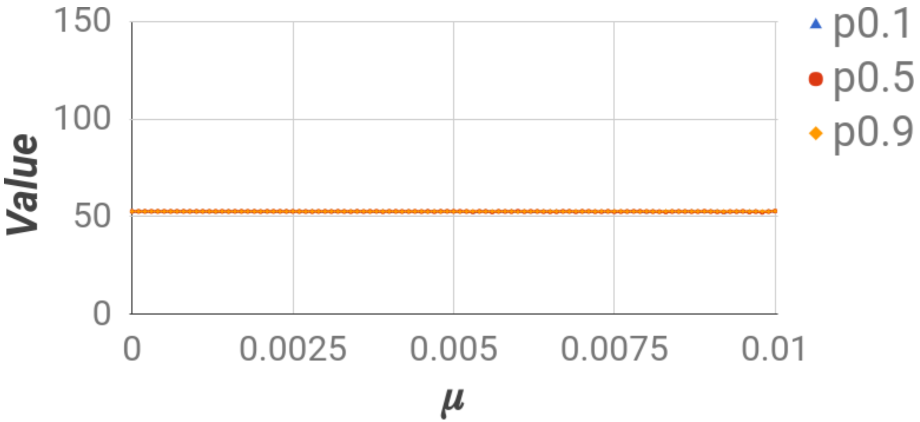
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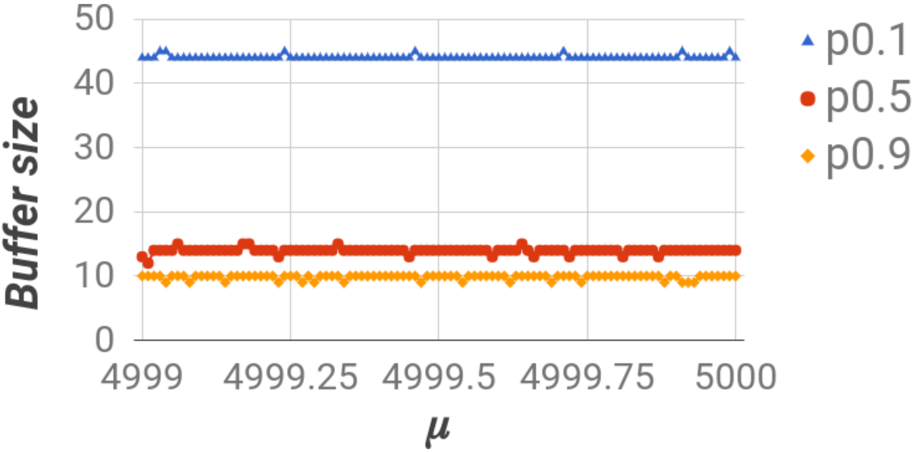
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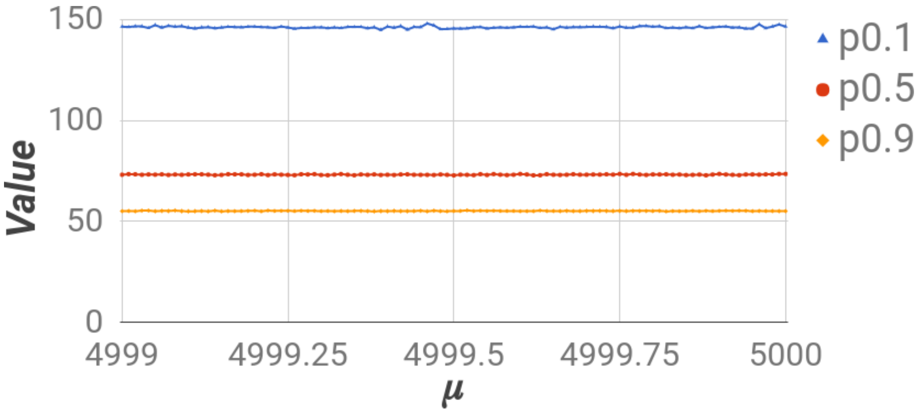
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