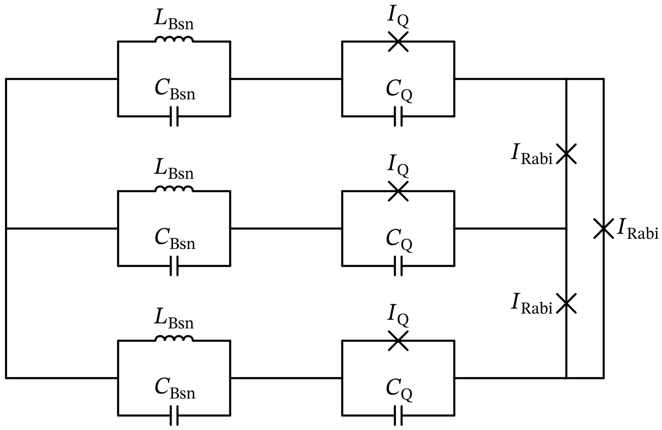

FIG. 1. Superconducting circuit implementation of the 2-mode $\mathbb{Z}_3$ Rabi model $\hat{H}_{\rm R2}$. The LC circuits ($L_{\rm B}, C_{\rm B}$) host the boson modes; the charge qubits ($I_{\rm Q}, C_{\rm Q}$) correspond to the qubit degrees of freedom; Josephson junctions $I_{\rm Rabi}$ are responsible for the interaction term in the qubit-boson ring. Altogether, the circuit is described by the Hamiltonian (**??**).

A. Superconducting circuit implementation

The obvious choice for boson modes is an LC circuit. For the qubit (σ_i) we consider superconducting charge qubits [**? ? ? ?**], because we want the qubit eigenstate to be a charge state. More extensive discussion on the charge qubits we use is provided in App. **??**. Here, we only want to comment that the qubit eigenstates are $|0\rangle, |1\rangle$ defined by $\hat{n}|0\rangle = 0$, $\hat{n}|1\rangle = |1\rangle$, where $\hat{n}$ is the capacitor charge operator. Below, to denote a linear span of these states we use $\mathcal{V}_k = \mathrm{Span}\{|0\rangle_k, |1\rangle_k\}$ where k is the number of the qubit.

Fig. 1 shows a superconducting realization of the three-site qubit–boson ring. Each horizontal branch of the circuit (Fig. 1) correspond to site $i = 0, 1, 2$ of the qubit-boson ring. The ith boson and qubit are implemented by the LC circuit and SC qubit on the ith branch respectively. The JJs on the on the vertical segments of the circuit on the right of Fig. 1 are responsible for the interaction term in H_{QB} (**??**) [**? ? ? ? ?**]. On Fig. 1 is, the characteristic parameters of every circuit element are also shown: LC circuit's conductor and inductor have capacitance $C_{\rm B}$ and inductance $L_{\rm B}$ respectively. The qubit's JJ and capacitance have critical current $I_{\rm Q}$ and capacitance $C_{\rm Q}$ respectively. The coupling JJs on the right have critical current $I_{\rm R}$. For clarity, we omit any elements that are necessary for an actual experimental implementation, e.g., read-out resonators, flux-bias lines, and filtering components.

Next, we describe why this circuit models the Hamiltonian (**??**). Without the coupling JJs, the LC circuits and qubits are independent from each other and are described by a non-interacting Hamiltonian:

$$\hat{H} = \sum_{i=0}^{2} \hat{H}_{LC,i}(\phi_i, q_i) + \sum_{i=0}^{2} \hat{H}_{Q,i}(\varphi_i, n_i), \qquad (1)$$

where ϕ_i, q_i are the magnetic flux and the capacitor charge of the ith LC circuit, while φ_i, n_i are the JJsuperconducting phase and capacitor charge of the qubit. The LC circuit Hamiltonian is obviously $H_{LC,i} = q_i^2/(2C) + \phi_i^2/(2L)$. Meanwhile, we leave the qubit Hamiltonian $\hat{H}_{Q,i}$ generic to allow different qubit types. However, we require the qubit to be a charge qubit and satisfy the following properties:

$$\hat{H}_Q\Big|_{\mathcal{V}} = \begin{pmatrix} \langle 0_i|\,\hat{H}_Q\,|0_i\rangle & \langle 0_i|\,\hat{H}_Q\,|1_i\rangle \\ \langle 1_i|\,\hat{H}_Q\,|0_i\rangle & \langle 1_i|\,\hat{H}_Q\,|1_i\rangle \end{pmatrix} = \epsilon\sigma_z, \qquad (2)$$

where the qubit states $|0_i\rangle, |1_i\rangle$ are the charge operator eigenstates $\hat{q}_i\,|0_i\rangle = Q\,|1_i\rangle$, $\hat{q}_i\,|1\rangle_i = (Q+1)\,|1\rangle_i$ with $Q \in \mathbb{Z}$. $\mathcal{V}_i = \mathrm{Span}\{|0_i\rangle, |1_i\rangle\}$ is a qubit Hilbert space, i.e, a computational subspace of the full Hilbert space of the qubit system $\hat{H}_{Q,i}$. Due to this property, the operator $e^{i\varphi_i}$ acts on the qubit subspace $\mathcal{V}_i$ as a raising operator:

$$e^{i\hat{\varphi}_i}\Big|_{\mathcal{V}_i} = \begin{pmatrix} \langle 0_i|\,e^{i\hat{\varphi}_i}\,|0_i\rangle & \langle 0_i|\,e^{i\hat{\varphi}_i}\,|1_i\rangle \\ \langle 1_i|\,e^{i\hat{\varphi}_i}\,|0_i\rangle & \langle 1_i|\,e^{i\hat{\varphi}_i}\,|1_i\rangle \end{pmatrix} = \sigma^+, \qquad (3)$$

because $[e^{i\phi}, n] = e^{i\phi}$.

If we now connect the ith and $(i+1)$th circuit branches with a JJ, it creates a cycle in the circuit. As a result, the fluxes and the superconducting phases have to satisfy a requirement:

$$\hat{\phi}_i + \hat{\varphi}_i + \hat{\varphi}_R - \hat{\varphi}_{i+1} - \hat{\phi}_{i+1} = 2\pi N, \text{ where } N \in \mathbb{Z}. \qquad (4)$$

In other words, the superconducting phase ϕ_R of the coupling JJ is not an independent degree of freedom. Therefore, for the corresponding term in the Hamiltonian we obtain:

$$\hat{V}_{\text{SC QB},j} = I_{\rm R}\cos\Big(\hat{\phi}_R\Big) = I_{\rm R}\cos\Big(\hat{\phi}_j + \hat{\varphi}_j - \hat{\phi}_{j+1} - \hat{\varphi}_{j+1}\Big). \qquad (5)$$

As we are only interested in qubit states $|0\rangle_k$, $|1\rangle_k$, we restrict the JJ term $\hat{V}_{\text{SC QB},j}$ to the qubit Hilbert subspace $\mathcal{V}_j \otimes \mathcal{V}_{j+1}$:

$$\begin{aligned} &\hat{V}_{\text{SC QB},j}\Big|_{\mathcal{V}_j\otimes\mathcal{V}_{j+1}} = \\ &= \frac{I_{\rm Rabi}}{2}\left(e^{i(\hat{\varphi}_j - \hat{\varphi}_{j+1})}e^{i(\hat{\phi}_j - \hat{\phi}_{j+1})} + h.c.\right)\Big|_{\mathcal{V}_j\otimes\mathcal{V}_{j+1}} \\ &= \frac{I_{\rm Rabi}}{2}\left(e^{i(\hat{\varphi}_j - \hat{\varphi}_{j+1})}\sigma_j^-\sigma_{j+1}^+ + h.c.\right). \end{aligned} \qquad (6)$$

Here we used the fact that $e^{\pm i\hat{\phi}}$ acts as a raising/lowering operator for the charge qubit: $e^{i\hat{\phi}}|_{\mathcal{V}} = \sigma^+$. Hence the need for charge qubits rather than flux or phase qubits.

The Josephson junction coupling between two pairs of an LC circuits and a charge qubits gives us precisely the interaction term we wanted. As a result, we conclude that the system depicted in Fig. 1 is described by the Hamiltonian (**??**). The qubit-boson ring couplings can be expressed by the circuit parameters:

$$\begin{aligned}&\epsilon = \frac{\delta}{2C_Q},\, \Omega_{QB} = \left(\sqrt{L_{\mathrm{B}} C_{\mathrm{B}}}\right)^{-1/2}, \\ &m_{QB} = C_{\mathrm{B}},\, g = \frac{I_{\mathrm{Rabi}}}{2}.\end{aligned} \tag{7}$$

For details of the ϵ computation, see App. **??**. Finally, applying Sec. **??** argument we deduce that the circuit is described by the $\mathbb{Z}_3$ Rabi model.