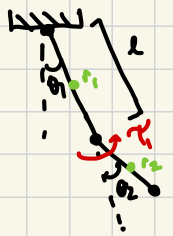
2-dof Acrobot



1) Derive Lagrangian

$\hookrightarrow L = T - V$

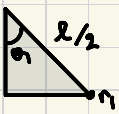
Kinetic Energy: $\frac{1}{2}mv^2$

Potential Energy: mgh

$\rightarrow T = \sum \frac{1}{2} J\omega^2$ where $I = \frac{1}{2}mr^2$

$= \sum \frac{1}{2} \dot{x} I \dot{x}^T$

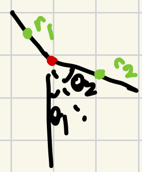
Link 1



$\Rightarrow \vec{r}_1 = \begin{bmatrix} l/_2 \sin\theta_1 \\ -l/_2 \cos\theta_1 \end{bmatrix}$

CoM of Link 1

Link 2



$\vec{r}_2 = \begin{bmatrix} l_1 \sin\theta_1 + l_2/_2 \sin(\theta_1+\theta_2) \\ -l_1\cos\theta_1 - l_2/_2 \cos(\theta_1+\theta_2) \end{bmatrix}$

Lagrangian Derivation

Kinetic

$T = T_1 + T_2 + T_3 \Rightarrow T_1 = \underbrace{\frac{1}{2}mv^2}_{\text{translational}} + \underbrace{\frac{1}{2}I\omega^2}_{\text{rotational}} \Rightarrow$ Link 1)

$\dot{r}_1 = \begin{bmatrix} l/_2 \cos\theta_1 \\ l/_2 \sin\theta_1 \end{bmatrix} \dot{\theta}_1$ $\quad T_1 = \frac{1}{2}\dot{r}^T m \dot{r} + \frac{1}{2} J \dot{\theta}_1^2$

$= \frac{1}{2} m \dot{\theta}_1^2 (l^2/4 \cos^2\theta_1 + l^2/4 \sin^2\theta_1) + \frac{1}{2} I \dot{\theta}_1^2$

$= \frac{1}{8} m \dot{\theta}_1^2 l^2 + \frac{1}{2}\left(\frac{1}{12} m l^2\right) \dot{\theta}_1^2$

$\frac{1}{8} m \dot{\theta}_1^2 l_1^2 +$

$= \frac{1}{8} m \dot{\theta}_1 l^2 + \frac{1}{24} m l^2 \dot{\theta}_1^2$

$= \boxed{\frac{1}{6} m_1 l_1^2 \dot{\theta}_1^2}$

$\vec{r}_2 = \begin{bmatrix} l_1 \sin\theta_1 + l_2/_2 \sin(\theta_1+\theta_2) \\ -l_1\cos\theta_1 - l_2/_2 \cos(\theta_1+\theta_2) \end{bmatrix}$

Link 2) $\dot{r}_2 = \begin{bmatrix} (l_1 \cos\theta_1)\dot{\theta}_1 + [l_2/_2 \cos(\theta_1+\theta_2)](\dot{\theta}_1+\dot{\theta}_2) \\ -l_1 \sin\theta_1 \dot{\theta}_1 - [l_2/_2 \sin(\theta_1+\theta_2)](\dot{\theta}_1+\dot{\theta}_2) \end{bmatrix}$

$\dot{r}_2 = \begin{bmatrix} l_1\cos\theta_1 \dot{\theta}_1 \\ -l_1 \sin\theta_1 \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} (\frac{1}{2} l_2 \cos(\theta_1+\theta_2))(\dot{\theta}_1+\dot{\theta}_2) \\ (-\frac{1}{2} l_2 \sin(\theta_1+\theta_2))(\dot{\theta}_1+\dot{\theta}_2) \end{bmatrix}$

Use: $(\vec{a}+\vec{b})^2 = \vec{a}^2 + 2(\vec{a}\cdot\vec{b}) + \vec{b}^2$

$\dot{r}_2^2 = l_1^2\cos^2\theta_1 \dot{\theta}_1^2 + l_1^2 \sin^2\theta_1 \dot{\theta}_1^2 +$
$2(\frac{1}{2} l_1 l_2 \cos\theta_1 \cos(\theta_1+\theta_2) \dot{\theta}_1 (\dot{\theta}_1+\dot{\theta}_2) + \frac{1}{2} l_1 l_2 \sin\theta_1 \sin(\theta_1+\theta_2) \dot{\theta}_1 (\dot{\theta}_1+\dot{\theta}_2)) +$
$\frac{1}{4} l_2^2 \cos^2(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)^2 + \frac{1}{4} l_2^2 \sin^2(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)^2$

$\dot{r}_2^2 = l_1^2 \dot{\theta}_1^2 + l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_2) + \frac{1}{4} l_2^2 (\dot{\theta}_1+\dot{\theta}_2)^2$

$T_2 = \frac{1}{2}\dot{r}^T m \dot{r} + \frac{1}{2} J\omega^2 \Rightarrow \frac{1}{2} m (l_1^2 \dot{\theta}_1^2 + l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_2) + \frac{1}{4} l_2^2 (\dot{\theta}_1+\dot{\theta}_2)^2) +$
$\frac{1}{2}(1/12\, m l_2^2)(\dot{\theta}_1+\dot{\theta}_2)^2$

$= \boxed{\frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_2) + \frac{1}{6} m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2}$

Potential Energy

$V_1 = mgh = m_1 g\left(-\frac{l}{2}\cos\theta_1\right)$ $\quad V_2 = m_2 g\left(-l_1\cos\theta_1 - l_2/2\cos(\theta_1+\theta_2)\right)$

$\hookrightarrow L = T - V \Rightarrow$

$$\boxed{\begin{aligned}&\tfrac{1}{2}m_2 l_1^2\dot\theta_1^2 + \tfrac{m_2}{2}l_1 l_2\dot\theta_1(\dot\theta_1+\dot\theta_2)\cos(\theta_2) + \tfrac{1}{6}m_2 l_2^2(\dot\theta_1+\dot\theta_2)^2 + \tfrac{1}{6}m_1 l_1^2\dot\theta_1^2 + \\ &\tfrac{1}{2}m_1 g l_1\cos\theta_1 + m_2 g\left(l_1\cos\theta_1 + l_2/2\cos(\theta_1+\theta_2)\right)\end{aligned}}$$

Euler-Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_i}\right) - \frac{\partial L}{\partial\theta_i} = \tau_i$$

$\rightarrow$ Non-conservative Force

$\Downarrow$

Set $\theta_1 = 0$ for natural eqns.

For θ_1

$$\frac{\partial L}{\partial\theta_1} = -m_1 g\frac{l_1}{2}\sin\theta_1 + m_2 g\left(-l_1\sin\theta_1 - l_2/2\sin(\theta_1+\theta_2)\right)$$

$$\frac{\partial L}{\partial\dot\theta_1} = m_2 l_1^2\dot\theta_1 + \tfrac{1}{2}m_2 l_1 l_2(\dot\theta_1+\dot\theta_2)\cos\theta_2 + \tfrac{1}{2}m_2 l_1 l_2\dot\theta_1\cos\theta_2 + \tfrac{1}{3}m_2 l_2^2(\dot\theta_1+\dot\theta_2) + \tfrac{1}{3}m_1 l_1^2\dot\theta_1$$

$$= \left(\tfrac{1}{3}m_1 l_1^2 + m_2 l_1^2\right)\dot\theta_1 + \tfrac{1}{2}m_2 l_1 l_2\cos\theta_2(2\dot\theta_1+\dot\theta_2) + \tfrac{1}{3}m_2 l_2^2(\dot\theta_1+\dot\theta_2)$$

$\downarrow \frac{d}{dt}$

$$= \left(\tfrac{1}{3}m_1 l_1^2 + m_2 l_1^2\right)\ddot\theta_1 - \tfrac{1}{2}m_2 l_1 l_2\sin\theta_2\,\dot\theta_2(2\dot\theta_1+\dot\theta_2) + \tfrac{1}{2}m_2 l_1 l_2\cos\theta_2(2\ddot\theta_1+\ddot\theta_2) + \tfrac{1}{3}m_2 l_2^2(\ddot\theta_1+\ddot\theta_2)$$

θ_1

$$\hookrightarrow \boxed{\begin{aligned}&\left(\tfrac{1}{3}m_1 l_1^2 + m_2 l_1^2\right)\ddot\theta_1 - \tfrac{1}{2}m_2 l_1 l_2\sin\theta_2\,\dot\theta_2(2\dot\theta_1+\dot\theta_2) + \tfrac{1}{2}m_2 l_1 l_2\cos\theta_2(2\ddot\theta_1+\ddot\theta_2) + \tfrac{1}{3}m_2 l_2^2(\ddot\theta_1+\ddot\theta_2) + \\ &m_1 g\frac{l_1}{2}\sin\theta_1 + m_2 g l_1\sin\theta_1 + m_2 g\frac{l_2}{2}\sin(\theta_1+\theta_2) = \tau_i = 0\end{aligned}}$$

acrobot doesn't actuate θ_1

For θ_2

$$\frac{\partial L}{\partial\theta_2} = -\frac{m_2}{2}l_1 l_2\dot\theta_1(\dot\theta_1+\dot\theta_2)\sin(\theta_2) - m_2 g\frac{l_2}{2}\sin(\theta_1+\theta_2)$$

$$\frac{\partial L}{\partial\dot\theta_2} = \frac{m_2}{2}l_1 l_2\dot\theta_1\cos(\theta_2) + \tfrac{1}{3}m_2 l_2^2(\dot\theta_1+\dot\theta_2)$$

$\Downarrow \frac{d}{dt}$

$$\frac{m_2}{2}l_1 l_2\ddot\theta_1\cos\theta_2 - \frac{m_2}{2}l_1 l_2\dot\theta_1\sin(\theta_2)\dot\theta_2 + \tfrac{1}{3}m_2 l_2^2(\ddot\theta_1+\ddot\theta_2)$$

$$\hookrightarrow \frac{m_2}{2}l_1 l_2\ddot\theta_1\cos\theta_2 - \frac{m_2}{2}l_1 l_2\dot\theta_1\sin(\theta_2)\dot\theta_2 + \tfrac{1}{3}m_2 l_2^2(\ddot\theta_1+\ddot\theta_2) +$$
$$\frac{m_2}{2}l_1 l_2\dot\theta_1(\dot\theta_1+\dot\theta_2)\sin(\theta_2) + m_2 g\frac{l_2}{2}\sin(\theta_1+\theta_2)$$

$= \frac{m_2}{2} l_1 l_2 \ddot{\theta}_1 \cos\theta_2 - \frac{m_2}{2} l_1 l_2 \dot{\theta}_1 \sin(\theta_2) \dot{\theta}_2 + \frac{1}{3} m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) +$

$\frac{m_2}{2} l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_2) + m_2 g \frac{l_2}{2} \sin(\theta_1 + \theta_2)$

$$\Rightarrow \boxed{\frac{m_2}{2} l_1 l_2 \ddot{\theta}_1 \cos\theta_2 + \frac{m_2}{2} l_1 l_2 \dot{\theta}_1^2 \sin\theta_2 + \frac{1}{3} m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 g \frac{l_2}{2} \sin(\theta_1 + \theta_2) = \tau_2}$$

↳ For joint 2 (θ_2)

↳ Manipulator Equation Form

$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$

$M(q)$ ↓ Inertia matrix; $C(q,\dot{q})$ ↓ Coriolis and Centripetal terms; $\tau_g(q)$ ↓ Torque to oppose gravity; Bu ↓ Control torques

$M(q) \Rightarrow$ inertia matrix $\Rightarrow$ how masses and geometry resist acceleration

$C(q,\dot{q})\dot{q} \Rightarrow$ Coriolis and Centripetal terms $\Rightarrow$ velocity-velocity couplings that can either help or fight motion depending on configuration

$\tau_g(q) \Rightarrow$ gravity terms → torques you'd have to apply to hold arm against gravity

$u \Rightarrow$ vector of joint torques → B: which joints to actuate

$\vec{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$

↳ $M(q) = \begin{bmatrix} \frac{1}{3} m_1 l_1^2 + m_2 l_1^2 + m_2 l_1 l_2 \cos\theta_2 + \frac{1}{3} m_2 l_2^2 & \frac{1}{2} m_2 l_1 l_2 \cos\theta_2 + \frac{1}{3} m_2 l_2^2 \\ \frac{1}{2} m_2 l_1 l_2 \cos\theta_2 + \frac{1}{3} m_2 l_2^2 & \frac{1}{3} m_2 l_2^2 \end{bmatrix}$

$C(q,\dot{q}) = \begin{bmatrix} -m_2 l_1 l_2 \sin\theta_2 \, \dot{\theta}_2 & -\frac{1}{2} m_2 l_1 l_2 \sin\theta_2 \, \dot{\theta}_2 \\ \frac{m_2}{2} l_1 l_2 \dot{\theta}_1 \sin\theta_2 & 0 \end{bmatrix}$

$\tau_g(q) = \begin{bmatrix} -m_1 g \frac{l_1}{2} \sin\theta_1 - m_2 g l_1 \sin\theta_1 - m_2 g \frac{l_2}{2} \sin(\theta_1 + \theta_2) \\ -m_2 g \frac{l_2}{2} \sin(\theta_1 + \theta_2) \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; u = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$