Exercises - answer

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 + 4x_2 + x_3 \\
\text{subject to:} \quad & 2x_1 + x_2 + x_3 \leq 10 \\
& x_1 + x_2 - x_3 \leq 4 \\
& 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 6,\ 1 \leq x_3 \leq 6
\end{aligned}
$$

Master Problem

$$\text{maximize} \quad z = \sum_{j=1}^{p_R} (c^\top v_j)\lambda_j \tag{1}$$

$$\text{subject to:} \quad \sum_{j=1}^{p_R} (A_1 v_j)\lambda_j \leq 10 \tag{2}$$

$$\sum_{j=1}^{p_R} (A_2 v_j)\lambda_j \leq 4 \tag{3}$$

$$\sum_{j=1}^{p_R} \lambda_j = 1 \tag{4}$$

Consider μ_1, μ_2 e ν the dual variables related to the constraints 2, 3 and 4 respectively. p_R are the columns of the restricted master problem.

Auxiliary Problem

$$
\begin{aligned}
\text{maximize} \quad & cr = (2 - 2\mu_1 - \mu_2)x_1 + (4 - \mu_1 - \mu_2)x_2 + (1 - \mu_1 + \mu_2)x_3 - \nu \\
\text{subject to:} \quad & 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 6,\ 1 \leq x_3 \leq 6
\end{aligned}
$$

Let $x_1 = x_2 = 0,\ x_3 = 1$ be the initial solution. Master problem for column 1:

$$
\begin{aligned}
\text{maximize} \quad & z = 1\lambda_1 \\
\text{subject to:} \quad & 1\lambda_1 \leq 10 \\
& -1\lambda_1 \leq 4 \\
& \lambda_1 = 1
\end{aligned}
$$

$\overline{z} = 1$, $\lambda_1 = 1$ e $\mu_1 = \mu_2 = \nu = 0$.

Auxiliary problem 1

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 + 4x_2 + 1x_3 \\
\text{subject to:} \quad & 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 6,\ 1 \leq x_3 \leq 6
\end{aligned}
$$

OF $= 38$, $x_1 = 4,\ x_2 = x_3 = 6$. UB $= 1 + 38 = 39$. Solving auxiliary problem for column 1, we have column 2, $v_2^\top = (4, 6, 6)$ and the coefficients for λ_2 at the objective function of the master problem of the second iteration are:

$$cv_2 = (2,4,1)^\top \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix} = 38,\ A_1 v_2 = (2,1,1)^\top \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix} = 20,\ A_2 v_2 = (1,1,-1)^\top \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix} = 4$$

Second iteration

$$
\begin{aligned}
\text{maximize} \quad & z = 1\lambda_1 + 38\lambda_2 \\
\text{subject to :} \quad & 1\lambda_1 + 20\lambda_2 \leq 10 \\
& -1\lambda_1 + 4\lambda_2 \leq 4 \\
& \lambda_1 + \lambda_2 = 1
\end{aligned}
$$

$\overline{z} = 18.526316,\ \lambda_1 = 0.526316,\ \lambda_2 = 0.473684,$
$\mu_1 = 1.947368,\ \mu_2 = 0,\ \nu = -0.947368.$

Auxiliary problem 2

$$
\begin{aligned}
\text{maximize} \quad & -1.8904736x_1 + 2.052632x_2 - 2.947368x_3 + 0.947368 \\
\text{subject to: } & 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 6,\ 1 \leq x_3 \leq 6
\end{aligned}
$$

OF $= 12.315792,\ x_1 = 0,\ x_2 = 6,\ x_3 = 1.$
UB $= 18.526316 + 12.315792 = 30.842108.$
Column 3 is: $v_3^\top = (0, 6, 1)$ and the coefficients of λ_3 at the objective function and at the constraints are, respectively:

$$
cv_3 = (2,4,1)^\top \begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix} = 25,\ A_1 v_3 = (2,1,1)^\top \begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix} = 7,\ A_2 v_3 = (1,1,-1)^\top \begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix} = 5.
$$

Third iteration

$$
\begin{aligned}
\text{maximize} \quad & z = 1\lambda_1 + 38\lambda_2 + 25\lambda_3 \\
\text{subject to:} \quad & 1\lambda_1 + 20\lambda_2 + 7\lambda_3 \leq 10 \\
& -1\lambda_1 + 4\lambda_2 + 5\lambda_3 \leq 4 \\
& \lambda_1 + \lambda_2 + \lambda_3 = 1
\end{aligned}
$$

$\overline{z} = 25.857143,\ \lambda_1 = 0.119048,\ \lambda_2 = 0.285714,\ \lambda_3 = 0.595238$
$\mu_1 = 1.214286,\ \mu_2 = 2.785714$ e $\nu = 2.571429$

Auxiliary Problem

$$
\begin{aligned}
\text{maximize} \quad & -3.214286x_1 + 0.571428x_3 - 2.571429 \\
\text{subject to: } & 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 6,\ 1 \leq x_3 \leq 6
\end{aligned}
$$

OF $=12.857139,\ x_1 = x_2 = 0,\ x_3 = 6.$
UB: 38.714282.
Column 4 is $v_4^\top = (0, 0, 6)$ and the coefficients for λ_4 at the objective function and at the constraints of the master problem are respectively:

$$
cv_4 = (2,4,1)^\top \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = 6,\ A_1 v_4 = (2,1,1)^\top \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = 6,\ A_2 v_4 = (1,1,-1)^\top \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = -6.
$$

Fourth iteration

$$
\begin{aligned}
\text{maximize} \quad & z = 1\lambda_1 + 38\lambda_2 + 25\lambda_3 + 6\lambda_4 \\
\text{subject to:} \quad & 1\lambda_1 + 20\lambda_2 + 7\lambda_3 + 6\lambda_4 \leq 10 \\
& -1\lambda_1 + 4\lambda_2 + 5\lambda_3 - 6\lambda_4 \leq 4 \\
& \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1
\end{aligned}
$$

$\overline{z} = 26.75,\ \lambda_1 = 0.0,\ \lambda_2 = 0.236111,\ \lambda_3 = 0.694444,\ \lambda_4 = 0.069444$
$\mu_1 = 1.125,\ \mu_2 = 1.625$ e $\nu = 9.0$

Auxiliary problem

$$\begin{aligned} &\text{maximize} \;\; cr = -1.875x_1 + 1.25x_2 + 1.5x_3 - 9 \\ &\text{subject to: } 0 \leq x_1 \leq 4,\; 0 \leq x_2 \leq 6,\; 1 \leq x_3 \leq 6 \end{aligned}$$

OF=7.5, $x_1 = 0$, $x_2 = 6$, $x_3 = 6$
UB: 34.25.
Column 5 is $v_4^\top = (0, 6, 6)$ and the coefficients for λ_5 at the objective function and at the constraints of the master problem are respectively:

$$cv_4 = (2,4,1)^\top \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix} = 30,\;\; A_1 v_4 = (2,1,1)^\top \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix} = 12,\;\; A_2 v_4 = (1,1,-1)^\top \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix} = 0.$$

Fifth iteration

$$\begin{aligned} \text{maximize } z = 1\lambda_1 + 38\lambda_2 + 25\lambda_3 + 6\lambda_4 + 30\lambda_5 \\ \text{subject to: } \;\; 1\lambda_1 + 20\lambda_2 + 7\lambda_3 + 6\lambda_4 + 12\lambda_5 \leq 10 \\ -1\lambda_1 + 4\lambda_2 + 5\lambda_3 - 6\lambda_4 + 0\lambda_5 \leq 4 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1 \end{aligned}$$

$\overline{z} = 28.0$, $\lambda_1 = 0.0$, $\lambda_2 = 0.0$, $\lambda_3 = 0.4$, $\lambda_4 = 0.0$, $\lambda_5 = 0.6$
$\mu_1 = 1.0$, $\mu_2 = 0.0$ e $\nu = 18.0$.

Auxiliary problem

$$\begin{aligned} &\text{maximize} \;\; 3x_2 - 18 \\ &\text{subject to: } 0 \leq x_1 \leq 4,\; 0 \leq x_2 \leq 6,\; 1 \leq x_3 \leq 6 \end{aligned}$$

OF=0, $x_1 = 0$, $x_2 = 6$, $x_3 = 1$.
UB: 28.0.

As UB $= \overline{z} = 28.0$ we are at the optimal solution.