Short Course on Column Generation - Part II

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Overview

Just remembering

- Last class we talked about the relationship between tighter formulations, smaller gaps and smaller branch and bound tree;
- Column generation works better for integer programming;
- One example is the generalized assignment problem (GAP) for which the relaxed restricted master problem (RMP) is tighter than the relaxation of the original problem, [1], p 317.

Branch and Price

- Branch and Price is a combination of Branch and Bound with column generation, [2];
- But there are some difficulties when applying the column generation, which is originally developed for linear programming problems, in integer programming problems;
- One of these problems is that the usual branching may destroy the structure of the pricing problem [1];
- If you want to read more about that subject, the suggestion is reading [3].

Branch and Price

- At Branch and Price we solve the integer Auxiliary Problem;
- As the Auxiliary Problem is as difficult to solve then the original problem, there are strategies used to solve many auxiliary problems of the literature, as the knapsack problem that will appear at our examples;
- The consequence is that there are specific Branch and Price algorithms adapted to many problems of the literature;
- Once we generate integer columns, we add these columns to the master problem until there is no column to add anymore;
- If we are in the case of no more columns to add and the solution of the master problem is not integer, we run a Branch and Bound at the master problem.

- We have m tasks to be assigned to n machines, $|m| \geq |n|$;

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Branch and Price - Generalized Assignment Problem (GAP)

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- Each task is assigned to exactly one machine;
- Each machine has its capacity constraint;

Branch and Price - Generalized Assignment Problem (GAP)

- We have m tasks to be assigned to n machines, $|m| \geq |n|$;
- Each task is assigned to exactly one machine;
- Each machine has its capacity constraint;
- Objective: maximize the profit assignment of the m tasks to the n machines.

Branch and Price - GAP

- We are going to use p_{ij} for the profit of assigning task i to machine j;
- w_{ij} is the amount of resource consumption of task i at machine j;
- d_j is the total resource of machine j;
- x_{ij} is the binary variable which indicates whether task i is assigned to machine j.

Branch and Price - GAP

$$
\begin{aligned}
\text{maximize} \quad & \sum_{1 \le i \le m} \sum_{1 \le j \le n} p_{ij} x_{ij} \\
\text{subject to:} \quad & \sum_{i \le j \le n} x_{ij} = 1, \qquad i = 1, ..., m \\
& \sum_{1 \le i \le m} w_{ij} x_{ij} \le d_j, \qquad j = 1, ..., n \\
& x_{ij} \in \{0, 1\}, \ \ i = 1, ..., m, \ \ j = 1, .., n
\end{aligned}
$$

Knapsack Problem

- Given a set of items, each item with a related weight (w) and value (p);
- Given a knapsack with a total weight capacity (d);
- The objective is to determine the number of each item to put at the knapsack respecting the total weight capacity of the knapsack;
- The knapsack problem is:

$$\begin{aligned} \text{maximize} \quad & \sum_{1 \le i \le m} \sum_{1 \le j \le n} p_j x_j \\ \text{subject to:} \quad & \sum_{1 \le i \le m} w_j x_j \le d, \qquad j = 1, \ldots, n \\ & x_j \in \{0, 1\}, \; j = 1, \ldots, m \end{aligned}$$

Branch and Price - GAP

- In order to rewrite GAP problem applying the Dantzig-Wolfe reformulation;
- Consider that the m entries of a column

$$y_j^k = (y_{1j}^k, y_{2j}^k, ..., y_{mj}^k) \tag{1}$$

  satisfy the knapsack constraint and also binary constraints of GAP problem.
- We are going to rewrite the GAP problem using these columns.

Branch and Price - GAP

Master Problem

$$
\begin{aligned}
\text{maximize} \quad & \sum_{1\le j\le n}\ \sum_{1\le k\le K_j} \Big(\sum_{1\le i\le m} p_{ij} y_{ij}^k\Big)\lambda_j^k \\
\text{subject to:} \quad & \sum_{1\le j\le n}\ \sum_{1\le k\le K_j} y_{ij}^k \lambda_j^k = 1, \qquad i = 1, \ldots, m \\
& \sum_{1\le k\le K_j} \lambda_j^k = 1,\ \ j = 1, \ldots, n \\
& \lambda_j^k \in \{0,1\},\ \ j = 1, \ldots, n, k = 1, \ldots, K_j
\end{aligned}
$$

Branch and Price - GAP

Auxiliary Problem

$$\begin{aligned} \text{maximize} \quad & \sum_i (p_{ij} - u_i) x_{ij} - \nu_j \\ \text{subject to:} \quad & \sum_i w_{ij} x_{ij} \leq d \\ & x_{ij} \in \{0, 1\}, \; i = 1, ..., m \end{aligned}$$

Numerical example GAP

This example is at [4].

$m = 3,\ n = 2,\ (d_1, d_2) = (11, 18)$

$$p_{ij} = \begin{pmatrix} 10 & 6 \\ 7 & 8 \\ 5 & 11 \end{pmatrix},\ w_{ij} = \begin{pmatrix} 9 & 5 \\ 6 & 7 \\ 3 & 9 \end{pmatrix}$$

Formulation

$$\begin{aligned} \text{maximize} \quad & 10x_{11} + 7x_{21} + 5x_{31} + 6x_{12} + 8x_{22} + 11x_{32} \\ \text{subject to:} \quad & 9x_{11} + 6x_{21} + 3x_{31} \leq 11 \\ & 5x_{12} + 7x_{22} + 9x_{32} \leq 18 \\ & x_{ij} \in \{0, 1\},\ i = 1, ..., 3,\ j = 1, 2. \end{aligned}$$

Numerical example GAP

For

$$9x_{11} + 6x_{21} + 3x_{31} \leq 11$$

which is related to the availability of machine 1, we have four ($= K_1$) possible results: $y_{i1}^k = (1,0,0), (0,1,0), (0,0,1), (0,1,1)$

- $(1,0,0)$ means task one at machine one, consuming 9 units of 11 units of resource available;
- $(0,1,0)$ means task two at machine one, consuming 6 units of 11 units of resource available;
- $(0,0,1)$ means task three at machine one, consuming 3 units of 11 units of resource available;
- $(0,1,1)$ means task 2 and 3 at machine one, consuming 9 units of 11 units of resource available.

Numerical example GAP

For

$$5x_{12} + 7x_{22} + 9x_{32} \leq 18$$

which is related to the availability of machine 2, we have six $(= K_2)$ possible results: $y_{i2}^k = (1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1)$

- $(1,0,0)$ means task one at machine two, consuming 5 units of 18 units of resource available;
- $(0,1,0)$ means task two at machine two, consuming 7 units of 18 units of resource available;
- $(0,0,1)$ means task three at machine two, consuming 9 units of 18 units of resource available;
- $(1,1,0)$ means task 1 and 2 at machine two, consuming 12 units of 18 units of resource available;
- $(0,1,1)$ means task 2 and 3 at machine two, consuming 16 units of 18 units of resource available;
- $(1,0,1)$ means task 1 and 3 at machine two, consuming 14 units of 18 units of resource available.

Numerical example GAP

So, for the first constraint of the master problem,

$$\sum_{1 \leq j \leq n} \sum_{1 \leq k \leq K_j} y_{ij}^k \lambda_j^k = 1, \qquad i = 1, ..., m$$

for $i = 1, ..., 3,\ j = 1, 2$ and we know that $K_1 = 1, ..., 4$ and $K_2 = 1, .., 6$, we have:

$$y_{11}^1\lambda_1^1 + y_{11}^2\lambda_1^2 + y_{11}^3\lambda_1^3 + y_{11}^4\lambda_1^4 + y_{12}^1\lambda_2^1 + y_{12}^2\lambda_2^2 + y_{12}^3\lambda_2^3 + y_{12}^4\lambda_2^4 + y_{12}^5\lambda_2^5 + y_{12}^6\lambda_2^6 = 1 \qquad (2)$$

$$y_{21}^1\lambda_1^1 + y_{21}^2\lambda_1^2 + y_{21}^3\lambda_1^3 + y_{21}^4\lambda_1^4 + y_{22}^1\lambda_2^1 + y_{22}^2\lambda_2^2 + y_{22}^3\lambda_2^3 + y_{22}^4\lambda_2^4 + y_{22}^5\lambda_2^5 + y_{22}^6\lambda_2^6 = 1 \qquad (3)$$

$$y_{31}^1\lambda_1^1 + y_{31}^2\lambda_1^2 + y_{31}^3\lambda_1^3 + y_{31}^4\lambda_1^4 + y_{32}^1\lambda_2^1 + y_{32}^2\lambda_2^2 + y_{32}^3\lambda_2^3 + y_{32}^4\lambda_2^4 + y_{32}^5\lambda_2^5 + y_{32}^6\lambda_2^6 = 1 \qquad (4)$$

Numerical example GAP

Taking the feasible solutions $y^k_{i1} = (1,0,0),(0,1,0),(0,0,1),(0,1,1)$ for machine 1 and $y^k_{i2} = (1,0,0),(0,1,0),(0,0,1),(1,1,0),(0,1,1),(1,0,1)$ for machine 2, we have:

$$y^1_{11} = 1,\ y^2_{11} = y^3_{11} = y^4_{11} = 0,\ y^1_{12} = 1,\ y^2_{12} = y^3_{12} = 0,\ y^4_{12} = 1,\ y^5_{12} = 0,\ y^6_{12} = 1,$$

$$y^1_{21} = 0,\ y^2_{21} = 1,\ y^3_{21} = 0,\ y^4_{21} = 1,\ y^1_{22} = 0,\ y^2_{22} = 1,\ y^3_{22} = 0,\ y^4_{22} = 1,\ y^5_{22} = 1,\ y^6_{22} = 0,$$

$$y^1_{31} = 0,\ y^2_{31} = 0,\ y^3_{31} = 1,\ y^4_{31} = 1,\ y^1_{32} = 0,\ y^2_{32} = 0,\ y^3_{32} = 1,\ y^4_{32} = 0,\ y^5_{32} = 1,\ y^6_{32} = 1.$$

So, the constraint (2) is:

$$\lambda^1_1 + 0 + 0 + 0 + \lambda^1_2 + 0 + 0 + \lambda^4_2 + 0 + \lambda^6_2 = 1.$$

The constraint (3) is:

$$0 + \lambda^2_1 + 0 + \lambda^4_1 + 0 + \lambda^2_2 + 0 + \lambda^4_2 + \lambda^5_2 + 0 = 1.$$

The constraint (4) is:

$$0 + 0 + \lambda^3_1 + \lambda^4_1 + 0 + 0 + \lambda^3_2 + 0 + \lambda^5_2 + \lambda^6_2 = 1.$$

Numerical example GAP

Now we are going to write the constraints related to the convexity property

$$\sum_{1 \le k \le K_j} \lambda_j^k = 1, \; j = 1, ..., n$$

We have $k_1 = 1, ..., 4, \; k_2 = 1, ..., 6$ and $j = 1, 2$, so we have:

$$\lambda_1^1 + \lambda_1^2 + \lambda_1^3 + \lambda_1^4 = 1$$

$$\lambda_2^1 + \lambda_2^2 + \lambda_2^3 + \lambda_2^4 + \lambda_2^5 + \lambda_2^6 = 1$$

Numerical example GAP

Now we have to calculate the coefficients of the objective function. For machine 1 $(1,0,0),(0,1,0),(0,0,1),(0,1,1)$ the first feasible solution has a cost of 10, which is the cost of task 1 at machine 1, solution 2 has a cost of 7, solution 3 has a cost 5 and solution 4 has a cost of 12 related to 7+5.
For machine 2 we have $(1,0,0),(0,1,0),(0,0,1),(1,1,0),(0,1,1),(1,0,1)$ and solution 1 has a cost of 6, solution 2 has a cost 8, solution 3 has a cost 11, solution 4 has a cost 14, solution 5 has a cost 19 and solution 6 has a cost 17.
So, the expression for the objective function is:

$$10\lambda_1^1 + 7\lambda_1^2 + 5\lambda_1^3 + 12\lambda_1^4 + 6\lambda_2^1 + 8\lambda_2^2 + 11\lambda_2^3 + 14\lambda_2^4 + 19\lambda_2^5 + 17\lambda_2^6.$$

Numerical example GAP

So, the master problem is given by:

$$
\begin{aligned}
\text{maximize} \quad & 10\lambda_1^1 + 7\lambda_1^2 + 5\lambda_1^3 + 12\lambda_1^4 + 6\lambda_2^1 + 8\lambda_2^2 + 11\lambda_2^3 + 14\lambda_2^4 + 19\lambda_2^5 + 17\lambda_2^6 \\
\text{subject to:} \quad & \lambda_1^1 + \lambda_2^1 + \lambda_2^4 + \lambda_2^6 = 1 \\
& \lambda_1^2 + \lambda_1^4 + \lambda_2^2 + \lambda_2^4 + \lambda_2^5 = 1 \\
& \lambda_1^3 + \lambda_1^4 + \lambda_2^3 + \lambda_2^5 + \lambda_2^6 = 1 \\
& \lambda_1^1 + \lambda_1^2 + \lambda_1^3 + \lambda_1^4 = 1 \\
& \lambda_2^1 + \lambda_2^2 + \lambda_2^3 + \lambda_2^4 + \lambda_2^5 + \lambda_2^6 = 1 \\
& \lambda_j^k \geq 0
\end{aligned}
$$

Numerical example GAP

Using two phase method for finding an initial basis, we have the fist basis given by $\lambda_B = (\lambda_1^2, \lambda_1^3, \lambda_1^4, \lambda_2^1, \lambda_2^3)$. So, the first phase Restricted Master Problem is:

$$
\begin{aligned}
\text{maximize} \quad & 7\lambda_1^2 + 5\lambda_1^3 + 12\lambda_1^4 + 6\lambda_2^1 + 11\lambda_2^3 \\
\text{subject to:} \quad & \lambda_2^1 = 1 \\
& \lambda_1^2 + \lambda_1^4 = 1 \\
& \lambda_1^3 + \lambda_1^4 + \lambda_2^3 = 1 \\
& \lambda_1^2 + \lambda_1^3 + \lambda_1^4 = 1 \\
& \lambda_2^1 + \lambda_2^3 = 1 \\
& \lambda_j^k \geq 0
\end{aligned}
$$

and the optimal solution is $\lambda_1^4 = \lambda_2^1 = 1,\ \lambda_1^2 = \lambda_1^3 = \lambda_2^3 = 0$ with $OF = 18$ and the value of the dual variables are: $\mu_1 = 0,\ \mu_2 = 7,\ \mu_3 = 5,\ \nu_1 = 0,\ \nu_2 = 6.$

Numerical example GAP

The Auxiliary problem for machine 1 of the first phase is:

$$\begin{aligned} \text{maximize} \quad & (10-0)x_{11} + (7-7)x_{21} + (5-5)x_{31} - 0 \\ \text{subject to:} \quad & 9x_{11} + 6x_{21} + 3x_{31} \le 11 \\ & x_{11}, x_{21}, x_{31} \in \{0,1\}. \end{aligned}$$

The auxiliary problem for machine 2 of the first phase is

$$\begin{aligned} \text{maximize} \quad & (6-0)x_{12} + (8-7)x_{22} + (11-5)x_{32} - 6 \\ \text{subject to:} \quad & 5x_{12} + 7x_{22} + 9x_{32} \le 18 \\ & x_{12}, x_{22}, x_{32} \in \{0,1\}. \end{aligned}$$

Numerical example GAP

The Auxiliary problem for machine 1 of the first phase is:

$$\begin{aligned} \text{maximize} \quad & 10x_{11} \\ \text{subject to:} \quad & 9x_{11} + 6x_{21} + 3x_{31} \leq 11 \\ & x_{11}, x_{21}, x_{31} \in \{0, 1\}. \end{aligned}$$

$x_{11} = 1, x_{21} = x_{31} = 0,\ OF = 10$. Solution (1,0,0) means λ_1^1.
The auxiliary problem for machine 2 of the first phase is

$$\begin{aligned} \text{maximize} \quad & 6x_{12} + x_{22} + 6x_{32} - 6 \\ \text{subject to:} \quad & 5x_{12} + 7x_{22} + 9x_{32} \leq 18 \\ & x_{12}, x_{22}, x_{32} \in \{0, 1\}. \end{aligned}$$

$x_{12} = 1, x_{22} = 0,\ x_{32} = 1,\ OF = 6$. Solution (1,0,1) means λ_2^6.

Numerical example GAP

We are going to insert columns λ_1^1 and λ_2^6 at the master problem. So, the second iteration Restricted Master Problem is:

$$\begin{aligned}
\text{maximize} \quad & 10\lambda_1^1 + 7\lambda_1^2 + 5\lambda_1^3 + 12\lambda_1^4 + 6\lambda_2^1 + 11\lambda_2^3 + 17\lambda_2^6 \\
\text{subject to:} \quad & \lambda_1^1 + \lambda_2^1 + \lambda_2^6 = 1 \\
& \lambda_1^2 + \lambda_1^4 = 1 \\
& \lambda_1^3 + \lambda_1^4 + \lambda_2^3 + \lambda_2^6 = 1 \\
& \lambda_1^1 + \lambda_1^2 + \lambda_1^3 + \lambda_1^4 = 1 \\
& \lambda_2^1 + \lambda_2^3 + \lambda_2^6 = 1 \\
& \lambda_j^k \geq 0
\end{aligned}$$

The optimal solution is: $\lambda_1^1 = 0, \lambda_1^2 = 1, \lambda_1^3 = \lambda_1^4 = \lambda_2^1 = \lambda_2^3 = 0, \lambda_2^6 = 1$.
$\mu_1 = 6, \mu_2 = 3, \mu_3 = 11, \nu_1 = 4, \nu_2 = 0$.

Numerical example GAP

The Auxiliary problem for machine 1 of the second iteration is:

$$\begin{aligned} \text{maximize} \quad & (10-6)x_{11} + (7-3)x_{21} + (5-11)x_{31} - 4 \\ \text{subject to:} \quad & 9x_{11} + 6x_{21} + 3x_{31} \leq 11 \\ & x_{11}, x_{21}, x_{31} \in \{0,1\}. \end{aligned}$$

The auxiliary problem for machine 2 of the second iteration is:

$$\begin{aligned} \text{maximize} \quad & (6-6)x_{12} + (8-3)x_{22} + (11-11)x_{32} - 0 \\ \text{subject to:} \quad & 5x_{12} + 7x_{22} + 9x_{32} \leq 18 \\ & x_{12}, x_{22}, x_{32} \in \{0,1\}. \end{aligned}$$

Numerical example GAP

The Auxiliary problem for machine 1 of the second iteration is:

$$\begin{aligned} \text{maximize} \quad & 4x_{11} + 4x_{21} - 6x_{31} - 4 \\ \text{subject to:} \quad & 9x_{11} + 6x_{21} + 3x_{31} \leq 11 \\ & x_{11}, x_{21}, x_{31} \in \{0, 1\}. \end{aligned}$$

$x_{11} = 1, x_{21} = x_{31} = 0,\ OF = 0$. Solution (1,0,0) means λ_1^1 which already is at the solution thus OF=0.

The auxiliary problem for machine 2 of the second iteration is:

$$\begin{aligned} \text{maximize} \quad & 5x_{22} \\ \text{subject to:} \quad & 5x_{12} + 7x_{22} + 9x_{32} \leq 18 \\ & x_{12}, x_{22}, x_{32} \in \{0, 1\}. \end{aligned}$$

$x_{12} = 0, x_{22} = 1,\ x_{32} = 0,\ OF = 5$. Solution (0,1,0) means λ_2^2.

Numerical example GAP

We are going to insert column λ_2^2 at the master problem. So, the third iteration Restricted Master Problem is:

$$
\begin{aligned}
\text{maximize} \quad & 10\lambda_1^1 + 7\lambda_1^2 + 5\lambda_1^3 + 12\lambda_1^4 + 6\lambda_2^1 + 8\lambda_2^2 + 11\lambda_2^3 + 17\lambda_2^6 \\
\text{subject to:} \quad & \lambda_1^1 + \lambda_2^1 + \lambda_2^6 = 1 \\
& \lambda_1^2 + \lambda_1^4 + \lambda_2^2 = 1 \\
& \lambda_1^3 + \lambda_1^4 + \lambda_2^3 + \lambda_2^6 = 1 \\
& \lambda_1^1 + \lambda_1^2 + \lambda_1^3 + \lambda_1^4 = 1 \\
& \lambda_2^1 + \lambda_2^2 + \lambda_2^3 + \lambda_2^6 = 1 \\
& \lambda_j^k \geq 0
\end{aligned}
$$

The optimal solution is: $\lambda_1^1 = 0, \lambda_1^2 = 1, \lambda_1^3 = \lambda_1^4 = \lambda_2^1 = \lambda_2^3 = 0, \lambda_2^6 = 1$.
$\mu_1 = 0, \mu_2 = -3, \mu_3 = 5, \nu_1 = 10, \nu_2 = 12$.

Numerical example GAP

The Auxiliary problem for machine 1 of the third iteration is:

$$\begin{aligned} \text{maximize} \quad & (10-0)x_{11} + (7+3)x_{21} + (5-5)x_{31} - 10 \\ \text{subject to:} \quad & 9x_{11} + 6x_{21} + 3x_{31} \leq 11 \\ & x_{11}, x_{21}, x_{31} \in \{0,1\}. \end{aligned}$$

The auxiliary problem for machine 2 of the third iteration is:

$$\begin{aligned} \text{maximize} \quad & (6-0)x_{12} + (8+3)x_{22} + (11-5)x_{32} - 12 \\ \text{subject to:} \quad & 5x_{12} + 7x_{22} + 9x_{32} \leq 18 \\ & x_{12}, x_{22}, x_{32} \in \{0,1\}. \end{aligned}$$

Numerical example GAP

The Auxiliary problem for machine 1 of the third iteration is:

$$\begin{aligned} \text{maximize} \quad & 10x_{11} + 10x_{21} - 10 \\ \text{subject to:} \quad & 9x_{11} + 6x_{21} + 3x_{31} \leq 11 \\ & x_{11}, x_{21}, x_{31} \in \{0, 1\}. \end{aligned}$$

$x_{11} = 1, x_{21} = x_{31} = 0,\ OF = 0$. Solution (1,0,0) means λ_1^1.
The auxiliary problem for machine 2 of the third iteration is:

$$\begin{aligned} \text{maximize} \quad & 6x_{12} + 11x_{22} + 6x_{32} - 12 \\ \text{subject to:} \quad & 5x_{12} + 7x_{22} + 9x_{32} \leq 18 \\ & x_{12}, x_{22}, x_{32} \in \{0, 1\}. \end{aligned}$$

$x_{12} = 1, x_{22} = 1,\ x_{32} = 0,\ OF = 5$. Solution (1,1,0) means λ_2^4.

Numerical example GAP

We are going to insert column λ_2^4 at the master problem. So, the fourth iteration Restricted Master Problem is:

$$\begin{aligned}
\text{maximize} \quad & 10\lambda_1^1 + 7\lambda_1^2 + 5\lambda_1^3 + 12\lambda_1^4 + 6\lambda_2^1 + 8\lambda_2^2 + 11\lambda_2^3 + 14\lambda_2^4 + 17\lambda_2^6 \\
\text{subject to:} \quad & \lambda_1^1 + \lambda_2^1 + \lambda_2^4 + \lambda_2^6 = 1 \\
& \lambda_1^2 + \lambda_1^4 + \lambda_2^2 + \lambda_2^4 = 1 \\
& \lambda_1^3 + \lambda_1^4 + \lambda_2^3 + \lambda_2^6 = 1 \\
& \lambda_1^1 + \lambda_1^2 + \lambda_1^3 + \lambda_1^4 = 1 \\
& \lambda_2^1 + \lambda_2^2 + \lambda_2^3 + \lambda_2^6 = 1 \\
& \lambda_j^k \geq 0
\end{aligned}$$

The optimal solution is: $\lambda_1^1 = 0, \lambda_1^2 = 1, \lambda_1^3 = \lambda_1^4 = \lambda_2^1 = \lambda_2^3 = 0, \lambda_2^6 = 1$.
$\mu_1 = 5, \mu_2 = 2, \mu_3 = 5, \nu_1 = 5, \nu_2 = 7$.

Numerical example GAP

The Auxiliary problem for machine 1 of the fourth iteration is:

$$\begin{aligned} \text{maximize} \quad & (10-5)x_{11} + (7-2)x_{21} + (5-5)x_{31} - 5 \\ \text{subject to:} \quad & 9x_{11} + 6x_{21} + 3x_{31} \leq 11 \\ & x_{11}, x_{21}, x_{31} \in \{0,1\}. \end{aligned}$$

The auxiliary problem for machine 2 of the fourth iteration is:

$$\begin{aligned} \text{maximize} \quad & (6-5)x_{12} + (8-2)x_{22} + (11-5)x_{32} - 7 \\ \text{subject to:} \quad & 5x_{12} + 7x_{22} + 9x_{32} \leq 18 \\ & x_{12}, x_{22}, x_{32} \in \{0,1\}. \end{aligned}$$

Numerical example GAP

The Auxiliary problem for machine 1 of the fourth iteration is:

$$\begin{aligned} \text{maximize} \quad & 5x_{11} + 5x_{21} - 5 \\ \text{subject to:} \quad & 9x_{11} + 6x_{21} + 3x_{31} \leq 11 \\ & x_{11}, x_{21}, x_{31} \in \{0, 1\}. \end{aligned}$$

$x_{11} = 1, x_{21} = x_{31} = 0,\ OF = 0$. Solution (1,0,0) means λ_1^1.
The auxiliary problem for machine 2 of the fourth iteration is:

$$\begin{aligned} \text{maximize} \quad & x_{12} + 6x_{22} + 6x_{32} - 7 \\ \text{subject to:} \quad & 5x_{12} + 7x_{22} + 9x_{32} \leq 18 \\ & x_{12}, x_{22}, x_{32} \in \{0, 1\}. \end{aligned}$$

$x_{12} = 0, x_{22} = 1,\ x_{32} = 1,\ OF = 5$. Solution (0,1,1) means λ_2^5.

Numerical example GAP

We are going to insert column λ_2^4 at the master problem. So, the fifth iteration Restricted Master Problem is:

$$\begin{array}{rl}
\text{maximize} & 10\lambda_1^1 + 7\lambda_1^2 + 5\lambda_1^3 + 12\lambda_1^4 + 6\lambda_2^1 + 8\lambda_2^2 + 11\lambda_2^3 + 14\lambda_2^4 + 19\lambda_2^5 + 17\lambda_2^6 \\
\text{subject to:} & \lambda_1^1 + \lambda_2^1 + \lambda_2^4 + \lambda_2^6 = 1 \\
& \lambda_1^2 + \lambda_1^4 + \lambda_2^2 + \lambda_2^4 + \lambda_2^5 = 1 \\
& \lambda_1^3 + \lambda_1^4 + \lambda_2^3 + \lambda_2^5 + \lambda_2^6 = 1 \\
& \lambda_1^1 + \lambda_1^2 + \lambda_1^3 + \lambda_1^4 = 1 \\
& \lambda_2^1 + \lambda_2^2 + \lambda_2^3 + \lambda_2^5 + \lambda_2^6 = 1 \\
& \lambda_j^k \geq 0
\end{array}$$

The optimal solution is: $\lambda_1^1 = 1, \lambda_1^2 = \lambda_1^3 = \lambda_1^4 = \lambda_2^1 = \lambda_2^3 = \lambda_2^4 = 0$ $\lambda_2^5 = 1,\ \lambda_2^6 = 0$. $\mu_1 = -2, \mu_2 = 0, \mu_3 = 0, \nu_1 = 12, \nu_2 = 19$.

Numerical example GAP

The Auxiliary problem for machine 1 of the fifth iteration is:

$$\begin{aligned} \text{maximize} \quad & (10+2)x_{11} + (7-0)x_{21} + (5-0)x_{31} - 12 \\ \text{subject to:} \quad & 9x_{11} + 6x_{21} + 3x_{31} \leq 11 \\ & x_{11}, x_{21}, x_{31} \in \{0,1\}. \end{aligned}$$

The auxiliary problem for machine 2 of the fifth iteration is:

$$\begin{aligned} \text{maximize} \quad & (6+2)x_{12} + (8-0)x_{22} + (11-0)x_{32} - 19 \\ \text{subject to:} \quad & 5x_{12} + 7x_{22} + 9x_{32} \leq 18 \\ & x_{12}, x_{22}, x_{32} \in \{0,1\}. \end{aligned}$$

Numerical example GAP

The Auxiliary problem for machine 1 of the fourth iteration is:

$$\begin{aligned} \text{maximize} \quad & 12x_{11} + 7x_{21} + 5x_{31} - 12 \\ \text{subject to:} \quad & 9x_{11} + 6x_{21} + 3x_{31} \leq 11 \\ & x_{11}, x_{21}, x_{31} \in \{0, 1\}. \end{aligned}$$

$x_{11} = 0, x_{21} = x_{31} = 1,\ OF = 0$. Solution (0,1,1) means λ_1^4.
The auxiliary problem for machine 2 of the fourth iteration is:

$$\begin{aligned} \text{maximize} \quad & 8x_{12} + 8x_{22} + 11x_{32} - 19 \\ \text{subject to:} \quad & 5x_{12} + 7x_{22} + 9x_{32} \leq 18 \\ & x_{12}, x_{22}, x_{32} \in \{0, 1\}. \end{aligned}$$

$x_{12} = 1, x_{22} = 0,\ x_{32} = 1,\ OF = 0$. Solution (1,0,1) means λ_2^6.
Thus, there is no more column to add to the Master Problem, so the current solution is optimal.

Numerical example GAP

Optimal solution: $\lambda_1^1 = \lambda_2^5 = 1$ and this means $(1, 0, 0), (0, 1, 1)$, thus task one at machine 1, tasks 2 and 3 at machine 2.

Example - cutting stock

Traditional formulation for cutting stock problem - parameters and variables

- $l_j, \; j = 1, ..., r$ are the lengths of the bars needed by the costumers;
- L is the total length of the bar;
- the right side $b_i, \; i = 1, ...m$ of the constraints is related to the demand;
- the variables x_{ij} represent be the number of times item i is cut on bar j;
- y_j is 1 if the bar j is cut and 0 otherwise;

Example - cutting stock

Traditional formulation for cutting stock problem (Kantarovich)

$$
\begin{aligned}
\text{minimize} \quad & z = \sum_{j=1}^{r} y_j \\
\text{subject to:} \quad & \sum_{j=1}^{r} x_{ij} \geq b_i \;\; \text{for } i = 1, \ldots, m \\
& \sum_{i=1}^{m} l_j x_{ij} \leq L y_j \; \text{for } j = 1, \ldots, r \\
& x_{ij} \geq 0, \text{integer}, \;\; i = 1, \ldots, m, \;\; j = 1, \ldots, r. \\
& y_j, \;\; \text{binary}, \; j = 1, \ldots, r.
\end{aligned} \tag{5}
$$

Example - cutting stock

The traditional formulation is not good because

- when we find integer values for one bar, the fractional values that might be found previously can appear again as the solution for another bar.
- fractional values are easily found because we can have unused parts of each bar.

Example cutting stock

Column generation formulation.

- very large number of patterns P;
- For each $p \in P$, $a_{ip} \in \mathbb{Z}_+$ denote the number of pieces of length l_i in a pattern p;
- λ_p is the number of bars cut in pattern p.

Example cutting stock

Restricted Master Problem (Gilmore-Gomory 1960)

$$
\begin{aligned}
(RMP)\ \text{minimize} \quad & z = \sum_{p \in P} c_p \lambda_p \\
\text{subject to:} \quad & \sum_{p \in P} a_{ip} \lambda_p \geq b_i, \ \ i = 1, \ldots m \\
& \lambda_p \geq 0, \text{integer}, \ \ p \in P.
\end{aligned}
$$

Example cutting stock

We solve a relaxed *RMP* with a small subset of patterns $P' \subset P$ and generate new patterns as needed.

$$
\begin{aligned}
(RMP) \text{ minimize} \quad & z = \sum_{p \in P'} c_p \lambda_p \\
\text{subject to:} \quad & \sum_{p \in P'} a_{ip} \lambda_p \geq b_i, \;\; i = 1, ...m \\
& \lambda_p \geq 0, \;\; p \in P'.
\end{aligned}
$$

This relaxed problem is a linear programming problem and as we have seen in the initial slides of the first class, the feasible region defined by a linear programming problem is a convex set, thus we don't need the convexity constraint $\sum_{p \in P'} \lambda_p = 1$.

Example - cutting stock

General auxiliary problem

$$\begin{aligned} (AP)\ \text{minimize} \quad & z = (c^\top - \pi * A)v \\ \text{subject to:} \quad & Av \leq b \\ & v \geq 0. \end{aligned}$$

which can be written as

$$\begin{aligned} (AP)\ \text{minimize} \quad & z = (c_N^\top - \pi * a_N)x_N \\ \text{subject to:} \quad & Ax \leq b \\ & x \geq 0. \end{aligned}$$

where N denotes the set of non basic variables. But this auxiliary problem, given in our first class is a general auxiliary problem.

Example - cutting stock

In cutting stock problem, the variables of auxiliary problem are the columns. So, we can write our auxiliary problem as:

$$\begin{aligned} (AP)\ \text{minimize} \quad & z = 1 - \pi * a \\ \text{subject to:} \quad & l_j a \leq L \\ & a \geq 0 \end{aligned}$$

that can be rewritten as:

$$\begin{aligned} 1 - (AP)\ \text{maximize} \quad & z = \pi_j * a_j \\ \text{subject to:} \quad & l_j a_j \leq L \\ & a_j \geq 0. \end{aligned}$$

So, our auxiliary problem is a knapsack problem. And the reduced cost is given by

$$1 - \text{objective value of } (AP).$$

Example - cutting Stock

A company produces steel bars with $L = 100m$ and cuts the bars for the costumers according to their necessities. Now, the company has to satisfy the following demand:

length (m)	number of pieces needed
22	45
42	38
52	25
53	11
78	12

The costs c_p are all equal to one.
We want to minimize the number of steel bars we need to cut in order to satisfy the demand.

Example - cutting stock

In order to find an initial solution, we compute how many pieces of each length fits in one bar.

$$\left\lfloor \frac{L}{l_j} \right\rfloor$$

So we make:

$$\left\lfloor \frac{100}{22} \right\rfloor = 4, \left\lfloor \frac{100}{42} \right\rfloor = 2, \left\lfloor \frac{100}{52} \right\rfloor = 1, \left\lfloor \frac{100}{53} \right\rfloor = 1, \left\lfloor \frac{100}{78} \right\rfloor = 1.$$

so our matrix A at the first iteration is:

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example - cutting stock

Our first *RMP* is:

$$\begin{aligned} (RMP1)\ \text{minimize} \quad & z = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ \text{subject to:} \quad & 4\lambda_1 \geq 45 \\ & 2\lambda_2 \geq 38 \\ & \lambda_3 \geq 25 \\ & \lambda_4 \geq 11 \\ & \lambda_5 \geq 12 \\ & \lambda_1, ..., \lambda_5 \geq 0. \end{aligned}$$

The solution is: $\lambda_1 = 11.25,\ \lambda_2 = 19,\ \lambda_3 = 25, \lambda_4 = 11,\ \lambda_5 = 12.$

The value of the dual variables are:
$\pi_1 = 0.25,\ \pi_2 = 0.5,\ \pi_3 = \pi_4 = \pi_5 = 1.$

Example - cutting stock

In our auxiliary problem we want to check if there exists a new column with negative reduced cost.

$$\begin{aligned} (AP1) \text{ maximize} \quad & z = 0.25x_1 + 0.5x_2 + 1x_3 + 1x_4 + 1x_5 \\ \text{subject to:} \quad & 22x_1 + 42x_2 + 52x_3 + 53x_4 + 78x_5 \leq 100 \\ & x_1, ..., x_5 \geq 0 \text{ and integer.} \end{aligned}$$

The solution of $(AP1)$ is $x_1 = x_3 = x_5 = 0, x_2 = x_4 = 1$. Hence, the reduced cost is $1 - 1.5 = -0.5$ and we have a new column a_6,

$$a_6 = \begin{pmatrix} a_{16} \\ a_{26} \\ a_{36} \\ a_{46} \\ a_{56} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Example - cutting stock

$$
\begin{aligned}
(RMP2)\ \text{minimize} \quad & z = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 \\
\text{subject to:} \quad & 4\lambda_1 \geq 45 \\
& 2\lambda_2 + \lambda_6 \geq 38 \\
& \lambda_3 \geq 25 \\
& \lambda_4 + \lambda_6 \geq 11 \\
& \lambda_5 \geq 12 \\
& \lambda_1, ..., \lambda_6 \geq 0.
\end{aligned}
$$

The solution is:
$\lambda_1 = 11.25,\ \ \lambda_2 = 13.5,\ \ \lambda_3 = 25, \lambda_4 = 0,\ \ \lambda_5 = 12, \lambda_6 = 11.$

The value of the dual variables are:
$\pi_1 = 0.25,\ \ \pi_2 = \pi_4 = 0.5,\ \ \pi_3 = \pi_5 = 1.$

$$
\begin{aligned}
(AP2)\ \text{maximize}\quad & z = 0.25x_1 + 0.5x_2 + 1x_3 + 0.5x_4 + 1x_5 \\
\text{subject to:}\quad & 22x_1 + 42x_2 + 52x_3 + 53x_4 + 78x_5 \leq 100 \\
& x_1, ..., x_5 \geq 0 \text{ and integer.}
\end{aligned}
$$

The solution of $(AP2)$ is $x_1 = x_4 = x_5 = 0, x_2 = x_3 = 1$. Hence, the reduced cost is $1 - 1.5 = -0.5$ so we have a new column a_7,

$$
a_7 = \begin{pmatrix} a_{17} \\ a_{27} \\ a_{37} \\ a_{47} \\ a_{57} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}
$$

Example - cutting stock

$$\begin{aligned} (RMP3) \text{ minimize} \quad & z = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 \\ \text{subject to:} \quad & 4\lambda_1 \geq 45 \\ & 2\lambda_2 + \lambda_6 + \lambda_7 \geq 38 \\ & \lambda_3 + \lambda_7 \geq 25 \\ & \lambda_4 + \lambda_6 \geq 11 \\ & \lambda_5 \geq 12 \\ & \lambda_1, ..., \lambda_7 \geq 0. \end{aligned}$$

The solution is:
$\lambda_1 = 11.25,\ \lambda_2 = 1,\ \lambda_3 = \lambda_4 = 0,\ \lambda_5 = 12, \lambda_6 = 11, \lambda_7 = 25.$
The value of the dual variables are:
$\pi_1 = 0.25,\ \pi_2 = \pi_3 = \pi_4 = 0.5, \pi_5 = 1.$

Example - cutting stock

$$\begin{aligned}(AP3)\ \text{maximize}\quad & z = 0.25x_1 + 0.5x_2 + 0.5x_3 + 0.5x_4 + 1x_5\\ \text{subject to:}\quad & 22x_1 + 42x_2 + 52x_3 + 53x_4 + 78x_5 \leq 100\\ & x_1, ..., x_5 \geq 0 \text{ and integer.}\end{aligned}$$

The solution of $(AP3)$ is $x_1 = x_5 = 1, x_2 = x_3 = x_4 = 0$. Hence, the reduced cost is $1 - 1.25 = -0.25$ so we have a new column a_8,

$$a_8 = \begin{pmatrix} a_{18} \\ a_{28} \\ a_{38} \\ a_{48} \\ a_{58} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Example - cutting stock

$$
\begin{aligned}
(RMP4)\ \text{minimize} \quad & z = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 \\
\text{subject to:} \quad & 4\lambda_1 + \lambda_8 \geq 45 \\
& 2\lambda_2 + \lambda_6 + \lambda_7 \geq 38 \\
& \lambda_3 + \lambda_7 \geq 25 \\
& \lambda_4 + \lambda_6 \geq 11 \\
& \lambda_5 + \lambda_8 \geq 12 \\
& \lambda_1, ..., \lambda_8 \geq 0.
\end{aligned}
$$

The solution is:
$\lambda_1 = 8.25,\ \ \lambda_2 = 1,\ \ \lambda_3 = \lambda_4 == \lambda_5 = 0, \lambda_6 = 11, \lambda_7 = 25, \lambda_8 = 12.$
The value of the dual variables are:
$\pi_1 = 0.25,\ \ \pi_2 = \pi_3 = \pi_4 = 0.5,\ \ \pi_5 = 0.75.$

Example - cutting stock

$$
\begin{aligned}
(AP4)\ \text{maximize} \quad & z = 0.25x_1 + 0.5x_2 + 0.5x_3 + 0.5x_4 + 0.75x_5 \\
\text{subject to:} \quad & 22x_1 + 42x_2 + 52x_3 + 53x_4 + 78x_5 \leq 100 \\
& x_1, ..., x_5 \geq 0 \text{ and integer.}
\end{aligned}
$$

The solution of $(AP4)$ is $x_1 = 4,\ x_2 = x_3 = x_4 = x_5 = 0$. In this case, the reduced cost is 0, thus we don1t have a new column and the optimal solution was found at $(RMP4)$.

Example - cutting stock

Rounding the solution of ($RMP4$) we have the following:

- cut 9 entire bars in pattern 1, which means 4 pieces of 22m;
- cut 1 entire bar in pattern 2, which means 2 pieces of 42m;
- cut 11 entire bars in pattern 6, which means 1 piece of 42m and 1 piece of 53m;
- cut 25 entire bars in pattern 7, which means 1 piece of 42m and 1 piece of 52m;
- 12 entire bars in pattern 8, which means 1 piece of 22m and 1 piece of 78m.

This solution gives us 48 pieces of 22m, 38 pieces of 42m, 25 pieces of 52m, 11 pieces of 53m and 12 pieces of 78m.

Thank you

I would like to thank you all.
The files of the classes are at https://github.com/anauzeda/ENSIIE-2022
Please, you can contact me in case of any doubt at my e-mail
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