Tunable Spin-Orbit Coupling using a 'Molmer-Sorensen' coupling scheme

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Consider 3 pairs of Raman beamsg coupling the three m_f states in the ^{87}Rb $F = 1$ manifold as shown in fig no... . The Hamiltonian describing the interaction is:

$$\hat{H}_R(t) = \{\Omega_{21}\cos(2k_Rx - \omega_{21}t + \frac{\phi_1}{2}) + \Omega_{31}\cos(2k_Rx - \omega_{31}t - \frac{\phi_1}{2}) + \Omega_{41}\cos(2k_Rx - \omega_{41}t + \phi_2)\}\hat{F}_x \tag{1}$$

where $\Omega_{ij} \propto \vec{E}_i \times \vec{E}_j^\star$ represents the coupling strenght associated to each pair of Raman beams and $\omega_{ij} = \omega_i - \omega_j$. The frequencies are chosen so that $\omega_{31} + \omega 21$ is at 4 photon resonance with the $m_f = +1 \to m_f = -1$. Under a rotation about the z axis $\hat{U} = e^{i\omega t\hat{F}_z}$ at frequency $\bar{\omega} = \frac{\omega_{21}+\omega_{31}}{2}$, and after applying the rotating wave approximation (RWA), the Hamiltonian transforms to

$$\begin{aligned}
\hat{\tilde{H}}_R(t) =& \hat{U}^\dagger \hat{H}_R(t)\hat{U} \\
=& \frac{1}{2}\{\Omega_{21}\cos(2k_Rx + (\bar{\omega} - \omega_{21})t + \frac{\phi_1}{2}) + \Omega_{31}\cos(2k_Rx + (\bar{\omega} - \omega_{31})t - \frac{\phi_1}{2}) \\
&+ \Omega_{41}\cos(2k_Rx + (\bar{\omega} - \omega_{41})t + \phi_2)\}\hat{F}_x \\
&- \frac{1}{2}\{\Omega_{21}\sin(2k_Rx + (\bar{\omega} - \omega_{21})t + \frac{\phi_1}{2}) + \Omega_{31}\sin(2k_Rx + (\bar{\omega} - \omega_{31})t - \frac{\phi_1}{2}) \\
&+ \Omega_{41}\sin(2k_Rx + (\bar{\omega} - \omega_{41})t + \phi_2)\}\hat{F}_y
\end{aligned} \tag{2}$$

where we have used

$$e^{-i\theta\hat{F}_z}\hat{F}_x e^{i\theta\hat{F}_z} = \cos\theta\hat{F}_x + \sin\theta\hat{F}_y \tag{3}$$

and applied the rotating wave approximation (RWA)to get rid of the fast terms.

I want the time dependent Hamiltonian to look like it has a constant offset and a frequency modulated term. This can be done by choosing $\Omega_{21} = \Omega_{31} = \Omega$, $\Omega_{41} = \Omega_0$, and $\omega_{41} = \frac{\omega_{21}+\omega_{31}}{2}$. The Hamiltonian is now reduced to

$$\hat{\tilde{H}}_R(t) = \frac{\Omega}{2}\cos(\delta\omega t)[\cos(2k_Rx)\hat{F}_x - \sin(2k_Rx)\hat{F}_y] + \frac{\Omega_0}{2}[\cos(2k_Rx + \phi_2)\hat{F}_x - \sin(2k_Rx + \phi_2)\hat{F}_y] \tag{4}$$

where $\delta\omega = \frac{\omega_{31}-\omega_{21}}{2}$. Also notice that by redefining the time origin I can get rid of the first phase.

Things are looking good so far!

Now I can apply the usual x dependent rotation $\hat{U} = e^{i2k_Rx\hat{F}_z}$. I won't work out all the algebra explicitly here, but the complete system's Hamiltonian (in the interaction picture) should be something like

$$
\begin{aligned}
\hat{H}(t) =&\frac{\hbar^2}{2m}(\hat{k}-2k_R\hat{F}_z)^2+\frac{1}{2}[\Omega_0\cos\phi_2+\Omega\cos(\delta\omega t)]\hat{F}_x-\frac{1}{2}\Omega_0\sin\phi_2\hat{F}_y+\Delta\hat{F}_z+\epsilon(\hat{F}_z^2-\mathbb{I}) \\
=&\frac{\hbar^2\hat{k}^2}{2m}-\frac{\hbar^2}{m}2k_R\hat{k}\hat{F}_z+4E_r\hat{F}_z^2+\frac{1}{2}[\Omega_0\cos\phi_2+\Omega(t)]\hat{F}_x-\frac{1}{2}\Omega_0\sin\phi_2\hat{F}_y+\Delta\hat{F}_z+\epsilon(\hat{F}_z^2-\mathbb{I}),
\end{aligned} \tag{5}
$$

with $\Omega(t)=\Omega\cos(\delta\omega t)$.

To get rid of the time dependence in the Hamiltonian and ultimately getting the 'tunable' spin-orbit coupling we can choose a transoformation of the Hamiltonian such that $\hat{U}^\dagger\frac{\partial\hat{U}}{\partial t}=-i\frac{\Omega(t)}{2}\hat{F}_x$. This will be satisfied for

$$
\hat{U}=e^{-i\frac{\Omega}{2}\int_0^t\cos(\delta\omega t')dt'}=e^{-i\frac{\Omega}{2\delta\omega}\sin(\delta\omega t)}. \tag{6}
$$

Under this time dependent transformation, the time evolution of the system will be given by

$$
\begin{aligned}
\hat{\tilde{H}} =&\hat{U}^\dagger\hat{H}(t)\hat{U}+i\hat{U}^\dagger\frac{\partial\hat{U}}{\partial t} \\
=&\frac{\hbar^2\hat{k}^2}{2m}-\frac{\hbar^2}{m}2k_R\hat{k}\hat{U}^\dagger\hat{F}_z\hat{U}+\frac{1}{2}\Omega_0\cos\phi_2\hat{F}_x-\frac{1}{2}\Omega_0\sin\phi_2\hat{U}^\dagger\hat{F}_y\hat{U}+\Delta\hat{U}^\dagger\hat{F}_z\hat{U}+\epsilon(\hat{U}^\dagger\hat{F}_z^2\hat{U}-\mathbb{I})+4E_r\hat{U}^\dagger\hat{F}_z^2\hat{U}.
\end{aligned} \tag{7}
$$

The transformations for the angular momentum operators that don't commute with $\hat{F}_x$ are given by:

$$
\begin{aligned}
e^{i\theta\hat{F}_x}\hat{F}_z e^{-i\theta\hat{F}_x} &=\cos\theta\hat{F}_z+\sin\theta\hat{F}_y \\
e^{i\theta\hat{F}_x}\hat{F}_y e^{-i\theta\hat{F}_x} &=-\sin\theta\hat{F}_z+\cos\theta\hat{F}_y \\
e^{i\theta\hat{F}_x}\hat{F}_z^2 e^{-i\theta\hat{F}_x} &=\cos^2\theta\hat{F}_z^2+\sin^2\theta\hat{F}_y^2+\sin\theta\cos\theta(\hat{F}_z\hat{F}_y+\hat{F}_y\hat{F}_z).
\end{aligned} \tag{8}
$$

The JacobiAnger expansion will be useful to write an (almost) exact expression for the transformed Hamiltonian:

$$
\cos(z\sin\theta)=J_0(z)+2\sum_{n=1}^{\infty}J_{2n}(z)\cos(2n\theta)\approx J_0(z) \tag{9}
$$

$$
\sin(z\sin\theta)=2\sum_{n=0}^{\infty}J_{2n+1}(z)\sin((2n+1)\theta)\approx 0, \tag{10}
$$

so we can write

$$
\begin{aligned}
\hat{\tilde{H}} =&\frac{\hbar^2\hat{k}^2}{2m}-\frac{\hbar^2}{m}2k_R J_0(\Omega/2\delta\omega)\hat{k}\hat{F}_z+\frac{\Omega_0}{2}(\cos\phi_2\hat{F}_x-\sin\phi_2 J_0(\Omega/2\delta\omega)\hat{F}_y)+\Delta J_0(\Omega/2\delta\omega)\hat{F}_z \\
&+\epsilon(J_0^2(\Omega/2\delta\omega)\hat{F}_z^2+\frac{1}{2}(1-J_0(\Omega/\delta))\hat{F}_y^2-\mathbb{I})+4E_r(J_0^2(\Omega/\delta)\hat{F}_z^2+\frac{1}{2}(1-J_0(\Omega/\delta\omega))\hat{F}_y^2 \\
=&\frac{\hbar^2}{2m}(\hat{k}-2k_R J_0(\Omega/2\delta\omega)\hat{F}_z)^2+\frac{\Omega_0}{2}(\cos\phi_2\hat{F}_x-\sin\phi_2 J_0(\Omega/2\delta\omega)\hat{F}_y)+\Delta J_0(\Omega/2\delta\omega)\hat{F}_z \\
&+\epsilon[J_0^2(\Omega/2\delta\omega)\hat{F}_z^2+\frac{1}{2}(1-J_0(\Omega/\delta\omega))\hat{F}_y^2-\mathbb{I}]+2E_r(1-J_0(\Omega/\delta\omega))\hat{F}_y^2
\end{aligned} \tag{11}
$$

The terms on the last line can be simplified:

$$\epsilon[J_0^2(\Omega/\delta)\hat{F}_z^2 + \frac{1}{2}(1 - J_0(2\Omega/\delta))\hat{F}_y^2 - \mathbb{I}] + 2E_r(1 - J_0(2\Omega/\delta))\hat{F}_y^2) = \tag{12}$$

This looks almost like a spin one spin-orbit coupled system with some extra weird terms and couplings between $m_f = 1$ and $m_f = -1$. Adiabatic elimination of the $m_f = 0$ (ground) state leads to an effective Hamiltonian

$$\hat{H}_{eff} = \frac{\hbar^2}{2m}(\hat{k} - 2k_R J_0(\Omega/\delta)\hat{F}_{2z})^2 + \Delta J_0(\Omega/\delta)\hat{F}_{2z} \tag{13}$$