Problem 1

(a) **Convergence and stability of a numerical scheme.** In this problem I consider the numerical scheme

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{\Delta t^2} - c^2 \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} = 0 \tag{1}$$

to solve the one-dimensional wave equation $u_{tt} - c^2 u_{xx} = 0$. Here, $c \in R$ and U_j^n is the numerical approximation of $u(j\Delta x, n\Delta t)$.

(a) The numerical scheme in Eq. 1 is second-order accurate, this can be seen if we re-write equation (1) differently by taylor expanding its terms. Before taylor expanding them, we want to re-write them in a different form. Meaning:

$$\begin{aligned} U_j^{n+1} &= U(t_n + \Delta t, x_j) \\ U_j^{n-1} &= U(t_n - \Delta t, x_j) \\ U_{j+1}^n &= U(t_n, x_j + \Delta x) \\ U_{j-1}^n &= U(t_n, x_j - \Delta x) \end{aligned}$$

Then I taylor expand these as:

U$_j^{n+1}$ $= U(t_n + \Delta t, x_j) = U(t_n, x_j) + \Delta t * U_t(t_n, x_j) + \frac{\Delta t^2}{2!} U_{tt}(t_n, x_j) + \frac{\Delta t^3}{3!} U_{ttt}(t_n, x_j) + \frac{\Delta t^4}{4!} U_{tttt}(t_n, x_j) + O(h^5)$

U$_j^{n-1}$ $= U(t_n - \Delta t, x_j) = U(t_n, x_j) - \Delta t * U_t(t_n, x_j) + \frac{\Delta t^2}{2!} U_{tt}(t_n, x_j) - \frac{\Delta t^3}{3!} U_{ttt}(t_n, x_j) + \frac{\Delta t^4}{4!} U_{tttt}(t_n, x_j) + O(h^5)$

U$_{j+1}^n$ $= U(t_n, x_j + \Delta x) = U(t_n, x_j) + \Delta x_j * U_x(t_n, x_j) + \frac{\Delta x_j^2}{2!} U_{xx}(t_n, x_j) + \frac{\Delta x j^3}{3!} U_{xxx}(t_n, x_j) + \frac{\Delta x j^4}{4!} U_{xxxx}(t_n, x_j) + O(h^5)$

U$_{j+1}^n$ $= U(t_n, x_j + \Delta x) = U(t_n, x_j) - \Delta x_j * U_x(t_n, x_j) + \frac{\Delta x_j^2}{2!} U_{xx}(t_n, x_j) - \frac{\Delta x j^3}{3!} U_{xxx}(t_n, x_j) + \frac{\Delta x j^4}{4!} U_{xxxx}(t_n, x_j) + O(h^5)$

Now I will plug these values taylor-expanded in equation (1). Most values will cancel out. Hence, the truncation error T will result in:

$$T = \frac{2\frac{\Delta t^2}{2*1} U_{tt}(t_n, x_j) + 2\frac{\Delta t^4}{4*3*2*1} U_{tttt}(t_n, x_j) + 2 * O(h^5)}{\Delta t^2} - \ldots$$

... - $c^2(2\frac{\Delta x_j^2}{2*1}U_{xx}(t_n,x_j) + 2\frac{\Delta x_j^4}{4*3*2*1}U_{xxxx}(t_n,x_j) + 2*\frac{O(h^5)}{\Delta x^2}\frac{}{\Delta x^2}$

Cancelling and rearranging other terms leads to:

$$T = \frac{\Delta t^2}{12}U_{tttt}(t_n,x_j)+U_{tt}(t_n,x_j)+2\frac{O^5}{\Delta t^2}-c^2(U_{xx}(t_n,x_j)+\frac{\Delta x^2}{12}U_{xxxx}(t_n,x_j)+2*\frac{O(h^5)}{\Delta x^2}\Delta x^2$$

$$T = \frac{\Delta t^2}{12}U_{tttt}(t_n,x_j)+U_{tt}(t_n,x_j)+h.o.t.-c^2(U_{xx}(t_n,x_j)+\frac{\Delta x^2}{12}U_{xxxx}(t_n,x_j)+h.o.t.$$

I know that $U_{tt}(t_n,x_j) - c^2U_{xx} = 0$, hence:

$$T = \frac{1}{12}(\Delta t^2 U_{tttt}(t_n,x_j) + c^2\Delta x^2 U_{xxxx}(t_n,x_j)) + h.o.t. \tag{2}$$

These are second order terms, hence this equation is second order accurate.

(b) To show that the numerical scheme is stable, I will use Fourier stability analysis, by subsituting in the ansatz $U_j^n = \lambda(k)^n e^{ijk\Delta x}$.
I want to rewrite equation (1). I can say that $v = \frac{\Delta t * c}{\Delta x}$. Consequently, equation (1) becomes:

$$T = U_j^{n+1} - 2U_j^n + U_j^{n-1} - v^2[U_{j+1}^n - 2U_j^n + U_{j-1}^n] = 0$$

Here, we can now plug in: $U_j^n = \lambda(k)^n e^{ijk\Delta x}$.

$$\lambda^{n+1}e^{ikj\Delta x} - 2\lambda^n e^{ikj\Delta x} + \lambda^{n-1}e^{ikj\Delta x} - v^2\lambda^n[e^{ik(j+1)\Delta x} - 2e^{ikj\Delta x} + e^{ik(j-1)\Delta x}] = 0 \tag{3}$$

Additionally, I know that $e^{ik(j+1)\Delta x} = e^{ikj\Delta x}e^{ik\Delta x}$.

Hence I will divide (3) for $e^{ijk\Delta x}$ on both sides. This will result into :

$$\lambda^{n+1} - 2\lambda^n + \lambda^{n-1} - v^2[e^{ik\Delta x} - 2 + e^{-ik\Delta x}]\lambda^n = 0$$

Note: $e^{ik\Delta x} = cos(k\Delta x) + isin(k\Delta x)$ and $e^{-ik\Delta x} = cos(k\Delta x) - isin(k\Delta x)$. For our equation above, we can then further reduce using: $e^{ik\Delta x} + e^{-ik\Delta x} = 2cos(k\Delta x)$. Then, we can divide out by λ^{n-1} to get:

$$\lambda^2 - 2\lambda + 1 - v^2\lambda[2cos(k\Delta x) - 2] = 0$$

Here we want to solve for λ and we can use the quadratic formula.

$$\lambda^2 - \lambda(2v^2cos(k\Delta x) - 2v^2 + 2) + 1 = 0$$

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$$\lambda^2 - \lambda(2v^2cos(k\Delta x) - 2v^2 + 2) + 1 = 0$$

Solving for λ it becomes:

$$\begin{aligned}\lambda &= \frac{2v^2cos(k\Delta x) - 2v^2 + 2 \pm \sqrt{4(v^2cos(k\Delta x) - v^2 + 1)^2 - 4}}{2} \\ &= v^2cos(k\Delta x) - v^2 + 1 \pm \sqrt{(v^2cos(k\Delta x) - v^2 + 1)^2 - 1}\end{aligned}$$

Where we know that $(v^2k\Delta x - v^2 + 1)^2 - 1$ is always negative. Hence λ is a complex number made of a real part a and an imaginary part b, i.e. $a + ib$. Where b is $1 - (v^2k\Delta x - v^2 + 1)^2$ and a is $v^2cos(k\Delta x) - v^2 + 1$.

$$\lambda = v^2cos(k\Delta x) - v^2 + 1 \pm i\sqrt{1 - (v^2cos(k\Delta x) - v^2 + 1)^2}$$

$$\begin{aligned}a &= v^2cos(k\Delta x) - v^2 + 1 \\ b &= 1 - (v^2k\Delta x - v^2 + 1)^2\end{aligned}$$

Now, we can solve for the roots of λ:

$$|\lambda| = a^2 + ib^2$$

$$|\lambda| = (v^2cos(k\Delta x) - v^2 + 1)^2 \pm (i\sqrt{1 - (v^2cos(k\Delta x) - v^2 + 1)^2})^2$$

$$|\lambda| = (v^2cos(k\Delta x) - v^2 + 1)^2 \pm 1 - (v^2cos(k\Delta x) - v^2 + 1)^2$$

$$|\lambda| \pm 1$$

$$\lambda = 1$$

Given this, our scheme is stable because it satisfies that $\lambda =< 1$.