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# Homework 3

AM 205
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In this Latex Write up I have written the solutions to problems 2a and 4a and 4b. For the solutions to 1, 2b, 3 and 6 please refer to my jupyter notebook file.

## Problem 2

(a) **3-points Gauss quadrature rule**

Given that the cubic Legendre polynomial is $P_3(x) = \frac{1}{2}x(5x^2 - 3)$, in this exercise I derive the 3-point Gauss quadrature rule on the interval $[-1, 1]$ by evaluating the relevant integrals by hand. Here I will demonstrate that this quadrature rule integrates all polynomials up to the expected degree exactly, the expected degree is degree 5. I.e. this will work up to degree 5 and will fail for degree 6.

I can use the Legendre polynomial given to find the points that can be used to solve the Gauss Quadrature. The equation is $\dfrac{1}{2}x(5x^2 - 3) = 0$. The three points that I get are $x_0 = -\sqrt{\dfrac{3}{5}}$, $x_1 = 0$, and $x_2 = \sqrt{\dfrac{3}{5}}$.

Next, I can use the Lagrange interpolant to find the Quadrature weights. The equation for the Lagrange is $L_k(x) = \prod_{j=0,j\neq k}^{n} = \dfrac{x - x_j}{x_k - x_j}$. This can be done three times for each combination of points to get the following three equations: $L_0(x) = \dfrac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$, $L_1(x) = \dfrac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$, $L_2(x) = \dfrac{(x - x_1)(x - x_0)}{(x_2 - x_1)(x_2 - x_0)}$.

Each of the points can then be placed in to find the equations that need to be integrated to find the weights. Those equations are then:

$$
\begin{aligned}
L_0(x) &= \frac{5}{6}x(x - \sqrt{\frac{3}{5}}), \\
L_1(x) &= \frac{-5}{3}(x^2 - \frac{3}{5}), \\
L_2(x) &= \frac{5}{6}x(x + \sqrt{\frac{3}{5}}).
\end{aligned}
$$

Lastly, by taking the integral of each of these equations in the form of $\int_{-1}^{1} L_k(x)\,dx$, I get the weights $w_0 = \dfrac{5}{9}$, $w_1 = \dfrac{8}{9}$, $w_2 = \dfrac{5}{9}$.

Now I start to check up to which degree can this integration estimation method be used to.

**Goal**: Compare the true integral $\int_{-1}^{1} f(x)\,dx$ with $\sum w_i f(x_i)$ and if the results of these is the same.

**Degree 1:** $f(x) = ax + b$

Solution for Integral: $\int_{-1}^{1} f(x)\,dx = a\dfrac{x^2}{2} + bx|_{-1}^{1} =$ 2b

Solution for Summation: $\sum w_i f(x_i) = w0 * (ax_0 + b) + w1 * (ax_1 + b) + w2 * (ax_2 + b)$

I plug in the values:

$$\sum w_i f(x_i) = \frac{5}{9}(a\sqrt{-\frac{3}{5}} + b) + \frac{8}{9}(a * 0 + b) + \frac{5}{9}(a * \sqrt{\frac{3}{5}} + b)$$
$$\sum w_i f(x_i) = 2b$$

The solutions are matching. Hence, it works for degree 1.

**Degree 2:** $f(x) = ax^2 + bx + c$

Solution for Integral: $\int_{-1}^{1} f(x)\,dx = \dfrac{2a}{3} + 2c$

Solution for Summation: $\sum w_i f(x_i) = w0 * (ax_0^2 + bx_0 + c) + w1 * (ax_1^2 + bx_1 + c) + w2 * (ax_2^2 + bx_2 + c)$

I plug in the values:

$$\sum w_i f(x_i) = \frac{5}{9}(a(\sqrt{-\frac{3}{5}})^2 + b\sqrt{-\frac{3}{5}} + c) + \frac{8}{9}(a * 0^2 + b * 0 + c) + \frac{5}{9}(a(\sqrt{\frac{3}{5}})^2 + b\sqrt{\frac{3}{5}} + c)$$
$$\sum w_i f(x_i) = \frac{2a}{3} + 2c$$

The solutions are matching. Hence, it works for degree 2.

**Degree 3:** $f(x) = ax^3 + bx^2 + cx + d$

Solution for Integral: $\int_{-1}^{1} f(x)\,dx = \dfrac{2}{3}(b) + 2d$

Solution for Summation: $\sum w_i f(x_i) = w0 * (ax_0^3 + bx_0^2 + cx_0 + d) + w1 * (ax_1^3 + bx_1^2 + cx_1 + d) + w2 * (ax_2^3 + bx_2^2 + cx_2 + d)$

I plug in the values:

$$\sum w_i f(x_i) = \frac{5}{9}(a(\sqrt{-\frac{3}{5}})^3 + b(\sqrt{-\frac{3}{5}})^2 + c\sqrt{-\frac{3}{5}} + d) +$$
$$\frac{8}{9}(a * 0^3 + b * 0^2 + c0 + d) + \frac{5}{9}(a(\sqrt{\frac{3}{5}})^3 + b(\sqrt{\frac{3}{5}})^2 + c\sqrt{\frac{3}{5}} + d$$
$$\sum w_i f(x_i) = \frac{2b}{3} + 2d$$

The solutions are matching. Hence, it works for degree 3.

**Degree 4:** $f(x) = ax^4 + bx^3 + cx^2 + dx + e$
Solution for Integral: $\int_{-1}^{1} f(x)\, dx = \dfrac{2}{5}a + \dfrac{2}{3}c + 2e$
Solution for Summation: $\sum w_i f(x_i) = w0 * (ax_0^4 + bx_0^3 + cx_0^2 + dx_0 + e) + w1 * (ax_1^4 + bx_1^3 + cx_1^2 + dx_1 + e) + w2 * (ax_2^4 + bx_2^3 + cx_2^2 + dx_2 + e)$
I plug in the values:

$$\begin{aligned}\sum w_i f(x_i) = \frac{5}{9}(a(\sqrt{-\frac{3}{5}})^4 + b(\sqrt{-\frac{3}{5}})^3 + c(\sqrt{-\frac{3}{5}})^2 + d\sqrt{-\frac{3}{5}} + e) + \\ \frac{8}{9}(a * 0^4 + b * 0^3 + c0^2 + d0 + e) + \\ \frac{5}{9}(a(\sqrt{\frac{3}{5}})^4 + b(\sqrt{\frac{3}{5}})^3 + c(\sqrt{\frac{3}{5}})^2 + d\sqrt{\frac{3}{5}} + e \\ \sum w_i f(x_i) = \frac{2a}{5} + \frac{2c}{3} + 2e\end{aligned}$$

The solutions are matching. Hence, it works for degree 4.

**Degree 5:** $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$
Solution for Integral: $\int_{-1}^{1} f(x)\, dx = \dfrac{2}{5}b + \dfrac{2}{3}d + 2f$
Solution for Summation: $\sum w_i f(x_i) = w0 * (ax_0^5 + bx_0^4 + cx_0^3 + dx_0^2 + ex_0 + f) + w1 * (ax_1^5 + bx_1^4 + cx_1^3 + dx_1^2 + ex_1 + f) + w2 * (ax_2^5 + bx_2^4 + cx_2^3 + dx_2^2 + ex_2 + f)$
I plug in the values:

$$\begin{aligned}\sum w_i f(x_i) = \frac{5}{9}(a(\sqrt{-\frac{3}{5}})^4 + b(\sqrt{-\frac{3}{5}})^3 + c(\sqrt{-\frac{3}{5}})^2 + d\sqrt{-\frac{3}{5}} + e) + \\ \frac{8}{9}(a * 0^4 + b * 0^3 + c0^2 + d0 + e) + \\ \frac{5}{9}(a(\sqrt{\frac{3}{5}})^4 + b(\sqrt{\frac{3}{5}})^3 + c(\sqrt{\frac{3}{5}})^2 + d\sqrt{\frac{3}{5}} + e \\ \sum w_i f(x_i) = 2f + \frac{2b}{5} + \frac{2d}{3}\end{aligned}$$

The solutions are matching. Hence, it works for degree 5.

**Degree 6:** $f(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$
Solution for Integral: $\int_{-1}^{1} f(x)\, dx = \dfrac{2}{5}a + \dfrac{2}{3}c + 2e$
Solution for Summation: $\sum w_i f(x_i) = \sum w_i(ax_i^6 + bx_i^5 + cx_i^4 + dx_i^3 + ex_i^2 + fx_i + g)$

I plug in the values:

$$\begin{aligned}\sum w_i f(x_i) = \frac{5}{9}(a(-\sqrt{\frac{3}{5}})^6 + b(-\sqrt{\frac{3}{5}})^5 + c(-\sqrt{\frac{3}{5}})^4 + d(-\sqrt{\frac{3}{5}})^3 \\ +e(-\sqrt{\frac{3}{5}})^2 + f(-\sqrt{\frac{3}{5}}) + g) \\ +\frac{8}{9}(a(0)^6 + b(0)^5 + c(0)^4 + d(0)^3 + e(0)^2 + f(0) + g)+ \\ \frac{5}{9}(a\sqrt{\frac{3}{5}}^6 + b\sqrt{\frac{3}{5}}^5 + c\sqrt{\frac{3}{5}}^4 + d\sqrt{\frac{3}{5}}^3 + e\sqrt{\frac{3}{5}}^2 + f\sqrt{\frac{3}{5}} + g) \\ \sum w_i f(x_i) = 2g + \frac{2c}{5} + \frac{2e}{3} + \frac{6}{25}a\end{aligned}$$

The solutions are not matching. Hence, it does not works for degree 6.

# Problem 4

(a) **Error analysis of a numerical integration rule**

Applying the midpoint quadrature rule (*i.e.* $n = 0$ Newton–Cotes with the quadrature point at the midpoint) on the interval $[t_k, t_{k+1}]$ to $y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(t, y(t))dt$ leads to the implicit *midpoint method*,

$$y_{k+1} = y_k + hf(t_{k+1/2}, (y_k + y_{k+1})/2), \tag{1}$$

where $t_{k+1/2} = t_k + \frac{h}{2}$.

In Problem 4a I use Taylor series expansions to show that the order of accuracy of this method is 2.

To asses the accuracy I take the error into consideration. Hence I look at the truncation error. This is defined as:

$$T_k = \frac{y(t_{k+1}) - y(t_k)}{h} - f(t_{k+1/2}, \frac{y(t_k) + y(t_{k+1})}{2}) \tag{2}$$

At the end of this proof, I want to have something looking like different $O(h^2)$ hence I need to re-phrase some terms and taylor expand some terms.

Let's start from rephrasing the last term on the right, i.e.:

$$f(t_{k+1/2}, \frac{y(t_k) + y(t_{k+1})}{2}) \tag{3}$$

I want to re-write (3) to have the form of:

$$f(t_k + a; y(t_k) + b) \tag{4}$$

To do this I equate:

$$\begin{aligned} t_k + a &= t_{k+1/2} \\ t_k + a &= t_k + \frac{h}{2} \\ a &= \frac{h}{2} \end{aligned}$$

and

$$\begin{aligned} y(t_k) + b &= \frac{y(t_k) + y(t_{k+1})}{2} \\ b &= -\frac{1}{2}y(t_k) + \frac{y(t_{k+1})}{2} \end{aligned}$$

Now I want to taylor-expand (3), which becomes:

$$f(t_k, y(t_k)) + af_t(t_k, y(t_k)) + bf_y(t_k, y(t_k)) + O(h^2) \tag{5}$$

Given this, I can now re-write (2) as:

$$T_k = \frac{y(t_{k+1}) - y(t_k)}{h} - [f(t_k, y(t_k)) + af_t(t_k, y(t_k)) + bf_y(t_k, y(t_k)) + O(h^2)] \tag{6}$$

On the side, I also know that $y(t_{k+1}) = y(t_k + h)$. The taylor expansion of this is:

$$y(t_k) + y'(t_k)h + y''(t_k)h^2\frac{1}{2} + O(h^3) \tag{7}$$

Now I can plug in in (6) the values of $a$ and $b$.

$$T_k = \frac{y(t_{k+1}) - y(t_k)}{h} - [f(t_k, y(t_k)) + \frac{h}{2}f_t(t_k, y(t_k)) + [-\frac{1}{2}y(t_k) + \frac{y(t_{k+1})}{2}]f_y(t_k, y(t_k)) + O(h^2)] \tag{8}$$

Now I plug in in (8) the taylor expansion of $y(t_{k+1})$ (i.e. (7)).

$$\begin{aligned} T_k &= \frac{[y(t_k) + y'(t_k)h + y''(t_k)h^2\frac{1}{2} + O(h^3)] - y(t_k)}{h} - ... \\ T_k &= \frac{[y'(t_k)h + y''(t_k)h^2\frac{1}{2} + O(h^3)]}{h} - ... \end{aligned}$$

and

$$T_k = ... - [f(t_k, y(t_k)) + \frac{h}{2} f_t(t_k, y(t_k)) + [-\frac{1}{2} y(t_k) +$$
$$\frac{[y(t_k) + y'(t_k)h + y''(t_k)h^2\frac{1}{2} + O(h^3)]}{2}] f_y(t_k, y(t_k)) + O(h^2)]$$

$$T_k = ... - [y'(t_k) + \frac{h}{2} f_t(t_k, y(t_k)) + [-\frac{1}{2} y(t_k) +$$
$$\frac{[y(t_k) + y'(t_k)h + y''(t_k)h^2\frac{1}{2} + O(h^3)]}{2}] f_y(t_k, y(t_k)) + O(h^2)]$$

... where the term $y'(t_k)$ will cancel out and will lead to:

$$T_k = \frac{[y''(t_k)h^2\frac{1}{2} + O(h^3)]}{h} - [+\frac{h}{2} f_t(t_k, y(t_k)) + [-\frac{1}{2} y(t_k) +$$
$$\frac{[y(t_k) + y'(t_k)h + y''(t_k)h^2\frac{1}{2} + O(h^3)]}{2}] f_y(t_k, y(t_k)) + O(h^2)]$$

Now I want to take the term $f_t(t_k, y(t_k))$ and re-write it differently. Given that $y''(t_k) = \frac{df}{dt} + \frac{df}{dy} y'(t_k)$, I can rewrite $f_t(t_k, y(t_k))$ as $y''(t_k) - \frac{df}{dy} y'(t_k)$, hence as $y''(t_k) - f_y y'(t_k)$. Hence, I continue:

$$T_k = \frac{[y''(t_k)h^2\frac{1}{2} + O(h^3)]}{h} - [+\frac{h}{2}(y''(t_k) - f_y y'(t_k)) + (-\frac{1}{2} y(t_k) +$$
$$\frac{[y(t_k) + y'(t_k)h + y''(t_k)h^2\frac{1}{2} + O(h^3)]}{2}) f_y(t_k, y(t_k)) + O(h^2)]$$

$$T_k = y''(t_k)\frac{h}{2} + O(h^2) - [+\frac{h}{2}(y''(t_k) - f_y y'(t_k)) + (-\frac{1}{2} y(t_k) +$$
$$\frac{[y(t_k) + y'(t_k)h + y''(t_k)h^2\frac{1}{2} + O(h^3)]}{2}) f_y(t_k, y(t_k)) + O(h^2)]$$

$$T_k = y''(t_k)\frac{h}{2} + O(h^2) - [\frac{h}{2} y''(t_k) - \frac{h}{2} f_y y'(t_k)) - \frac{1}{2} y(t_k) f_y$$
$$+ \frac{y(t_k)}{2} f_y + f_y \frac{y'(t_k)h}{2} + \frac{y''(t_k)\frac{1}{2}h^2}{2} f_y + \frac{O(h^3)}{2} f_y + O(h^2)]$$

...where terms will cancel out to:

$$T_k = y''(t_k)\frac{h}{2} + O(h^2) - [\frac{h}{2} y''(t_k) + \frac{y''(t_k)\frac{1}{2}h^2}{2} f_y + \frac{O(h^3)}{2} f_y + O(h^2)]$$

$$T_k = O(h^2) - [\frac{y''(t_k)\frac{1}{2}h^2}{2} f_y + \frac{O(h^3)}{2} f_y + O(h^2)]$$
$$T_k = O(h^2) - [\frac{y''(\theta)\frac{1}{2}h^2}{2} f_y + O(h^2)]$$
$$T_k = O(h^2) - [O(h^2) + O(h^2)]$$

...which leads $T_k$ to have the accuracy of $O(h^2)$, i.e. second order accurate.

(b) **Stability region of the method for the equation: $y' = \lambda y$.**

In this exercise I want to find for what values of $\bar{h} = h\lambda \in$ the method is stable. I know that:

$$y' = \lambda y$$
$$f(t_k, y(t_k)) = \lambda y(t_k)$$
$$f(t_{k+\frac{1}{2}}, y(t_{k\frac{1}{2}})) = \lambda y(t_{k\frac{1}{2}})$$

Hence:

$$y_{k+1} = y_k + hf(t_{k+1/2}, (y_k + y_{k+1})/2)$$
$$y_{k+1} = y_k + hy'$$
$$y_{k+1} = y_k + h\lambda y(t_{k\frac{1}{2}})$$
$$y_{k+1} = y_k + h\lambda\frac{(y_k + y_{k+1})}{2}$$

...which continues to:

$$y_{k+1} - \frac{h\lambda}{2} y_{k+1} = y_k + \frac{h\lambda}{2} y_k$$
$$...$$
$$\frac{y_{k+1}}{y_k} = \frac{2 + \lambda h}{2 - \lambda h}$$

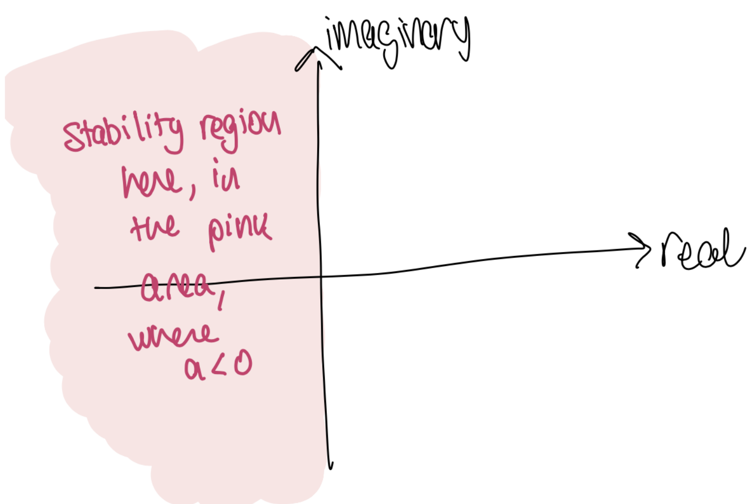
Hence my amplification factor is $\frac{2+\lambda h}{2-\lambda h}$.

I know that $\bar{h} = a + bi$. Hence my amplification factor is: $\frac{2+\lambda(a+bi)}{2-\lambda(a+bi)}$. Now I want to check where this is stable, which is for $\frac{2+\lambda(a+bi)}{2-\lambda(a+bi)} < 1$.

Hence:

$$
\begin{aligned}
(2 + (a + bi) &< (2 - (a + bi) \\
(2 + a)^2 + b^2 &< (2 - a)^2 + b^2 \\
4 + 4a + a^2 + b^2 &< 4 - 4a + a^2 + b^2 \\
8a &< 0 \\
a &< 0
\end{aligned}
$$

Which means that this method is stable for $a < 0$, which is everywhere we need it to be stable for the mathematical stability, which is all to the left of the y axis. Given that $\bar{h}$ is made of an imaginary part $bi$ and of a real part $a$, meaning $\bar{h} = a + bi$, and given that the imaginary numbers are represented across the y axis and given that the real numbers are represented across the x axis, this method is stable everywhere to the left of the y axis.




$\bar{h}$ = real + imaginary
$= a + bi$

if $a<0$, it means real$<0$, hence all to the left of the imaginary (y) axis is stable.