CSCI-GA.2560-001, Artificial Intelligence April 4, 2022

Solutions to Problem 1 of Homework 7 (1 Points)

Name: Anav Prasad (ap7152) *Due: 5PM on Monday, March 28*

Collaborators:

You have a box with 3 coins.
1 of the coins is weighted so that it comes up heads with probability 0.1. (Category 1)
1 is weighted so that it comes up heads with probability 0.4. (Category 2)
1 is weighted so that it come up heads with probability 0.8. (Category 3)

What is the probability of heads, if you pick a coin at random and flip it?

Solution:
Let the coins of categories $1, 2$, and 3 be represented respectively by C_1, C_2, and C_3.
Therefore, in view of the above given information, we can conclude the following:
$P(H \mid C_1) = 0.1$
$P(H \mid C_2) = 0.4$
$P(H \mid C_3) = 0.8$

Also, trivially, the probability of randomly choosing one of these coins is $1/3$. As such:
$P(C_1) = 1/3 = 0.\overline{3}$
$P(C_2) = 1/3 = 0.\overline{3}$
$P(C_3) = 1/3 = 0.\overline{3}$

The probability of a randomly selected coin coming up heads, represented by $P(H)$, is given as follows:

$$\begin{aligned}
P(H) &= P(H, C_1) + P(H, C_2) + P(H, C_3) \\
&= P(H \mid C_1) \cdot P(C_1) + P(H \mid C_2) \cdot P(C_2) + P(H \mid C_3) \cdot P(C_3) \\
&= 0.1 \cdot 0.\overline{3} + 0.4 \cdot 0.\overline{3} + 0.8 \cdot 0.\overline{3} \\
&= 0.0\overline{3} + 0.1\overline{3} + 0.2\overline{6} \\
\implies P(H) &= 0.4\overline{3}
\end{aligned}$$

$\square$

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Solutions to Problem 2 of Homework 7 (2 Points)

Name: Anav Prasad (ap7152) *Due: 5PM on Monday, March 28*

Collaborators:

You randomly select a coin and flip it: it comes up tails. What is the probability that it comes up tails again when you flip the same coin again?

Solution:
First, let's consider some probabilities that may be of need later on:

$$\begin{aligned}
P(T \mid C_1) &= 1 - P(H \mid C_1) = 0.9 \\
P(T \mid C_2) &= 1 - P(H \mid C_2) = 0.6 \\
P(T \mid C_3) &= 1 - P(H \mid C_3) = 0.2
\end{aligned}$$

$$\begin{aligned}
P(T) &= P(T, C_1) + P(T, C_2) + P(T, C_3) \\
&= P(T \mid C_1) \cdot P(C_1) + P(T \mid C_2) \cdot P(C_2) + P(T \mid C_3) \cdot P(C_3) \\
&= 0.9 \cdot \frac{1}{3} + 0.6 \cdot \frac{1}{3} + 0.2 \cdot \frac{1}{3} \\
\therefore P(T) &= \frac{17}{30}
\end{aligned}$$

$$P(C_1 \mid T) = \frac{P(T \mid C_1) \cdot P(C_1)}{P(T)} = \frac{0.9 \cdot \frac{1}{3}}{\frac{17}{30}} = \frac{9}{17}$$

$$P(C_2 \mid T) = \frac{P(T \mid C_2) \cdot P(C_2)}{P(T)} = \frac{0.6 \cdot \frac{1}{3}}{\frac{17}{30}} = \frac{6}{17}$$

$$P(C_3 \mid T) = \frac{P(T \mid C_3) \cdot P(C_3)}{P(T)} = \frac{0.2 \cdot \frac{1}{3}}{\frac{17}{30}} = \frac{2}{17}$$

Now, the probability of a randomly selected coin coming up tails given that the same coin came up tails in the previous flip is as follows:

$$\begin{aligned}
P(T \mid T) &= P(T, C_1 \mid T) + P(T, C_2 \mid T) + P(T, C_3 \mid T) \\
&= P(T \mid C_1, T) \cdot P(C_1 \mid T) + P(T \mid C_2, T) \cdot P(C_2 \mid T) + P(T \mid C_3, T) \cdot P(C_3 \mid T)
\end{aligned}$$

At this point, it can be said that the probability of a coin flip coming up tails given the coin selection already is equal to probability of a coin flip coming up tails given the coin selection and that the coin gave tails in a previous flip. That is so because, given the coin selection, two coin flips are independent events. Therefore,

$$\begin{aligned} P(T \mid T) &= P(T \mid C_1) \cdot P(C_1 \mid T) + P(T \mid C_2) \cdot P(C_2 \mid T) + P(T \mid C_3) \cdot P(C_3 \mid T) \\ &= 0.9 \cdot \frac{9}{17} + 0.6 \cdot \frac{6}{17} + 0.2 \cdot \frac{2}{17} \\ \therefore P(T \mid T) &= \frac{121}{170} \approx 0.71 \end{aligned}$$

$\square$

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Solutions to Problem 3 of Homework 7 (2 Points)

Name: Anav Prasad (ap7152) *Due: 5PM on Monday, March 28*

Collaborators:

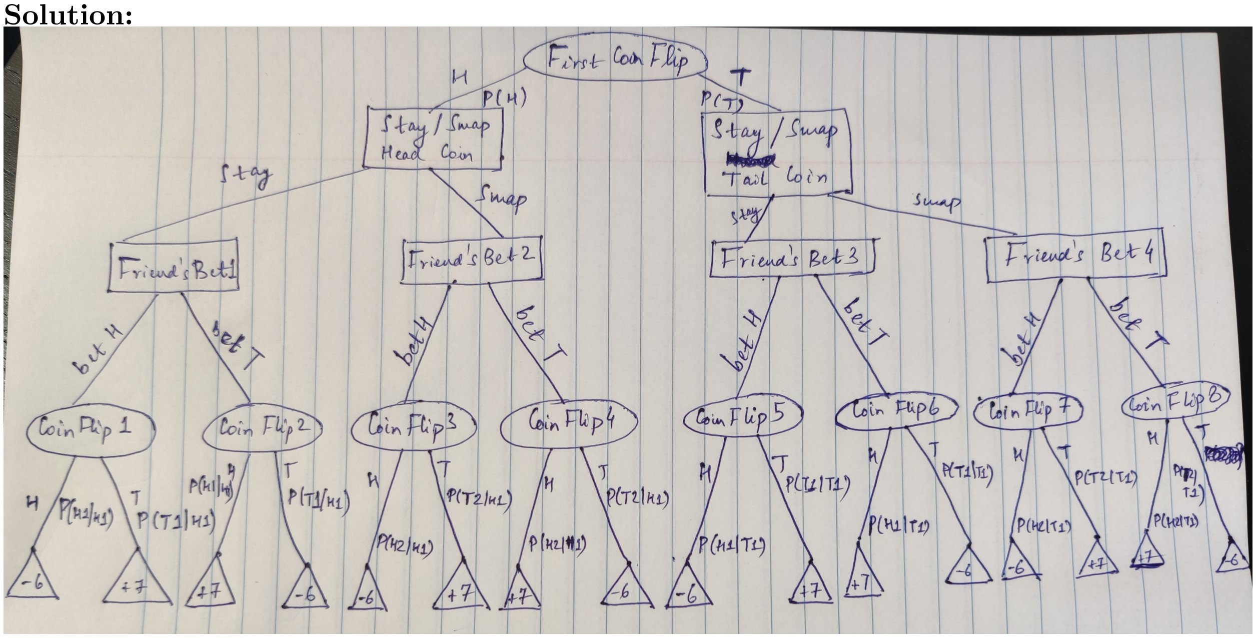
A friend offers you a game: A coin will be randomly selected and flipped.

After you both observe the outcome, you may choose to keep the coin, or set aside the first coin and switch to a new random coin from the remaining in the box.

Once the choice is made, your friend now gets to bet on heads or tails and the updated coin is flipped. If correct, your friend wins \$6, otherwise you win \$7

Draw the decision tree for this game, you should use conditional probabilities without solving for actual values. You may submit a photograph of a hand-drawn diagram for this question only.

Solution:



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Solutions to Problem 4 of Homework 7 (5 Points)	
Name: Anav Prasad (ap7152)	*Due: 5PM on Monday, March 28*
Collaborators:	

Should you play this game? Show your work and explain your answer by solving the tree for expected values.

Solution:
First of all, let us compute the probabilities in the decision tree (along with the auxiliary probabilities):

$$
\begin{aligned}
P(H) &= \frac{13}{30} = 0.4\overline{3} \qquad \text{(computed in Problem 1)} \\
P(T) &= \frac{17}{30} = 0.5\overline{6} \qquad \text{(computed in Problem 2)} \\
P_1(C_1 \mid H) &= \text{ probability of the first coin being } C_1 \text{ given head was the flip's result} \\
&= \frac{P(H \mid C_1) \cdot P(C_1)}{P(H)} \\
&= \frac{0.1 \cdot 1/3}{13/30} \\
\therefore P_1(C_1 \mid H) &= \frac{1}{13} \approx 0.078 \\
P_1(C_1 \mid T) &= \text{ probability of the first coin being } C_1 \text{ given tail was the flip's result} \\
&= \frac{P(T \mid C_1) \cdot P(C_1)}{P(T)} \\
&= \frac{0.9 \cdot 1/3}{17/30} \\
\therefore P_1(C_1 \mid T) &= \frac{9}{17} \approx 0.529 \\
P_1(C_2 \mid H) &= \text{ probability of the first coin being } C_2 \text{ given head was the flip's result} \\
&= \frac{P(H \mid C_2) \cdot P(C_2)}{P(H)} \\
&= \frac{0.4 \cdot 1/3}{13/30} \\
\therefore P_1(C_2 \mid H) &= \frac{4}{13} \approx 0.308
\end{aligned}
$$

$$
\begin{aligned}
P_1(C_2 \mid T) &= \text{ probability of the first coin being } C_2 \text{ given tail was the flip's result} \\
&= \frac{P(T \mid C_2) \cdot P(C_2)}{P(T)} \\
&= \frac{0.6 \cdot 1/3}{17/30} \\
\therefore P_1(C_2 \mid T) &= \frac{6}{17} \approx 0.353 \\
P_1(C_3 \mid H) &= \text{ probability of the first coin being } C_3 \text{ given head was the flip's result} \\
&= \frac{P(H \mid C_3) \cdot P(C_3)}{P(H)} \\
&= \frac{0.8 \cdot 1/3}{13/30} \\
\therefore P_1(C_3 \mid H) &= \frac{8}{13} \approx 0.615 \\
P_1(C_3 \mid T) &= \text{ probability of the first coin being } C_3 \text{ given tail was the flip's result} \\
&= \frac{P(T \mid C_3) \cdot P(C_3)}{P(T)} \\
&= \frac{0.2 \cdot 1/3}{17/30} \\
\therefore P_1(C_3 \mid T) &= \frac{2}{17} \approx 0.118 \\
P_2(C_1 \mid H) &= \text{ probability of the second coin being } C_1 \text{ given head in the first flip} \\
&= \text{probability of choosing } C_1 \text{ for the second coin} \cdot (1 - P_1(C_1 \mid H)) \\
&= \frac{1}{2} \cdot \left(1 - \frac{1}{13}\right) \\
\therefore P_2(C_1 \mid H) &= \frac{6}{13} \approx 0.461 \\
P_2(C_1 \mid T) &= \text{ probability of the second coin being } C_1 \text{ given tail in the first flip} \\
&= \text{probability of choosing } C_1 \text{ for the second coin} \cdot (1 - P_1(C_1 \mid T)) \\
&= \frac{1}{2} \cdot \left(1 - \frac{9}{17}\right) \\
\therefore P_2(C_1 \mid T) &= \frac{4}{17} \approx 0.235 \\
P_2(C_2 \mid H) &= \text{ probability of the second coin being } C_2 \text{ given head in the first flip} \\
&= \text{probability of choosing } C_2 \text{ for the second coin} \cdot (1 - P_1(C_2 \mid H)) \\
&= \frac{1}{2} \cdot \left(1 - \frac{4}{13}\right) \\
\therefore P_2(C_2 \mid H) &= \frac{9}{26} \approx 0.346
\end{aligned}
$$

$$
\begin{aligned}
P_2(C_2 \mid T) &= \text{ probability of the second coin being } C_2 \text{ given tail in the first flip} \\
&= \text{probability of choosing } C_2 \text{ for the second coin} \cdot (1 - P_1(C_2 \mid T)) \\
&= \frac{1}{2} \cdot \left(1 - \frac{6}{17}\right) \\
\therefore P_2(C_2 \mid T) &= \frac{11}{34} \approx 0.324 \\
P_2(C_3 \mid H) &= \text{ probability of the second coin being } C_3 \text{ given head in the first flip} \\
&= \text{probability of choosing } C_3 \text{ for the second coin} \cdot (1 - P_1(C_3 \mid H)) \\
&= \frac{1}{2} \cdot \left(1 - \frac{8}{13}\right) \\
\therefore P_2(C_3 \mid H) &= \frac{5}{26} \approx 0.192 \\
P_2(C_3 \mid T) &= \text{ probability of the second coin being } C_3 \text{ given tail in the first flip} \\
&= \text{probability of choosing } C_3 \text{ for the second coin} \cdot (1 - P_1(C_3 \mid T)) \\
&= \frac{1}{2} \cdot \left(1 - \frac{2}{17}\right) \\
\therefore P_2(C_3 \mid T) &= \frac{15}{34} \approx 0.441 \\
P(H1 \mid H1) &= \text{ probability of getting head given that a head had been obtained in the flip} \\
&\quad\ \text{before with the same coin} \\
&= P(H, C_1 \mid H) + P(H, C_2 \mid H) + P(H, C_3 \mid H) \\
&= P(H \mid C_1, H) \cdot P(C_1 \mid H) + P(H \mid C_2, H) \cdot P(C_2 \mid H) + P(H \mid C_3, H) \cdot P(C_3 \mid H) \\
&= \frac{1}{10} \cdot \frac{1}{13} + \frac{4}{10} \cdot \frac{4}{13} + \frac{8}{10} \cdot \frac{8}{13} \\
\therefore P(H1 \mid H1) &= \frac{81}{130} \approx 0.62 \\
P(H1 \mid T1) &= \text{ probability of getting head given that a tail had been obtained in the flip} \\
&\quad\ \text{before with the same coin} \\
&= P(H, C_1 \mid T) + P(H, C_2 \mid T) + P(H, C_3 \mid T) \\
&= P(H \mid C_1, T) \cdot P(C_1 \mid T) + P(H \mid C_2, T) \cdot P(C_2 \mid T) + P(H \mid C_3, T) \cdot P(C_3 \mid T) \\
&= \frac{1}{10} \cdot \frac{9}{17} + \frac{4}{10} \cdot \frac{6}{17} + \frac{8}{10} \cdot \frac{2}{17} \\
\therefore P(H1 \mid T1) &= \frac{49}{170} \approx 0.288 \\
P(T1 \mid H1) &= \text{ probability of getting tail given that a head had been obtained in the flip} \\
&\quad\ \text{before with the same coin} \\
&= P(T, C_1 \mid H) + P(T, C_2 \mid H) + P(T, C_3 \mid H) \\
&= P(T \mid C_1, H) \cdot P(C_1 \mid H) + P(T \mid C_2, H) \cdot P(C_2 \mid H) + P(T \mid C_3, H) \cdot P(C_3 \mid H) \\
&= \frac{9}{10} \cdot \frac{1}{13} + \frac{6}{10} \cdot \frac{4}{13} + \frac{2}{10} \cdot \frac{8}{13} \\
\therefore P(T1 \mid H1) &= \frac{49}{130} \approx 0.377
\end{aligned}
$$

$$
\begin{aligned}
P(T1 \mid T1) = &\text{ probability of getting tail given that a tail had been obtained in the flip} \\
&\text{ before with the same coin} \\
&= P(T, C_1 \mid T) + P(T, C_2 \mid T) + P(T, C_3 \mid T) \\
&= P(T \mid C_1, T) \cdot P(C_1 \mid T) + P(T \mid C_2, T) \cdot P(C_2 \mid T) + P(T \mid C_3, T) \cdot P(C_3 \mid T) \\
&= \frac{9}{10} \cdot \frac{9}{17} + \frac{6}{10} \cdot \frac{6}{17} + \frac{2}{10} \cdot \frac{2}{17} \\
\therefore P(T1 \mid T1) &= \frac{121}{170} \approx 0.712 \\
P(H2 \mid H1) = &\text{ probability of getting head given that a head had been obtained in the flip} \\
&\text{ before with a different coin} \\
&= P(H2, C_1 \mid H1) + P(H2, C_2 \mid H1) + P(H2, C_3 \mid H1) \\
&= P(H \mid C_1) \cdot P_2(C_1 \mid H) + P(H \mid C_2) \cdot P_2(C_2 \mid H) + P(H \mid C_3) \cdot P_2(C_3 \mid H) \\
&= \frac{1}{10} \cdot \frac{6}{13} + \frac{4}{10} \cdot \frac{9}{26} + \frac{8}{10} \cdot \frac{5}{26} \\
\therefore P(H2 \mid H1) &= \frac{44}{130} \approx 0.338 \\
P(H2 \mid T1) = &\text{ probability of getting head given that a tail had been obtained in the flip} \\
&\text{ before with a different coin} \\
&= P(H2, C_1 \mid T1) + P(H2, C_2 \mid T1) + P(H2, C_3 \mid T1) \\
&= P(H \mid C_1) \cdot P_2(C_1 \mid T) + P(H \mid C_2) \cdot P_2(C_2 \mid T) + P(H \mid C_3) \cdot P_2(C_3 \mid T) \\
&= \frac{1}{10} \cdot \frac{4}{17} + \frac{4}{10} \cdot \frac{11}{34} + \frac{8}{10} \cdot \frac{15}{34} \\
\therefore P(H2 \mid T1) &= \frac{86}{170} \approx 0.506 \\
P(T2 \mid H1) = &\text{ probability of getting tail given that a head had been obtained in the flip} \\
&\text{ before with a different coin} \\
&= P(T2, C_1 \mid H1) + P(T2, C_2 \mid H1) + P(T2, C_3 \mid H1) \\
&= P(T \mid C_1) \cdot P_2(C_1 \mid H) + P(T \mid C_2) \cdot P_2(C_2 \mid H) + P(T \mid C_3) \cdot P_2(C_3 \mid H) \\
&= \frac{9}{10} \cdot \frac{6}{13} + \frac{6}{10} \cdot \frac{9}{26} + \frac{2}{10} \cdot \frac{5}{26} \\
\therefore P(T2 \mid H1) &= \frac{86}{130} \approx 0.662 \\
P(T2 \mid T1) = &\text{ probability of getting tail given that a tail had been obtained in the flip} \\
&\text{ before with a different coin} \\
&= P(T2, C_1 \mid T1) + P(T2, C_2 \mid T1) + P(T2, C_3 \mid T1) \\
&= P(T \mid C_1) \cdot P_2(C_1 \mid T) + P(T \mid C_2) \cdot P_2(C_2 \mid T) + P(T \mid C_3) \cdot P_2(C_3 \mid T) \\
&= \frac{9}{10} \cdot \frac{4}{17} + \frac{6}{10} \cdot \frac{11}{34} + \frac{2}{10} \cdot \frac{15}{34} \\
\therefore P(T2 \mid T1) &= \frac{84}{170} \approx 0.494
\end{aligned}
$$

Computing the expected values now:

$$\begin{aligned}
\text{Coin Flip 1} &= P(H1 \mid H1) \cdot -6 + P(T1 \mid H1) \cdot 7 \\
&= \frac{81}{130} \cdot -6 + \frac{49}{130} \cdot 7 \\
\therefore \textbf{Expected Value Coin Flip 1} &= \frac{-143}{130} = -1.1 \\
\text{Coin Flip 2} &= P(H1 \mid H1) \cdot 7 + P(T1 \mid H1) \cdot -6 \\
&= \frac{81}{130} \cdot 7 + \frac{49}{130} \cdot -6 \\
\therefore \textbf{Expected Value Coin Flip 2} &= \frac{273}{130} = 2.1 \\
\text{Coin Flip 3} &= P(H2 \mid H1) \cdot -6 + P(T2 \mid H1) \cdot 7 \\
&= \frac{44}{130} \cdot -6 + \frac{86}{130} \cdot 7 \\
\therefore \textbf{Expected Value Coin Flip 3} &= \frac{338}{130} = 2.6 \\
\text{Coin Flip 4} &= P(H2 \mid H1) \cdot 7 + P(T2 \mid H1) \cdot -6 \\
&= \frac{44}{130} \cdot 7 + \frac{86}{130} \cdot -6 \\
\therefore \textbf{Expected Value Coin Flip 4} &= \frac{-208}{130} = -1.6 \\
\text{Coin Flip 5} &= P(H1 \mid T1) \cdot -6 + P(T1 \mid T1) \cdot 7 \\
&= \frac{49}{170} \cdot -6 + \frac{121}{170} \cdot 7 \\
\therefore \textbf{Expected Value Coin Flip 5} &= \frac{553}{170} \approx 3.25 \\
\text{Coin Flip 6} &= P(H1 \mid T1) \cdot 7 + P(T1 \mid T1) \cdot -6 \\
&= \frac{49}{170} \cdot 7 + \frac{121}{170} \cdot -6 \\
\therefore \textbf{Expected Value Coin Flip 6} &= \frac{-383}{170} \approx -2.25 \\
\text{Coin Flip 7} &= P(H2 \mid T1) \cdot -6 + P(T2 \mid T1) \cdot 7 \\
&= \frac{86}{170} \cdot -6 + \frac{84}{170} \cdot 7 \\
\therefore \textbf{Expected Value Coin Flip 7} &= \frac{72}{170} \approx 0.42 \\
\text{Coin Flip 8} &= P(H2 \mid T1) \cdot 7 + P(T2 \mid T1) \cdot -6 \\
&= \frac{86}{170} \cdot 7 + \frac{84}{170} \cdot -6 \\
\therefore \textbf{Expected Value Coin Flip 8} &= \frac{98}{170} \approx 0.58
\end{aligned}$$

$$
\begin{aligned}
\text{Friend's Bet 1} &= \min(\text{Expected Value Coin Flip 1},\\
&\qquad \text{Expected Value Coin Flip 2})\\
&= \min(-1.1, 2.1)\\
\therefore \textbf{Expected Value Friend's Bet 1} &= -1.1\\
\text{Friend's Bet 1} &= \min(\text{Expected Value Coin Flip 3},\\
&\qquad \text{Expected Value Coin Flip 4})\\
&= \min(2.6, -1.6)\\
\therefore \textbf{Expected Value Friend's Bet 2} &= -1.6\\
\text{Friend's Bet 3} &= \min(\text{Expected Value Coin Flip 5},\\
&\qquad \text{Expected Value Coin Flip 6})\\
&= \min\left(\frac{553}{170}, \frac{-383}{170}\right)\\
\therefore \textbf{Expected Value Friend's Bet 3} &= \frac{-383}{170} \approx -2.25\\
\text{Friend's Bet 4} &= \min(\text{Expected Value Coin Flip 7},\\
&\qquad \text{Expected Value Coin Flip 8})\\
&= \min\left(\frac{72}{170}, \frac{98}{170}\right)\\
\therefore \textbf{Expected Value Friend's Bet 4} &= \frac{72}{170} \approx 0.42\\
\text{Stay/Swap Head Coin} &= \max(\text{Expected Value Friend's Bet 1},\\
&\qquad \text{Expected Value Friend's Bet 2})\\
&= \max\left(-1.1, -1.6\right)\\
\therefore \textbf{Expected Value Stay/Swap Head Coin} &= -1.1\\
\text{Stay/Swap Tail Coin} &= \max(\text{Expected Value Friend's Bet 3},\\
&\qquad \text{Expected Value Friend's Bet 4})\\
&= \max\left(\frac{-383}{170}, \frac{72}{170}\right)\\
\therefore \textbf{Expected Value Stay/Swap Tail Coin} &= 72/170 \approx 0.42\\
\text{First Coin Flip} &= P(H) \cdot (\text{Expected Value Stay/Swap Head Coin}) +\\
&\quad P(T) \cdot (\text{Expected Value Stay/Swap Tail Coin})\\
&= \frac{13}{30} \cdot -1.1 + \frac{17}{30} \cdot \frac{72}{170}\\
&= \frac{-2431 + 1224}{22100}\\
\therefore \textbf{Expected Value First Coin Flip} &= \frac{-1207}{22100} \approx -0.0546
\end{aligned}
$$

Therefore, for us, the expected value of the entire game is negative and, as such, we should *not* play the game. $\square$