Guide to Expressing Facts in a First-Order Language

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There is no cookbook method for taking a fact expressed in natural language or any other form and expressing it in first-order logic. You have to think though the logical structure of what it is you want to say. However, the following list of suggestions and common errors may be helpful.

1 Syntax.

You must obey the syntactic restrictions of first-order logic. In particular, the arguments to functions and predicates must be terms. Predicates, Boolean operators, and quantifiers cannot appear with in the scope of a function or predicate. For instance, in a language where `Child(x,y)` and `Male(x)` are predicates, the sentences "Sam has a male child" cannot be expressed in the form

(1.1) $\mathtt{Male}(\exists_{\mathtt{x}}\ \mathtt{Child(x,Sam)})$!!! WRONG !!!

The sentence "Mary and Ed are children of Anne" cannot be expressed in the form

(1.2) $\mathtt{Child(Mary} \wedge \mathtt{Ed,\ Anne).}$!!! WRONG !!!

(I have seen things like this, and worse, not only in students' homeworks but in published papers.)

The correct formulations of these are

(1.3) $\exists_{\mathtt{x}}\ \mathtt{Child(x,Sam)} \wedge \mathtt{Male(x).}$
(1.4) $\mathtt{Child(Mary,Anne)} \wedge \mathtt{Child(Ed,Anne).}$

2 Restricted quantification.

Two common form of sentence is "All α's are β" and "Some α's are β." In mathematical notation, these are sometimes written using restricted quantification:

(2.1) $\forall_{\mathtt{x}|\alpha(\mathtt{x})}\ \beta(\mathtt{x})$.
(2.2) $\exists_{\mathtt{x}|\alpha(\mathtt{x})}\ \beta(\mathtt{x})$.

These translate into standard first-order logic, respectively, as

(2.3) $\forall_{\mathtt{x}}\ \alpha(\mathtt{x}) \Rightarrow \beta(\mathtt{x})$.
(2.4) $\exists_{\mathtt{x}}\ \alpha(\mathtt{x}) \wedge \beta(\mathtt{x})$.

For instance, the sentences "All crows are black" and "Some squirrels are black" are represented, respectively

(2.5) $\forall_{\mathtt{x}}$ `Crow(x)` $\Rightarrow$ `Black(x).`
(2.6) $\exists_{\mathtt{x}}$ `Squirrel(x)` $\wedge$ `Black(x).`

3 Two easily detected errors

If you have written a formula of the form $\forall\alpha(\mathtt{x}) \wedge \beta(\mathtt{x})$ then you have probably made a mistake. To check, translate it to the logically equivalent form $[\forall_{\mathtt{x}}\alpha(\mathtt{rxX})] \wedge [\forall_{\mathtt{y}}\beta(\mathtt{y})]$" and see if it still looks good. Both forms mean "Everything is both α and β" or equivalently "Everything is α and everything is β."

If you have written a formula of the form $\exists_{\mathtt{x}}\alpha(\mathtt{x})\Rightarrow\beta(\mathtt{x})$ then it is 100 to 1 that you have made a mistake. Keep in mind that this is equivalent to $\exists_{\mathtt{x}}\neg\alpha(\mathtt{x}) \vee \beta(\mathtt{x})$, which is in turn equivalent to $[\exists_{\mathtt{x}}\neg\alpha(\mathtt{x})] \vee [\exists_{\mathtt{y}}\beta(\mathtt{y})]$; i.e. either there exists something that is not α or there exists something that is β. For instance the formula

(3.1) $\exists_{\mathtt{x}}$ `Crow(x)` $\Rightarrow$ `Black(x).` !!! WRONG !!!

means "Either [there exists something in the universe is not a crow] or [there exists something that is black]".

The only correct uses of the second form above I have seen have been where the variable `x` does not actually appear in α, the left hand side of the implication. For instance, the sentence "If Y is even then there exists X such that $Y = X + X$ can be correctly written

(3.2) $\forall_{\mathtt{y}}$ $\exists_{\mathtt{x}}$ `Even(x)` $\Rightarrow$ `x=y+y.`

But, unless there is some strong reason to wish all the quantifiers to be prenex, it is in my opinion much clearer to write quantifiers with the smallest possible scope, and therefore rewrite this sentence,

(3.3) $\forall_{\mathtt{x}}$ `Even(x)` $\Rightarrow$ $\exists_{\mathtt{y}}$ $\mathtt{x} = \mathtt{y} + \mathtt{y}$.

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4 Chained implications.

In mathematical writing, it is common to say "α implies β implies γ", meaning that α implies β which implies γ, and sometimes to notate this $\alpha\Rightarrow\beta\Rightarrow\gamma$. Likewise, the statement that α, β and γ all have the same truth value is sometimes notates $\alpha\Leftrightarrow\beta\Leftrightarrow\gamma$. It is important to keep in mind that what the first of these really means is

(4.1) $[\alpha\Rightarrow\beta] \wedge [\beta\Rightarrow\gamma]$

and what the second really means is

(4.2) $[\alpha\Leftrightarrow\beta] \wedge [\beta\Leftrightarrow\gamma]$

For instance, "Squirrels are rodents, which are mammals," is represented

(4.3) $\forall_x$ `[Squirrel(x)` $\Rightarrow$ `Rodent(x)]` $\wedge$ `[Rodent(x)` $\Rightarrow$ `Mammal(x)]`.

In particular (4.1) is NOT equivalent to either

(4.4) $\alpha \Rightarrow (\beta \Rightarrow \gamma)$, nor to
(4.5) $(\alpha \Rightarrow \beta) \Rightarrow \gamma$,

and (4.2) is not equivalent to

(4.6) $\alpha \Leftrightarrow (\beta \Leftrightarrow \gamma)$, nor to
(4.7) $(\alpha \Leftrightarrow \beta) \Leftrightarrow \gamma$,

5 Don't trust the English: Quantifiers.

Natural language were not designed by logicians, and a single English form can have very different logical meanings in different sentences.

For instance, it's often the case that words like "some", "something", "someone", and "a" correspond to an existential quantifiers. For example "Some squirrels are black," or "Jack saw a squirrel" can be represented

(5.1) $\exists_x$ `Squirrel(x)` $\wedge$ `Black(x)`.
(5.2) $\exists_x$ `Squirrel(x)` $\wedge$ `Saw(Jack,x)`.

However, in the sentence "If someone is eighteen, then they are allowed to vote," the meaning is that this implication is true of everyone. The representation is therefore

(5.3) $\forall_x$ `Geq(Age(x),Times(18,Year))` $\Rightarrow$ `Legal(Do(x,Vote))`.

NOT (5.4) $\exists_x$ `Geq(Age(x),Times(18,Year))` $\Rightarrow$ `Legal(Do(x,Vote))`. WRONG!!!
(See #3 above)

STILL LESS (5.5) [$\exists_x$ `Geq(Age(x),Times(18,Year))]` $\Rightarrow$ `Legal(Do(x,Vote))`. WRONG!!!

Formula 5.4, as discussed in section means "Either there is someone who is not at least 18, or there is someone who can vote." Formula 5.5 means "If there is someone is at least 18 then `x` can vote." (Since the `x` inside `Legal(Do(x,Vote))` is outside the scope of the quantifier, it is a different variable than the `x` in the first part of 5.5.)

Likewise, in the sentence "A squirrel is a rodent", the meaning is that this is true of all squirrels.

(5.6) $\forall_x$ `Squirrel(x)` $\Rightarrow$ `Rodent(x)`.

In the sentence "A vegetarian is a person who eats no meat," the meaning of "a vegetarian" is "all vegetarians," and the sentence as a whole is definitional; it is represented by the biconditional,

(5.7) $\forall_x$ `Vegetarian(x)` $\Leftrightarrow$ `[`$\neg$ $\exists_y$ `Meat(y)` $\wedge$ `Eats(x,y)]`.

Different meanings of "a" may appear together in the same sentence. For instance, the sentence "If a person has a licence, then they can legally drive," is represented

(5.8) $\forall_x$ `[`$\exists_y$ `License(y)` $\wedge$ `Owns(x,y)]` $\Rightarrow$ `Legal(Do(x,Drive))`.

The statement is true of all people, but each person has only one license. Similarly, the sentence (the implicit content of car ads) "A worthwhile person owns an expensive car," is represented

(5.9) $\forall_x$ `Worthwhile(x)` $\Rightarrow$ $\exists_y$ `Expensive(y)` $\wedge$ `Car(y)` $\wedge$ `Owns(x,y)`.

The statement is asserted about all people and asserts that each person, if worthwhile, owns at least one car.

The sentence, "Lucy owns a parrot that is larger than a cat," the parrot is existentially quantified (she owns one parrot) but the cat is universally quantified (the parrot is larger than any cat.) (More precisely, the meaning is probably that the parrot is larger than a *typical* cat; but that would not be expressible in first-order logic, and the universal quantifier is closer to what is meant than the existential quantifier.)

(5.10) $\exists_x$ `Parrot(x)` $\wedge$ `Owns(Lucy,x)` $\wedge$ $\forall_y$ `Cat(y)` $\Rightarrow$ `Larger(x,y)`.

6 In designing a language, don't shortchange the repreentation!

Especially: If you need to represent a world in which things change, you need to include a representation of time.

For example "If the light is off, and you flip the switch, the light will be on." It is tempting to represent this as

(6.1) `Off(Light)` $\wedge$ `Flip(Switch)` $\Rightarrow$ `On(Light)`. !!WRONG!!

But then given the axiom

(6.2) `Off(Light)` $\wedge$`Flip(Switch)`

you can deduce that the light is both on and off.

Adding an axiom that the light cannot be both on and off,

(6.3) $\neg$`[Off(Light)` $\wedge$ `On(Light)]`

makes things worse. This would not mean that you should cancel `Off(Light)` when `On(Light)` becomes true; there is no way in predicate calculus to cancel anything. Rather it would mean that (6.2) `Off(Light)` $\wedge$ `Flip(Switch)` is inconsistent; or in other words, if the light is off, then you cannot flip the switch.

Also, there is no relation between the time when a fact is inferred in a reasoning system and the time it becomes true. The fact `On(Light)` does not *become* true when you infer it; it is always, timelessly, true.

To handle this correctly, you need to have a language that indicates the times of things. There are a number of different ways to do this. Here is one:

(6.4) $\forall_{t1,t2}$ `Holds(t1,Off(Light))` $\wedge$ `Occurs(t1,t2,Flip(Switch))` $\Rightarrow$
`Holds(t2,On(Light))`

(6.5) $\forall_t$ $\neg$`[Holds(t,Off(Light))` $\wedge$ `Holds(t,On(Light))]`.

7 Alternative correct representation.

Obviously, there are multiple logically equivalent ways of representing a given sentence. For instance "A vegetarian is someone who eats no meat" can be represented in any of the logically equivalent forms:

(7.1) $\forall_x$ `Vegetarian(x)` $\Leftrightarrow$ `[`$\neg\exists_y$ `Meat(y)` $\wedge$ `Eats(x,y)]`.
(7.2) $\forall_x$ `Vegetarian(x)` $\Leftrightarrow$ `[`$\forall_y$ `Eats(x,y)` $\Rightarrow$ $\neg$`Meat(y)]`.
(7.3) $\forall_x$ `[`$\exists_y$ `Meat(y)` $\wedge$ `Eats(x,y)]` $\Leftrightarrow$ $\neg$`Vegetarian(x)`.

and many others.

Less obviously, there may be multiple correct ways of representing a given statement relative to an implicit body of background knowledge. For example, the statement "Pamela has a long nose" may be represented either as

(7.4) $\exists_x$ `NoseOf(x,Pamela)` $\wedge$ `Long(x)` or as
(7.5) $\forall_x$ `NoseOf(x,Pamela)` $\wedge$ `Long(x)`

The two are equivalent given that people have exactly one nose. Absent that information, (6.4) is consistent with the possibility that Pamela has some number of short noses in addition to at least one long nose and (6.5) is consistent with the possibility that Pamela has no nose, or multiple noses, all of which are long.

Another example: the statement "Alabama borders Mississippi" can be represented either as

`(7.6) Borders(Alabama, Mississippi).` or as
`(7.7) Borders(Mississippi, Alabama).`

given that the `Borders` relation is symmetric.

8 Limits on First-Order Representation

You should keep in mind, also, that most natural language sentences *cannot* be translated exactly to first-order logic. In many cases you have to sacrifice some of the nuance of the original text. In some cases, there is not even any reasonable approximation in first-order logic.

9 Check your answer.

When you have written down a representation in first-order logic, read it back and make sure that it means what you intended. Keep in mind that quantifiers apply to everything in the universe, and that $\alpha \Rightarrow \beta$ means just "either [not α] or β". For example if you have translated, "A squirrel is a rodent" as $\forall_{\mathtt{x}}$ `Squirrel(x)` $\Rightarrow$ `Rodent(x)`, read that formula back as "For every x in the universe, either x is not a squirrel or x is a rodent" and then figure out whether this means the same thing as the original sentence.