Examples of expressions in FOL

Example

Every man is mortal	$\forall x(man(x) \rightarrow mortal(x))$
Socrate is a man	$man(Socrate)$
Socrate is mortal	$mortal(Socrate)$

Components of First Order Logic

Objects, Relations, Functions

Whereas propositional logic assumes world contains facts, first-order logic (like natural language) assumes the world contains: **Objects, Relations, Functions**.

- Objects: people, houses, numbers, theories, colors, football games, wars, centuries $\cdots$
- Relations: red, round, multistoried $\cdots$, brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, $\cdots$
- Functions: father of, best friend, second half of, one more than, beginning of $\cdots$

The Language: Logical Symbols

Logical Symbols

A first order language $\mathcal{L}$ is built upon the following sets of symbols:

- propositional connectives: $\neg, \wedge, \vee$
  (plus the shortcuts $\rightarrow$ and $\leftrightarrow$);
- propositional constants $\top$ and $\perp$
  (represent **True** and **False** respectively);
- equality $=$
  (not always included);
- a denumerable set of individual variable symbols:
  $x_1, x_2, \cdots$;
- universal quantification $\forall$;
- existential quantification $\exists$;

The Language: Parameters

Parameters

- A denumerable set of predicate symbols, each associated with a positive integer n, arity. A predicate with arity n is called n-ary;
- A denumerable set of function symbols, each associated with a positive integer n, arity. A function with arity n is called n-ary;
- A denumerable set of constant symbols.

Note

The parameters characterise different first order languages, while logical symbols are always the same.
Therefore parameters are often called the Signature of a First Order Language.

Example I

The language of pure predicates

n-ary predicate symbols: $P_1^n, P_2^n, \cdots$;

constant symbols: $c_1, c_2, \cdots$;

no function symbols, no equality.

Example

The Book is on the table:

- OnTable(Book)
- On(Table,Book)

Example II

The language of set theory

- Equality;
- predicate symbols: only the binary predicate $\in$;
- constant symbols: $\{\,\}$;
- no function symbols.

Example

There exists no set such that all other sets are its element

$$\neg\exists x \forall y (y \in x)$$

Example III

The language of elementary number theory

Equality;

predicate symbols: only the binary predicate $<$;

constant symbols: 0;

function symbols: a unary function symbol s, successor function, and the binary function symbols $+$ and $\times$, addition and multiplication

Example

There exists no number greater than all others

$$\neg \exists x \forall y (y < x)$$

Definition of FOL Formulas

FOL formulas

Inductive definition of basic components

1. Terms
2. Atomic Formulas

Terms

FOL terms

The set Term of the terms of $\mathcal{L}$ is inductively defined as follows:

1. Every constant is a term;
2. Every variable symbol is a term
3. If $t_1 \ldots t_n$ are terms and f is a n-ary function symbol, $f(t_1, \ldots, t_n)$ is a term (functional term).
4. All terms are generated by applying the above rules

Example (Terms for FOL)

$c, \quad x, \quad f(x, y), \quad f(g(c), y), \quad plus(plus(x, 1), 3), \ \ldots$

Atomic Formulas

Atoms

The set Atom of the atomic formulae is inductively defined as follows:

1. $\bot$ and $\top$ are atoms;
2. If t_1 and t_2 are terms then $t_1 = t_2$ is an atom;
3. If $t_1, \cdots, t_n$ are terms and P is a n-ary predicate symbol $P(t_1, \cdots, t_n)$ is an atom;
4. All atomic formulas are defined by applying the above rules

Example (Atoms in FOL)

$P(x)$, $Q(x, c)$, $R(x, f(x, y + c))$, $\cdots$

Scope of Quantifiers

Definition (Scope of quantifiers)

The scope of a quantifier occurring in a formula is the formula to which the quantifier applies

Example (Scope of quantifiers)

$\forall x(Q(x) \to R(x))$ the scope of $\forall$ is $(Q(x) \to R(x))$
$\forall x(Q(x) \to \exists y\ R(y))$ the scope of $\forall$ is $(Q(x) \to \exists y\ R(y))$ and the scope of $\exists$ is $R(y)$

Free and bounded variables

Definition (Free occurence of a variable)

An occurrence of a variable in a formula is free if the variable is not in the scope of any quantifier. An occurence of a variable which is not free is bound

Definition (Free variable)

A variable in a formula is free if at least one occurrence of the variable is free. A variable is bound if at least one occurrence is bound.

Examples of free and bound variables

Example (Free occurences and free variables)

$\forall x(Q(x,y) \rightarrow R(x,y))$ the occurence of y is free while the occurence of x is bound, therefore y is free while x is bound
$\forall x(Q(x,y) \rightarrow \exists y \; R(x,y))$ the occurrence of y in Q is free while the occurence of y in R is bound, the occurrences of x in both formulas are bound. Therefore, the variable x is bound while the variable y is both free and bound

First Order Formulas

Well-Formed Formulas

The set of formulae of $\mathcal{L}$ is inductively defined as follows:

- Every atom is a formula;
- If A is a formula $\neg A$ is a formula;
- If $\circ$ is a binary operator, A and B are formulas, then $A \circ B$ is a formula;
- If A is a formula, x is a free variable in A then $\forall x A$ and $\exists x A$ are formulas
- All formulas are generated by a finite number of applications of the above rules.

Example (FOL Formulas)

$P(x), \quad \exists x Q(x, c), \quad \forall x R(x, f(x, y + c)), \cdots$

Operator Precedence

Operator Precedence

Precedence among logical operators is defined as follows:

$$\forall, \exists, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$$

convention: all operators are right associative (as in propositional logic).

Example

$$\forall x P(x) \rightarrow \exists y \exists z Q(y, z) \wedge \neg \exists x R(x)$$

$$(\forall x P(x)) \rightarrow \exists y (\exists z (Q(y, z) \wedge \neg (\exists x (R(x))).$$

Note

The inner occurrence of x is bound to the innermost existential quantifier

Ground and Closed Formulas

Definition (Ground Formula)

A formula F is ground if it does not contain variables

Definition (Closed Formula)

A formula F is closed if it does not contain free variables

Example (Ground and Closed Formulas)

$Boring(GrandeFratello)$	(ground)
$\forall x(Reality(x) \rightarrow Boring(x))$	(closed, not ground)
$\forall x(Reality(x) \rightarrow BetterProgram(y, x))$	(not closed, not ground)

Example of FOL formalisation

Example (Basic axioms of natural language)

- A_1: for every number there is one and only one immediate successor
- A_2: there is no number for which 0 is the immediate successor
- A_3: for every number other than 0 there is one and only one immediate predecessor

Assume:

- $s(x)$ is function for immediate successor
- $p(x)$ is function for immediate predecessor
- $E(x, y)$ is true iff x is equal to y

Example contd.

- $A_1 \triangleq \forall x \exists y (E(s(x), y) \wedge (\forall z)(E(s(x), z) \rightarrow E(z, y)))$
- $A_2 \triangleq \neg((\exists x) E(s(x), 0))$
- $A_3 \triangleq \forall x(\neg E(x, 0) \rightarrow \exists y(E(p(x), y) \wedge (\forall z)(E(p(x), z) \rightarrow E(z, y)))$

Interpretations in FOL

- In Prop. Logic an Interpretation for a formula G is an assignment of truth values to each atoms occuring in the formula
- In FOL we have to do more than that:
  1. Specify a domain of interest (e.g., real numbers)
  2. An assignment to constants, function symbols and predicate symbols

Example (Interpretation)

Consider the set of formulas: $\{\forall x P(x), \exists x Q(x)\}$;
An interpretation will need to specify a domain, e.g. $D = \{1, 2\}$ and an assignment for all predicate symbol from D to the set $\{T, F\}$, for example $\{P(1) = T, P(2) = F\}$ and $\{Q(1) = F, Q(2) = T\}$.

Interpretation: Formal Definition

Definition of Interpretation

An Interpretation for the language $\mathcal{L}$ is a pair $I = \langle D, A \rangle$ where:

- D is a non empty set called domain of I;
- A is a function that maps:
  - every constant symbol c into an element $c^A \in D$;
  - every n-ary function symbol f into a function $f^A : D^n \to D$;
  - every n-ary predicate symbol P into a n-ary relation $P^A : D^n \to \{\top, \bot\}$.

Interpretation: Example

Example (Interpretation)

$$\forall x \exists y P(x, y)$$

- D, the set of human beings
  $P^A(a, b) = true$ iff b is father of a
  All human beings have a father
- D, the set of human beings
  $P^{A'}(a, b) = true$ iff b is mother of a
  All human beings have a mother
- D the set of natural numbers
  $P^{A''}(a, b) = true$ iff $a < b$
  For every nat number there is a greater one