Radiation Balance Between Parallel Plates

Assignment is completed by Mikkel Jaedicke (mijae12) & Anders Bæk (anbae12).

Equation 1 is known to be true for Saturn's double moon. Where $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$ denotes moon i's position in a cartesian coordinate system with Saturn as origo.

$$\begin{aligned} m_1 \frac{d^2}{dt^2} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= -\frac{m_1 \cdot M \cdot g}{r_1^3} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \frac{m_1 \cdot m_2 \cdot g}{r_{12}^3} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \\ m_2 \frac{d^2}{dt^2} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= -\frac{m_2 \cdot M \cdot g}{r_2^3} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \frac{m_1 \cdot m_2 \cdot g}{r_{12}^3} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \end{aligned} \tag{1}$$

where: $m_1 = m_2 = 9.20 \cdot 10^{18}$ [kg], $M = 5.68 \cdot 10^{26}$ [kg], $g = 4.98 \cdot 10^{-10} \left[\frac{\text{km}^3}{\text{kg} \cdot \text{døgn}}\right]$.

The initial values in equation 2 are also known to be true.

$$\begin{aligned} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 152870 \end{bmatrix}, \quad \frac{d}{dt} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1360278.1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ -153130 \end{bmatrix}, \quad \frac{d}{dt} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1359122.8 \\ 0 \end{bmatrix} \end{aligned} \tag{2}$$

---

1 Calculations

Equation 3 is obtained by dividing equation 1 by m_i.

$$\begin{aligned} \frac{d^2}{dt^2} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= -\frac{M \cdot g}{r_1^3} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \frac{m_2 \cdot g}{r_{12}^3} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \\ \frac{d^2}{dt^2} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= -\frac{M \cdot g}{r_2^3} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \frac{m_1 \cdot g}{r_{12}^3} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \end{aligned} \tag{3}$$

Equation 3 is seen to be a second-order conservative equation, which in general can be written in the form $\frac{d^2y}{dt} = f(x,y)$. Second-order conservative equations can efficiently be solved by using Stoermer's rule[1]. Stoermer's rule can be used in NR LIB by using the `StepperStoerm<rhs>`. Equation 4 shows the y vector and equation 5 shows the function $f(x,y)$, where x is not used.

$$y = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \frac{dx_1}{dt} \\ \frac{dy_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} \tag{4}$$

[1] Chap. 17.4. Numerical Recipes: The Art of Scientific Computing, ISBN-13: 978-0521880688

$$f(x,y) = \begin{bmatrix} \frac{-M\cdot g}{\left(\left(x_1^2+y_1^2\right)^{0.5}\right)^3}\cdot y\,[0] + \frac{m_2\cdot g}{\left(\left((x_1-x_2)^2+(y_1-y_2)^2\right)^{0.5}\right)^3}\cdot (y\,[2]-y\,[0]) \\ \frac{-M\cdot g}{\left(\left(x_1^2+y_1^2\right)^{0.5}\right)^3}\cdot y\,[1] + \frac{m_2\cdot g}{\left(\left((x_1-x_2)^2+(y_1-y_2)^2\right)^{0.5}\right)^3}\cdot (y\,[3]-y\,[1]) \\ \frac{-M\cdot g}{\left(\left(x_2^2+y_2^2\right)^{0.5}\right)^3}\cdot y\,[2] + \frac{m_1\cdot g}{\left(\left((x_1-x_2)^2+(y_1-y_2)^2\right)^{0.5}\right)^3}\cdot (y\,[2]-y\,[0]) \\ \frac{-M\cdot g}{\left(\left(x_2^2+y_2^2\right)^{0.5}\right)^3}\cdot y\,[3] + \frac{m_1\cdot g}{\left(\left((x_1-x_2)^2+(y_1-y_2)^2\right)^{0.5}\right)^3}\cdot (y\,[3]-y\,[1]) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

The position of the moons in polar coordinates are practical to have. The polar coordinates are calculated by equation 6.

$$\begin{aligned} r_1 &= \sqrt{x_1^2+y_1^2}\ [\text{km}] \\ \theta_1 &= \arctan\left(\frac{y_1}{x_1}\right)\ [\text{rad}] \\ r_2 &= \sqrt{x_2^2+y_2^2}\ [\text{km}] \\ \theta_2 &= \arctan\left(\frac{y_2}{x_2}\right)\ [\text{rad}] \end{aligned} \tag{6}$$

2 Code

Code Snippet 1 is a struct which contains the implementation of equation 5.

```
struct rhs{
    void operator() (const Doub x, VecDoub &y, VecDoub &f){
        f[0] = -M*g/(pow(pow(y[0]*y[0]+y[1]*y[1],0.5),3))*y[0] + m2*g/(pow(pow(pow(y[0]-y[2],2)+pow(y[1]-y[3],2),0.5),3))*(y[2]-y[0]);
        f[1] = -M*g/(pow(pow(y[0]*y[0]+y[1]*y[1],0.5),3))*y[1] + m2*g/(pow(pow(pow(y[0]-y[2],2)+pow(y[1]-y[3],2),0.5),3))*(y[3]-y[1]);
        f[2] = -M*g/(pow(pow(y[2]*y[2]+y[3]*y[3],0.5),3))*y[2] - m1*g/(pow(pow(pow(y[0]-y[2],2)+pow(y[1]-y[3],2),0.5),3))*(y[2]-y[0]);
        f[3] = -M*g/(pow(pow(y[2]*y[2]+y[3]*y[3],0.5),3))*y[3] - m1*g/(pow(pow(pow(y[0]-y[2],2)+pow(y[1]-y[3],2),0.5),3))*(y[3]-y[1]);
        f[4] = 0 ;
        f[5] = 0 ;
        f[6] = 0 ;
        f[7] = 0 ;
    }
};
```

Code Snippet 1: Struct used by `StepperStoerm<rhs>`.

Code Snippet 2 line 38-57 contains the solving process of the ODE. Line 58-64 can be modified to different output types.

```
38 int main() {
39     // Initialize y start values
40     VecDoub y_start(8);
41     y_start[0] = 0;
42     y_start[1] = 152870;
43     y_start[2] = 0;
44     y_start[3] = -153130;
```

```
    y_start[4] = -1360278.1;
    y_start[5] = 0;
    y_start[6] = 1359122.8;
    y_start[7] = 0;

    // Set stepper configurations
    const Doub atol=1.0e-3, rtol=0, h1=0.01, hmin=0.0, x1=0.0, x2=500.0;

    Output out(500);
    rhs func;

    Odeint<StepperStoerm<rhs> > ode(y_start,x1,x2,atol,rtol,h1,hmin,out,func);
    ode.integrate();
    /* Do what you need whit the output...
    for (int i=0;i<out.count;i++){
        Doub r1 = pow(pow(out.ysave[0][i],2) + pow(out.ysave[1][i],2),0.5)
        Doub r2 = pow(pow(out.ysave[2][i],2) + pow(out.ysave[3][i],2),0.5);
        Doub dTheta = atan2(out.ysave[1][i], out.ysave[0][i]) - atan2(out.ysave
            [3][i], out.ysave[2][i]);
    }
     */
    return 0;
}
```

Code Snippet 2: mian.cpp

3 Results

In figure 1 the moons's distance to Saturn r_1 and r_2 is depicted. It is clearly seen that the moons change orbit at around day 140 and day 430.
The lower figure shows the difference in the moons's angle θ_1 and θ_2 is depicted. It is clearly seen that the angle difference is 0 or 2π at day 140 and day 430.

3.1 Precision of the results

The local error at each iteration is set in $atol = 10^{-3}$ and $rtol = 0$. The `StepperStoerm<>` method in NR LIB uses the adaptive step size approach, meaning that the method will adapt the stepsize to have a smaller local error than 10^{-3}. When Output out(-1) is set the method outputs the steps that it uses to obtain a smaller local error than specified.
out.count gives the steps used by the method. In this exercise with $atol = 10^{-3}$, $rtol = 0$, $x1 = 0.0$ and $x2 = 500.0$ it gives 4166 steps.
The maximum global error at the last point must therefore be
$4166 \cdot 10^{-3} = 4.166[km]$.

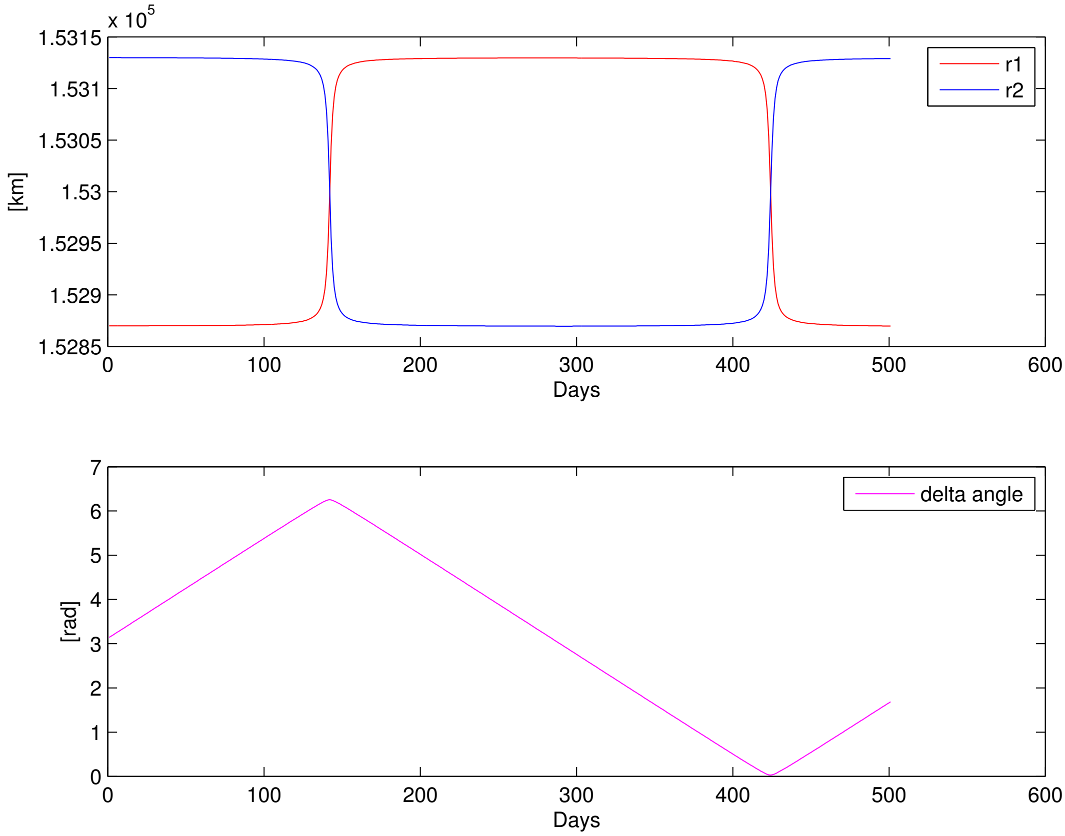

Figure 1: The upper plot depicts the moons's distance to Saturn r_1 and r_2 as a function of time. The lower plot depicts the difference in the moons's angle θ_1 and θ_2.