Proving 6174 as the unique Kaprekar's constant

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Let $K(\alpha)$ be an operation on a 4-digit positive integer with condition that not all digits are the same. The operation is as follows, if $\alpha = \overline{abcd}$ then $K(\alpha) = \overline{a_1a_2a_3a_4} - \overline{a_4a_3a_2a_1}$ with $a_1 \geq a_2 \geq a_3 \geq a_4$ and $\{a, b, c, d\} = \{a_1, a_2, a_3, a_4\}$. Examples:

- $$K(1234) = K(1243) = K(1324) = ... = K(4321)$$
  $$= 4321 - 1234 = 3087$$

- $$K(2192) = 9221 - 1229 = 7992$$

- $$K(1000) = 1000 - 0001 = 1000 - 1 = 999$$

A Kaprekar's constant $\overline{abcd}$ is such that if $K(\overline{abcd}) = \overline{ABCD}$ and $\{A, B, C, D\} = \{a, b, c, d\}$, meaning that the resulting digits are the same as before the operation. We will show that the only Kaprekar's constant is 6174.

Proof:

Let us check $K(6174)$, it is $7641 - 1467 = 6174$. Now let us prove that it is the only 4-digit number with that behavior. Let $\overline{abcd}$ be such that $K(\overline{abcd}) = \overline{ABCD}$ and $\{A, B, C, D\} = \{a, b, c, d\}$.

$$K(\overline{abcd}) = \overline{a_1a_2a_3a_4} - \overline{a_4a_3a_2a_1}$$

so

$$D = a_4 - a_1 + 10$$

$$C = a_3 - a_2 + 9$$

Clearly $a_1 > a_4$ because not all digits are same. Now if $a_2 = a_3$ then

$$B = a_2 - a_3 + 9$$

$$A = a_1 - a_4 - 1$$

so $B = C = 9$, but 9 is the largest digit, so a_1 must be 9 too. If $D = a_1 = 9$, we have $D = 9 = a_4 - a_1 + 10 = a_4 + 1$ which means $a_4 = 8$. So $A = 8$. But $A = a_1 - a_4 - 1 = 0$, a contradiction. Now if $A = a_1 = 9$, we have $A = 9 = a_1 - a_4 - 1 = 8 - a_4$, so $a_4 = -1$ also a contradiction. So we cannot have $a_2 = a_3$ in order for $\overline{abcd}$ to be Kaprekar. So $a_1 \geq a_2 > a_3 \geq a_4$ and

$$D = a_4 - a_1 + 10$$
$$C = a_3 - a_2 + 9$$
$$B = a_2 - a_3 - 1$$
$$A = a_1 - a_4$$

Notice that $B + C = 8$ and $A + D = 10$. Since $a_1 - a_4 \geq a_2 - a_3$, then $A \geq B$. And similarly, $C \geq D - 1$. Now if $C = D - 1$, then this is true if and only if $a_3 - a_2 = a_4 - a_1$, which means $a_2 = a_1$, $a_3 = a_4$. So we have $\overline{a_1a_2a_3a_4} = \overline{xxyy}$, so we have two pairs of same digit. Also $A = B + 1$. So clearly either $A = C, B = D$ or $A = D$, $B = C$. If $A = C, B = D$ then $A = D - 1 = B - 1$ contradicts $A = B + 1$. So $A = D, B = C$. And from $A + D = 10, B + C = 8$ we get $A = D = 5, B = C = 4$. So $a_3 = a_4 = 4$, and $a_1 = a_2 = 5$, but then $B = a_2 - a_3 - 1 = 1 - 1 = 0 \neq 4$ contradiction. So we cannot have $C = D - 1$ for Kaprekar solution. So we must have $A \geq B$ and $C \geq D$. Then the possibilities are:

$$\overline{ABCD} = \overline{a_1a_2a_3a_4} \tag{1}$$

$$\overline{ABCD} = \overline{a_1a_3a_2a_4} \tag{2}$$

$$\overline{ABCD} = \overline{a_1a_4a_2a_3} \tag{3}$$

$$\overline{ABCD} = \overline{a_2a_3a_1a_4} \tag{4}$$

$$\overline{ABCD} = \overline{a_2a_4a_1a_3} \tag{5}$$

$$\overline{ABCD} = \overline{a_3a_4a_1a_2} \tag{6}$$

Let us check for each case,

- Case (1) will have $A = a_1$ so $a_1 = a_1 - a_4$ or $a_4 = 0$, then $D = 0 = -a_1 + 10$ implies $a_1 = 10$ contradiction. Similarly for case (2).
- Case (3) will have $A = a_1 = a_1 - a_4 \implies a_4 = 0$ also, and $a_2 - a_3 - 1 = B = a_4 = 0$ implies $a_2 - a_3 = 1$, so we have $C = a_2 = 8$, then it means $D = a_3 = 7$. Then $D = 7 = a_4 - a_1 + 10 = -a_1 + 10$, or $a_1 = 3$ clearly a contradiction because $a_1 > a_3$.
- Case (4) will have $D = a_4$, subtituting to the system we get $a_4 = D = a_4 - a_1 + 10$, so $a_1 = 10$ a contradiction.
- Case (5), we have

$$a_3 = a_4 - a_1 + 10$$
$$a_1 = a_3 - a_2 + 9$$
$$a_4 = a_2 - a_3 - 1$$
$$a_2 = a_1 - a_4$$

For a_1, a_4 we get

$$2a_1 - 3a_4 = 11$$

$$a_1 + a_4 = 8$$

so $-5a_4 = -5$ or $a_4 = 1$, $a_1 = 7$. $a_2 = 6$, $a_3 = 4$. The digits are $6, 1, 7, 4$, which is what we want to prove.

- Case (6), we have

$$a_2 = a_4 - a_1 + 10$$

$$a_1 = a_3 - a_2 + 9$$

$$a_4 = a_2 - a_3 - 1$$

$$a_3 = a_1 - a_4$$

For a_1, a_4 we get

$$a_4 - 2a_1 = 1$$

$$a_4 + a_1 = 8$$

so $3a_1 = 7$ and a_1 is not positive integer, a contradiction.

So we can only get a Kaprekar solution when case (5) and the solution is unique that is with digits $6, 1, 7, 4$.