NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2021-2022

EE7401 – PROBABILITY AND RANDOM PROCESSES

November / December 2021 | Time Allowed: 3 hours

INSTRUCTIONS

1. This paper contains 5 questions and comprises 4 pages.
2. Answer all 5 questions.
3. All questions carry equal marks.
4. This is a closed book examination.
5. Unless specifically stated, all symbols have their usual meanings.

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1. Two discrete random variables X and Y have possible values of 4, 5 and 6. Table 1 specifies one of their probability-mass functions (PMFs): $p_{X,Y}(x,y)$, $p_{Y|X}(y\,|\,x)$ or $p_{X|Y}(x\,|\,y)$. Another PMF is specified by Table 2.

Table 1

y \ x	4	5	6
4	***a***	**0.1**	**0.5**
5	**0.2**	***b***	**0.4**
6	**0.3**	**0.2**	***c***

Table 2

4	5	6
0.3	**0.4**	***d***

(a) Give the conditions (values of a, b, and c), under which the Table 1 specifies PMFs, $p_{Y|X}(y\,|\,x)$, $p_{X|Y}(x\,|\,y)$ and $p_{X,Y}(x,y)$, respectively.

(6 Marks)

Note: Question No. 1 continues on page 2.

(b) If Table 2 (on page 1) specifies $p_Y(y)$ and Table 1 (on page 1) and Table 2 together fully specify the two random variables X, and Y, what is the value of d and what PMF is specified by Table 1?

(4 Marks)

(c) Under the conditions of 1 (b), produce 3 tables to specify the other 3 PMFs.

(10 Marks)

2. Two continuous random variables X and Y have joint probability density function (pdf)

$$p_{X,Y}(x,y) = \lambda y^2 e^{-xy}, \quad x \geq 0,\ 1 \leq y \leq 5,$$

where λ is a constant.

(a) Express the minimal mean square error estimation of X as a function of the random variable Y. If we observed that Y=2, what is the estimate of X that has the minimal mean square error?

(12 Marks)

(b) For unknown value of X, find the estimate of Y that has the minimal mean square error. What is the mean square error of the estimate?

(8 Marks)

3. (a) Suppose the probability of the head showing up is p in flipping a coin.

(i) You flip the coin until the head shows up. Assuming independent coin flips, derive the probability that you need to flip the coin m times.

(4 Marks)

(ii) You flip the coin until the head shows up twice. Assuming independent coin flips, derive the probability that you need to flip the coin n times.

(6 Marks)

(b) $\mathbf{x}(t)$ is a wide sense stationary (WSS) real random process with mean $E\{\mathbf{x}(t)\} = \mu$ and autocorrelation $R_{xx}(\tau) = e^{-|\tau|}$. Another process is defined as $\mathbf{y}(t) = \mathbf{x}(t) + 2$.

(i) Find $E\{\mathbf{y}(t)\}$, the mean of $\mathbf{y}(t)$.

(ii) Find $R_{yy}(\tau)$, the autocorrelation of $\mathbf{y}(t)$.

(iii) Is it possible that $R_{xx}(\tau) = R_{yy}(\tau)$? If yes, find the required condition for both autocorrelations to be equal.

(10 Marks)

4. (a) *Fully vaccinated:* Assume that all 7,000,000,000 persons will become fully vaccinated at a random time $\mathbf{t}$, where $\mathbf{t}$ is uniform between time $t = 0$ years and time $t = 3$ years. Let a random process $\mathbf{x}(t)$ model the fully vaccinated world as $\mathbf{x}(t) = 7{,}000{,}000{,}000\ U(t - \mathbf{t})$, where $U(t)$ is the unit step function that is 0 for $t < 0$, and is 1 for $t \geq 0$.

(i) Determine $E\{\mathbf{x}(t)\}$, the possibility of full vaccination, for all time t. Note that $\mathbf{x}(t)$ is non-stationary.

(ii) Impact of full vaccination on the pandemic is exponential rather that linear. Assume that the impact of the possibility of full vaccination is $a[\mathbf{x}(t)]^b$ for some constants a, b. Determine $E\{a[\mathbf{x}(t)]^b\}$, the expected impact of full vaccination, for all time t.

(10 Marks)

(b) *A mosquito sensor:* A sensor triggers an oscillator as soon as a mosquito lands on it. The oscillator having an output $p + \cos(qt)$ produces an audible tone for a duration of $20\pi/q$. Its inventor, Amos Skittow, vouches that mosquito arrivals are Poisson arrivals with λ mosquitoes arriving per unit time. Thus, Mr. Skittow confirms that the oscillator output $\mathbf{s}(t)$ is a shot noise observed at the output of a linear system having the following impulse response:

$$h(t) = \begin{cases} p + \cos(qt) & 0 \leq t \leq 20\pi/q \\ 0 & elsewhere \end{cases}$$

(i) Determine $E\{\mathbf{s}(t)\}$, the mean output from Amos Skittow's oscillator.

(ii) Determine the expected power requirement of the oscillator, or the variance of $\mathbf{s}(t)$, given by $\sigma_s^2 = R_{ss}(0) - E^2\{\mathbf{s}(t)\}$.
[Hint: For a shot noise, $R_{ss}(\tau) = \lambda^2 H^2(0) + \lambda h(\tau) * h(-\tau)$ where $H(0)$ may be found from the frequency response $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$, and $*$ means convolution, $f(t) * g(t) = \int_{-\infty}^{\infty} f(t - \alpha)h(\alpha)d\alpha$.]

(10 Marks)

5. (a) A continuous-time homogeneous Markoff chain has 2 states and the following transition probability matrix,

$$\Pi(\tau) = \begin{bmatrix} 0.6 + 0.4e^{-\tau} & ? \\ ? & 0.6 + 0.6e^{-\tau} \end{bmatrix}$$

where $\tau = t_2 - t_1$ is the time difference.

(i) Find both missing entries in the above transition probability matrix.

(ii) Find the transition probability rate matrix $\Pi'(0^+)$ for this Markoff chain.

(iii) If the state probability vector at time $t = 0$ is $P(0) = [1 \quad 0]$, then find the state probability vector at time $t = 0.693$.

(8 Marks)

Note: Question No. 5 continues on page 4.

(b) *Old problem, young solution:* Prof. Oldman asked his student Eve R. Young to design a detector based on integrating the received signal from 0 to T and thresholding. Let $\mathbf{s}(t)$ be the received signal which is real, WSS, and let $R_{SS}(\tau)$ be its autocorrelation. Its desired integral is a random variable $\mathbf{z} = \int_0^T \mathbf{s}(t)dt$. Rather than designing an integrator circuit, Ms. Eve R. Young built a simpler circuit using value(s) of $\mathbf{s}(t)$ to estimate $\mathbf{z}$.

(i) Using just one value of $\mathbf{s}(t)$, Eve R. Young guessed the middle value would be the best choice, and designed an estimator as $\hat{\mathbf{z}} = a\mathbf{s}(T/2)$ (estimator-1). Show that the optimal MS estimator-1 has $a = \frac{2}{R_{SS}(0)}\int_0^{T/2} R_{SS}(x)dx$.

(ii) Prof. Oldman was not happy with the estimator-1 performance, so Eve R. Young decided to improve the estimation by using two extreme values, or $\hat{\mathbf{z}} = b\mathbf{s}(0) + c\mathbf{s}(T)$ (estimator-2). Find the values of b and c in terms of $R_{SS}(\tau)$ for the optimal MS estimator-2. (For typical autocorrelation functions, the LMS error of estimator-2 is indeed smaller than estimator-1, making the professor happy.)

(iii) Over dinner, Eve R. Young's friend told her to use the average value obtained from both extremes, or $\hat{\mathbf{z}} = d\frac{\mathbf{s}(0)+\mathbf{s}(T)}{2}$ (estimator-3). Without solving d but using your solution of estimator-2, comment on whether estimator-3 performs better than, same as, or worse than estimator-2.

(12 Marks)

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.