EE7401 Probability and Random Processes

Random Processes

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Reference Athanasios Papoulis, (S Unnikrishna Pillai,) ***Probability, Random Variables, and Stochastic Processes***, McGraw-Hill.

Course Outline

Random process and correlation function

Basic concepts. Statistics of stochastic processes. Definition of autocorrelation function. Properties of autocorrelation function. Poisson process. Definition and properties of cross correlation function. Correlation coefficient. White noise process. Normal process. Stationary random process. Wide-sense stationary process.

Random process in linear system and power spectrum

System with stochastic input. Examples, square law detector. Linear time-invariant system. Time domain analysis of linear system – input/output relationship between correlation functions. Definition of power spectrum, relationship between power spectrum and autocorrelation function. Property of power spectrum. Cross-power spectrum. Existence theorem. Frequency domain analysis of linear system – input/output relationship between power spectrums. White noise. Hilbert transform of random process. Wiener-Khinchin theorem. Discrete-time process. Correlation function related to discrete-time random process. Discrete-time linear time-invariant system. Power spectrum for discrete-time process. AR(1) process.

Basic application

Random walk and Wiener process. Thermal noise. Shot noise. Modulation, bandlimited process, sampling expansion.

Optimum linear system
Systems that maximize signal-to-noise ratio - Matched filter in the presence of white noise and colored noise. *Systems that minimize mean-square error - Smoothing.*

Ergodicity
Time average. Mean ergodic process, Slutsky's theorem. Discrete-time case. Covariance ergodic process. Distribution ergodic process. Measurement of power spectrum, autocorrelation estimate of power spectrum, periodogram estimate.

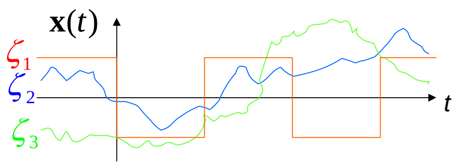
Markoff Chain and Markoff Process
Discrete-time Markoff Chains, Continuous-time Markoff Chains.

topics in red italics = not for examination purpose (pages marked by ● ● ● ● ● ● ● ● ● ● ● ●)

Stochastic Process

Random variable $\mathbf{x}$ takes a value for every outcome of an experiment.
Stochastic process $\mathbf{x}(t)$ takes a function of time for every outcome ζ of an experiment.

$\mathbf{x}(t)$: an ensemble of time functions



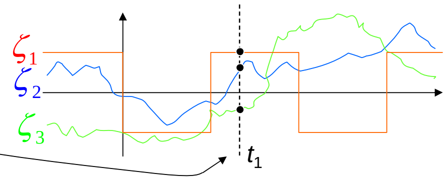
$\mathbf{x}(t), t \in$ real axis → continuous-time process
$\mathbf{x}[n], n \in$ integer axis → discrete-time process

Equality $\quad \mathbf{x}(t)=\mathbf{y}(t)$ if $\mathbf{x}(t,\zeta)=\mathbf{y}(t,\zeta)\ \forall t,\zeta$

$\mathbf{x}(t)$ and $\mathbf{y}(t)$ are equal in the MS sense if $E\{|\mathbf{x}(t)-\mathbf{y}(t)|^2\}=0\ \forall t$

First Order Statistics of Stochastic Processes

At a given t_1, $\mathbf{x}(t_1)$
is a random variable



For a specific t, $\quad F(x,t)=P\{\mathbf{x}(t)\le x\}\quad$ first order distribution

$$f(x,t)=\frac{\partial F(x,t)}{\partial x}\quad \text{first order density}$$

mean $\quad \eta(t)=E\{\mathbf{x}(t)\}=\int_{-\infty}^{\infty} x f(x,t)dx$

Second Order Statistics of Stochastic Processes

At t_1, $\mathbf{x}(t_1)=\mathbf{y}_1$ is a random variable
At t_2, $\mathbf{x}(t_2)=\mathbf{y}_2$ is another random variable

second order distribution $F(x_1,x_2;t_1,t_2)=P\{\mathbf{x}(t_1)\le x_1,\mathbf{x}(t_2)\le x_2\}$

second order density $\quad f(x_1,x_2;t_1,t_2)=\dfrac{\partial^2 F(x_1,x_2;t_1,t_2)}{\partial x_1\partial x_2}$

$\mathbf{x}(t)$ is in general a complex process, $\quad \mathbf{x}(t)=\mathbf{a}(t)+j\mathbf{b}(t)$
where $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are real processes

autocorrelation

$$R_{xx}(t_1,t_2)=E\{\mathbf{x}(t_1)\mathbf{x}^*(t_2)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2^* f(x_1,x_2;t_1,t_2)dx_1dx_2$$

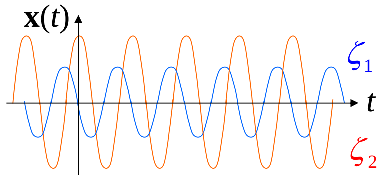
by definition $R_{xx}(t_2,t_1) = R^*_{xx}(t_1,t_2)$

for real processes $R_{xx}(t_2,t_1) = R_{xx}(t_1,t_2)$

for $t_1 = t_2 = t$, $R_{xx}(t,t) = E\{|\mathbf{x}(t)|^2\} =$ average power ≥ 0

Example 1 $\mathbf{x}(t,\zeta_i) = \mathbf{r}(\zeta_i)\cos\left(\omega t + \boldsymbol{\varphi}(\zeta_i)\right)$

$\mathbf{r}$, $\boldsymbol{\varphi}$ independent real random variables, $\boldsymbol{\varphi}$ uniform in $(-\pi, \pi)$



$\eta(t) = E\{\mathbf{x}(t)\} = E\{\mathbf{r}\}E\{\cos(\omega t + \boldsymbol{\varphi})\}$

but $E\{\cos(\omega t + \boldsymbol{\varphi})\} = \int_{-\pi}^{\pi} \cos(\omega t + \phi)\frac{1}{2\pi}\,d\phi = 0$

$\Rightarrow \eta(t) = 0$

$R_{xx}(t_1,t_2) = E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\}$

$= E\{\mathbf{r}^2 \cos(\omega t_1 + \boldsymbol{\varphi})\cos(\omega t_2 + \boldsymbol{\varphi})\}$

$= \tfrac{1}{2} E\{\mathbf{r}^2\}E\{\cos(\omega t_1 + \omega t_2 + 2\boldsymbol{\varphi}) + \cos(\omega t_1 - \omega t_2)\}$

$= \tfrac{1}{2} E\{\mathbf{r}^2\}[E\{\cos(\omega t_1 + \omega t_2 + 2\boldsymbol{\varphi})\} + E\{\cos(\omega t_1 - \omega t_2)\}]$

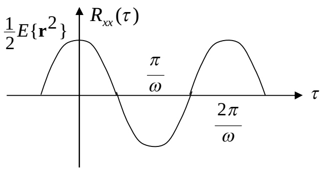
but $E\{\cos(\omega(t_1 + t_2) + 2\boldsymbol{\varphi})\} = 0$ as before

$= \frac{1}{2}\cos(\omega t_1 - \omega t_2)E\{\mathbf{r}^2\}$

average power $= R_{xx}(t,t) = \frac{1}{2}E\{\mathbf{r}^2\} =$ variance let $\tau = t_2 - t_1$

then $R_{xx}(\tau) = \frac{1}{2}\cos(\omega\tau)E\{\mathbf{r}^2\}$

autocorrelation depends only on τ

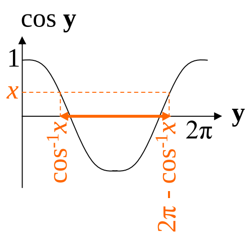


First order statistics for $\mathbf{r}$ = 1 (or a constant)

$$F(x,t) = P\{\mathbf{x}(t) \le x\} = P\{\cos(\omega t + \boldsymbol{\varphi}) \le x\} = P\{\cos\mathbf{y} \le x\}$$

where $\mathbf{y} = \omega t + \boldsymbol{\varphi}$ is uniform in $(\omega t - \pi, \omega t + \pi)$

Consider a period from 0 to 2π.



Now

$$P\{\cos\mathbf{y} \le x\}$$

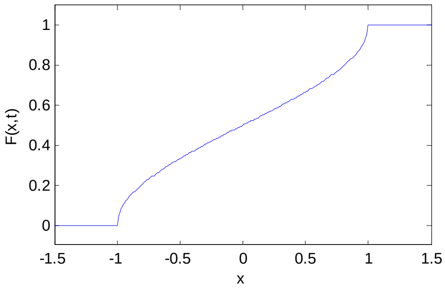
$$= P\{\cos^{-1}x \le \mathbf{y} \le 2\pi - \cos^{-1}x\}$$

$$\text{for } 0 \le \cos^{-1}x \le \pi, \quad -1 \le x \le 1$$

$$= \frac{2\pi - 2\cos^{-1}x}{2\pi} \quad \text{since uniform}$$

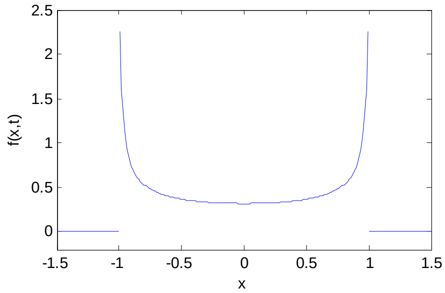
$$= 1 - \frac{\cos^{-1}x}{\pi}$$

$$\Rightarrow F(x,t) = \begin{cases} 0 & x < -1 \\ 1 - \frac{\cos^{-1} x}{\pi} & -1 \le x \le 1 \\ 1 & 1 < x \end{cases}$$



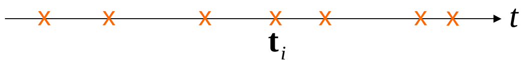
$F(x,t)$ independent of t

$$f(x,t) = \begin{cases} 0 & 1 < |x| \\ \dfrac{1}{\pi\sqrt{1-x^2}} & |x| \le 1 \end{cases}$$



Poisson Process

Place points $\mathbf{t}_i$ at random on the entire t axis such that on an average there are λ points per unit time.

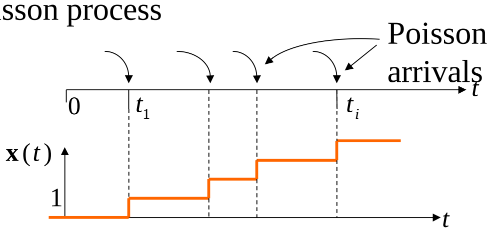


The number of points $\mathbf{n}(t_1,t_2)$ in an interval (t_1,t_2) is a Poisson random variable

$$P\{\mathbf{n}(t_1,t_2) = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad \text{where } t = t_2 - t_1 \text{ is interval length}$$

If the intervals (t_1,t_2) and (t_3,t_4) are non-overlapping, then the random variables $\mathbf{n}(t_1,t_2)$ and $\mathbf{n}(t_3,t_4)$ are independent.

$\mathbf{x}(t) = \mathbf{n}(0,t)$ is the Poisson process



$\eta(t)$ = mean of Poisson process

= average number of points in interval $t = E\{\mathbf{x}(t)\} = E\{\mathbf{n}(0,\ t)\} = \lambda t$

$R_{xx}(t,t) = E\{\mathbf{x}^2(t)\}$ = average power of Poisson process

$$= \sum_{k=1}^{\infty} \frac{e^{-a} a^k}{k!} k^2 \qquad \text{for } a = \lambda t$$

$$= e^{-a} \sum_{k=1}^{\infty} \frac{a^k}{k!} k^2$$

using Taylor series expansion, $e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$

$$\frac{d^2 e^a}{da^2} = \sum_{k=1}^{\infty} \frac{k(k-1)a^{k-2}}{k!}$$

$$\Rightarrow e^a = \frac{1}{a^2} \sum_{k=1}^{\infty} k^2 \frac{a^k}{k!} - \frac{1}{a^2} \sum_{k=1}^{\infty} k \frac{a^k}{k!}$$

but $\sum_{k=1}^{\infty} k \frac{a^k}{k!} = e^a \sum_{k=1}^{\infty} k \frac{e^{-a} a^k}{k!} = e^a a$

$$\Rightarrow e^a = \frac{1}{a^2} \sum_{k=1}^{\infty} k^2 \frac{a^k}{k!} - \frac{1}{a^2} e^a a$$

$$\Rightarrow \sum_{k=1}^{\infty} k^2 \frac{a^k}{k!} = e^a a + e^a a^2$$

therefore $R_{xx}(t,t) = E\{\mathbf{x}^2(t)\} = a + a^2 = \lambda t + \lambda^2 t^2$

autocorrelation $R_{xx}(t_1,t_2) = E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\}$

$$= E\{\mathbf{x}(t_1)[\mathbf{x}(t_1) + \mathbf{x}(t_2) - \mathbf{x}(t_1)]\}$$

$$= E\{\mathbf{x}^2(t_1)\} + E\{\mathbf{x}(t_1)[\mathbf{x}(t_2) - \mathbf{x}(t_1)]\}$$

But $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2) - \mathbf{x}(t_1)$ are independent because the intervals $(0, t_1)$ and (t_1, t_2) are non-overlapping

$$= E\{\mathbf{x}^2(t_1)\} + E\{\mathbf{x}(t_1)\}E\{\mathbf{x}(t_2) - \mathbf{x}(t_1)\}$$

$$= \lambda t_1 + \lambda^2 t_1^2 + \lambda t_1 \cdot \lambda(t_2 - t_1) = \lambda t_1 + \lambda^2 t_1 t_2$$

using $R_{xx}(t_1,t_2) = R_{xx}(t_2,t_1)$

$$R_{xx}(t_1,t_2) = \begin{cases} \lambda t_2 + \lambda^2 t_1 t_2 & t_1 \ge t_2 \\ \lambda t_1 + \lambda^2 t_1 t_2 & t_1 \le t_2 \end{cases}$$

or $R_{xx}(t_1,t_2) = \lambda \min(t_1,t_2) + \lambda^2 t_1 t_2$

n-th Order Statistics

$$F(x_1,\ldots,x_n;t_1,\ldots,t_n) = P\{\mathbf{x}(t_1) \le x_1,\ldots,\mathbf{x}(t_n) \le x_n\}$$

Second Order Statistics (continued)

Autocorrelation is a positive definite function.

For any $[a_0 \quad a_1 \quad \cdots \quad a_m]$, $\sum_{i=0}^{m}\sum_{j=0}^{m} a_i a_j^* R_{xx}(t_i,t_j) \ge 0$

Proof: $E\left\{\left|\sum_{i=0}^{m} a_i \mathbf{x}(t_i)\right|^2\right\} \ge 0$ since the argument is non-negative

$$\Rightarrow E\left\{\sum_i \sum_j a_i \mathbf{x}(t_i) a_j^* \mathbf{x}^*(t_j)\right\} \ge 0$$

$\Rightarrow \sum_i \sum_j a_i a_j^* E\{\mathbf{x}(t_i)\mathbf{x}^*(t_j)\} \ge 0$ from which the result follows.

autocovariance $C_{xx}(t_1,t_2) = R_{xx}(t_1,t_2) - \eta(t_1)\eta^*(t_2)$

for $t_1 = t_2 = t$, $C_{xx}(t,t) =$ variance of $\mathbf{x}(t)$

correlation coefficient $r(t_1,t_2) = \dfrac{C_{xx}(t_1,t_2)}{\sqrt{C_{xx}(t_1,t_1)C_{xx}(t_2,t_2)}}$

It follows that $|r(t_1,t_2)| \le 1$ and $r(t,t) = 1$

Cross-correlation of two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ is

$$R_{xy}(t_1,t_2) = E\{\mathbf{x}(t_1)\mathbf{y}^*(t_2)\} = R^*_{yx}(t_2,t_1)$$

cross-covariance $C_{xy}(t_1,t_2) = R_{xy}(t_1,t_2) - \eta_x(t_1)\eta^*_y(t_2)$

If $R_{xy}(t_1,t_2) = 0 \;\; \forall t_1,t_2$ then $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are orthogonal

If $C_{xy}(t_1,t_2) = 0 \;\; \forall t_1,t_2$ then $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are uncorrelated

White Noise Process

If $C_{xx}(t_1,t_2) = 0 \;\; \forall t_1 \ne t_2$ then $\mathbf{x}(t)$ is a white noise
[$\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are uncorrelated for every $t_1 \ne t_2$]

However, $C_{xx}(t,t)$ = variance of $\mathbf{x}(t) \ne 0$ for a nontrivial process

therefore, $C_{xx}(t_1,t_2) = q(t_1)\delta(t_1 - t_2)$ for some $q(t_1) \ge 0$
$\delta(t)$ = impulse function

Normal Process

If $\mathbf{x}(t_1), \mathbf{x}(t_2), \ldots, \mathbf{x}(t_n)$ are jointly normal for any $n, t_1, t_2, \ldots, t_n$ then $\mathbf{x}(t)$ is a normal process (real)

It's first order density is normal with mean $\eta(t)$, variance $C_{xx}(t,t)$

Stationary Process

Strict-Sense Stationary (SSS): statistics does not change with time

$$f(x_1, \ldots, x_n; t_1, \ldots, t_n) = f(x_1, \ldots, x_n; t_1 + c, \ldots, t_n + c) \quad \forall c$$

Thus, first-order statistics is independent of time, $f(x,t) = f(x)$

second-order statistics depends on time difference only,

$$f(x_1, x_2; t_1, t_2) = f(x_1, x_2; \tau) \quad \tau = t_1 - t_2$$

Our example 1, $\mathbf{x}(t) = \mathbf{r}\cos(\omega t + \boldsymbol{\varphi})$ is a SSS process

Wide-Sense Stationary (WSS): mean, autocorrelation does not change with time

$$E\{\mathbf{x}(t)\} = \eta(t) = \eta, \quad \text{constant}$$

$$E\{\mathbf{x}(t+\tau)\mathbf{x}^*(t)\} = R_{xx}(t+\tau, t) = R_{xx}(\tau)$$

Also implies constant average power $E\{|\mathbf{x}(t)|^2\} = R_{xx}(0)$

Two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are called jointly WSS if each is WSS and $R_{xy}(t+\tau, t) = R_{xy}(\tau)$

$$\text{SSS} \Rightarrow \text{WSS, but WSS} \not\Rightarrow \text{SSS}$$

However, if $\mathbf{x}(t)$ is a normal process, then $\mathbf{x}(t)$ WSS $\Rightarrow \mathbf{x}(t)$ SSS.

Example 1 (contd.)

Show that $\mathbf{x}(t)=\cos(\omega t+\boldsymbol{\varphi})$ with $\Phi(\lambda)=E\{e^{j\lambda\boldsymbol{\varphi}}\}$, $\Phi(1)=\Phi(2)=0$ is WSS. Find $E\{\mathbf{x}(t)\}$ and $R_{xx}(\tau)$.

Solution: $E\{\mathbf{x}(t)\}=E\{\cos(\omega t+\boldsymbol{\varphi})\}=E\{\text{Re}[e^{j(\omega t+\boldsymbol{\varphi})}]\}$

$$=\text{Re}[e^{j\omega t}E\{e^{j\boldsymbol{\varphi}}\}]=\text{Re}[e^{j\omega t}\Phi(1)]=0$$

hence, mean independent of time

$$R_{xx}(\tau)=E\{\mathbf{x}(t)\mathbf{x}(t+\tau)\}=E\{\cos(\omega t+\boldsymbol{\varphi})\cos(\omega t+\boldsymbol{\varphi}+\omega\tau)\}$$

$$=\frac{1}{2}E\{\cos(2\omega t+2\boldsymbol{\varphi}+\omega\tau)+\cos(\omega\tau)\}$$

$$=\frac{1}{2}E\{\cos(2\omega t+2\boldsymbol{\varphi}+\omega\tau)\}+\frac{1}{2}\cos(\omega\tau)$$

$$=\frac{1}{2}\text{Re}\left[e^{2j\omega t+j\omega\tau}E\{e^{2j\boldsymbol{\varphi}}\}\right]+\frac{1}{2}\cos(\omega\tau)$$

$$=\frac{1}{2}\text{Re}\left[e^{2j\omega t+j\omega\tau}\Phi(2)\right]+\frac{1}{2}\cos(\omega\tau)=0+\frac{1}{2}\cos(\omega\tau)$$

These answers match with earlier answers taking $E(\mathbf{r}^2)=1$ since $\mathbf{r}=1$

This is because, if $\boldsymbol{\varphi}$ is uniform in $(-\pi,\pi)$, then

$$\Phi(\lambda)=\int_{-\pi}^{\pi}e^{j\lambda\phi}\frac{1}{2\pi}d\phi=\frac{\sin(\lambda\pi)}{\lambda\pi}$$

and $\Phi(1)=\Phi(2)=0$ since $\sin(\pi)=\sin(2\pi)=0$

Example 2:

If $\mathbf{x}(t)$ is normal process with zero mean and $\mathbf{y}(t) = Ie^{a\mathbf{x}(t)}$

Then $\eta_y(t) = E\{\mathbf{y}(t)\} = E\left\{Ie^{a\mathbf{x}(t)}\right\} = \int_{-\infty}^{\infty} Ie^{ax} f(x,t)dx$

But $f(x,t) = \dfrac{1}{\sqrt{2\pi R_{xx}(t,t)}} e^{-x^2/2R_{xx}(t,t)}$

since variance = $R_{xx}(t,t)$ because mean is zero,

So $\quad \eta_y(t) = \int\limits_{-\infty}^{\infty} Ie^{ax} \dfrac{1}{\sqrt{2\pi R_{xx}(t,t)}} e^{-x^2/2R_{xx}(t,t)} dx$

$$= I \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi R_{xx}(t,t)}} e^{-[x-aR_{xx}(t,t)]^2/2R_{xx}(t,t)} e^{\frac{1}{2}a^2 R_{xx}(t,t)} dx$$

$$= Ie^{\frac{1}{2}a^2 R_{xx}(t,t)}$$

If $\mathbf{x}(t)$ is stationary, then $R_{xx}(t_1,t_2) = R_{xx}(\tau)$, and $\eta_y = Ie^{\frac{1}{2}a^2 R_{xx}(0)}$

For $\mathbf{x}(t)$ stationary, we will now find $R_{yy}(t_1,t_2)$.

Since it is a memoryless system, $\mathbf{y}(t)$ is SSS since $\mathbf{x}(t)$ is.

Therefore, $\quad R_{yy}(t_1,t_2) = R_{yy}(\tau)$

Now, $\quad f(x_1,x_2;\tau) = \dfrac{1}{2\pi R_{xx}(0)\sqrt{1-r^2}} e^{-\frac{(x_1^2 - 2rx_1x_2 + x_2^2)}{2(1-r^2)R_{xx}(0)}}$

since it is a zero-mean joint normal density with equal variance $R_{xx}(0)$

Here, r = correlation coefficient = $R_{xx}(\tau)/R_{xx}(0)$ since zero-mean

So $R_{yy}(\tau) = E\{\mathbf{y}(t)\mathbf{y}(t+\tau)\} = E\{I^2 e^{a[\mathbf{x}(t)+\mathbf{x}(t+\tau)]}\}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I^2 e^{a(x_1+x_2)} f(x_1,x_2;\tau)dx_1dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{I^2}{2\pi R_{xx}(0)\sqrt{1-r^2}} e^{a(x_1+x_2)-\frac{x_1^2-2rx_1x_2+x_2^2}{2(1-r^2)R_{xx}(0)}} dx_1dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{I^2}{2\pi R_{xx}(0)\sqrt{1-r^2}} e^{-\frac{(x_1-\alpha)^2-2r(x_1-\alpha)(x_2-\alpha)+(x_2-\alpha)^2}{2(1-r^2)R_{xx}(0)}+a^2R_{xx}(0)(1+r)} dx_1dx_2$$

for $\alpha = aR_{xx}(0)(1+r)$

$$= I^2 e^{a^2R_{xx}(0)(1+r)} = I^2 e^{a^2[R_{xx}(0)+R_{xx}(\tau)]}$$

System with Stochastic Input

$\mathbf{y}(t) = T[\mathbf{x}(t)]$

$$\mathbf{x}(t) \longrightarrow \boxed{T} \longrightarrow \mathbf{y}(t)$$

Deterministic system
$\Downarrow$
T is deterministic

If $\mathbf{y}(t) = g(\mathbf{x}(t))$, then output $\mathbf{y}(t)$ depends only on the present input $\mathbf{x}(t) \Rightarrow$ memoryless system

Since $\mathbf{y}(t_1) = g(\mathbf{x}(t_1)),\cdots,\mathbf{y}(t_n) = g(\mathbf{x}(t_n))$, $f_2(y_1,\cdots,y_n;t_1,\cdots,t_n)$ can be determined from $f_1(x_1,\cdots,x_n;t_1,\cdots,t_n)$ using earlier methods:

If $\mathbf{y}_1 = g_1(\mathbf{x}_1,..,\mathbf{x}_n),\cdots,\mathbf{y}_n = g_n(\mathbf{x}_1,..,\mathbf{x}_n)$

and random variables $\mathbf{x}_1,..,\mathbf{x}_n$ have density $f_x(x_1,..,x_n)$

then density of $\mathbf{y}_1,..,\mathbf{y}_n$ at values $y_1',..,y_n'$ may be found as follows.

If $g_1(x_1,..,x_n) = y_1', \cdots, g_n(x_1,..,x_n) = y_n'$ has one solution $x_1',..,x_n'$

then $$f_y(y_1',..,y_n') = \frac{f_x(x_1',..,x_n')}{|J(x_1',..,x_n')|}$$

where J is the jacobian of the transformation.

If there are several solutions, add the corresponding terms.

[for systems with memory, this task is very complex]

For a memoryless system, if $\mathbf{x}(t)$ is SSS, then $\mathbf{y}(t)$ is also SSS.

However, if $\mathbf{x}(t)$ is WSS, then $\mathbf{y}(t)$ may or may not be WSS.

Example 3: Square Law Detector

$\mathbf{y}(t) = \mathbf{x}^2(t)$ (real)

For $y > 0$, $y = x^2$ has two solutions $x = \pm\sqrt{y}$

jacobian $\frac{dy}{dx} = \pm 2\sqrt{y}$

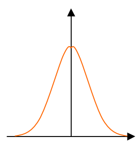
$$\Rightarrow f_y(y;t) = \frac{1}{2\sqrt{y}}\left[f_x(\sqrt{y};t) + f_x(-\sqrt{y};t) \right]$$ First order density

(no solution for $y < 0$)

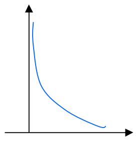
If $\mathbf{x}(t)$ is SSS, then $f_x(x;t) = f_x(x)$ $\Rightarrow f_y(y;t) = f_y(y)$

For example, take $\mathbf{x}(t)$ to be a normal stationary zero mean process,

$$f_x(x) = \frac{1}{\sqrt{2\pi R_{xx}(0)}} e^{-x^2/2R_{xx}(0)}$$



$$\Rightarrow f_y(y) = \frac{1}{\sqrt{2\pi R_{xx}(0)y}} e^{-y/2R_{xx}(0)} \times U(y)$$



$U(y)$ = unit step function

$$E\{\mathbf{y}(t)\} = R_{xx}(0)$$

Now for $y_1 > 0, y_2 > 0$, the system $y_1 = x_1^2, y_2 = x_2^2$ has four solutions $\left(\pm\sqrt{y_1}, \pm\sqrt{y_2}\right)$

$$\text{jacobian} \quad \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \pm 4\sqrt{y_1 y_2}$$

$$\Rightarrow f_y(y_1, y_2; t_1, t_2) = \frac{1}{4\sqrt{y_1 y_2}} \sum f_x(\pm\sqrt{y_1}, \pm\sqrt{y_2}; t_1, t_2)$$

Second order density

If $\mathbf{x}(t)$ is SSS, then

$$f_x(x_1, x_2; t_1, t_2) = f_x(x_1, x_2; \tau)$$

$$\Rightarrow f_y(y_1, y_2; t_1, t_2) = f_y(y_1, y_2; \tau)$$

For $\mathbf{x}(t)$ to be a normal stationary zero mean process, the second order density may be similarly found.

Since $\mathbf{x}(t+\tau)$ and $\mathbf{x}(t)$ are jointly normal zero mean,

$$R_{yy}(\tau) = E\{\mathbf{x}^2(t+\tau)\mathbf{x}^2(t)\} = E\{\mathbf{x}^2(t+\tau)\}E\{\mathbf{x}^2(t)\} + 2E^2\{\mathbf{x}(t+\tau)\mathbf{x}(t)\}$$

$$= R_{xx}^2(0) + 2R_{xx}^2(\tau)$$

$$E\{\mathbf{y}^2(t)\} = R_{yy}(0) = 3R_{xx}^2(0) \qquad \Rightarrow \sigma_y^2 = 2R_{xx}^2(0)$$

Linear Systems

$$\mathbf{y}(t) = L\left[\mathbf{x}(t)\right] \qquad \mathbf{x}(t) \longrightarrow \boxed{L} \longrightarrow \mathbf{y}(t)$$

Superposition holds $L\left[\mathbf{a}_1\mathbf{x}_1(t) + \mathbf{a}_2\mathbf{x}_2(t)\right] = \mathbf{a}_1 L\left[\mathbf{x}_1(t)\right] + \mathbf{a}_2 L\left[\mathbf{x}_2(t)\right]$

for any $\mathbf{a}_1, \mathbf{a}_2, \mathbf{x}_1(t), \mathbf{x}_2(t)$

<u>Time-invariant</u> if $L\left[\mathbf{x}(t+c)\right] = \mathbf{y}(t+c), \quad \forall c$

$h(t)$ = impulse response of a linear (LTI) system = $L\left[\delta(t)\right]$

Then $\mathbf{y}(t) = \mathbf{x}(t) * h(t) = \int_{-\infty}^{\infty} \mathbf{x}(t-\alpha) h(\alpha)\, d\alpha$

<u>Causal</u> if $h(t) = 0$ for all $t < 0$

If $\mathbf{x}(t)$ is a normal process, then $\mathbf{y}(t)$ is also a normal process.

If $\mathbf{x}(t)$ is SSS, then $\mathbf{y}(t)$ is also SSS.

If $\mathbf{x}(t)$ is WSS, then $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are jointly WSS: $\left(\eta_x, \eta_y, R_{xx}, R_{yy}, R_{xy}\right)$

Fundamental theorem:

$$E\left\{L\left[\mathbf{x}(t)\right]\right\} = E\left\{\int_{-\infty}^{\infty} \mathbf{x}(t-\alpha) h(\alpha)\, d\alpha\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x h(\alpha) f(x, t-\alpha)\, d\alpha\, dx$$

$$= \int_{-\infty}^{\infty} h(\alpha)\left[\int_{-\infty}^{\infty} x f(x, t-\alpha)\, dx\right] d\alpha$$

but $\int_{-\infty}^{\infty} x f(x, t-\alpha)\, dx = E\left\{\mathbf{x}(t-\alpha)\right\} = \eta_x(t-\alpha)$

$$= \int_{-\infty}^{\infty} \eta_x(t-\alpha) h(\alpha)\, d\alpha$$

$$= L\left[\eta_x(t)\right] = L\left\{E\{\mathbf{x}(t)\}\right\}$$

1. $R_{xy}(t_1,t_2) = L_2^*[R_{xx}(t_1,t_2)]$ where $L_2[\ ]$ operates on t_2

$$= \int_{-\infty}^{\infty} R_{xx}(t_1, t_2 - \alpha) h^*(\alpha) d\alpha$$

2. $R_{yy}(t_1,t_2) = L_1\left[R_{xy}(t_1,t_2)\right]$

Proof: $\mathbf{y}^*(t_2) = L_2^*[\mathbf{x}^*(t_2)]$

Using superposition, $\mathbf{x}(t_1)\mathbf{y}^*(t_2) = L_2^*[\mathbf{x}(t_1)\mathbf{x}^*(t_2)]$

Therefore $E\{\mathbf{x}(t_1)\mathbf{y}^*(t_2)\} = L_2^*[E\{\mathbf{x}(t_1)\mathbf{x}^*(t_2)\}]$

or $R_{xy}(t_1,t_2) = L_2^*[R_{xx}(t_1,t_2)]$

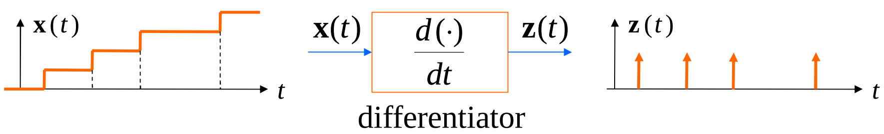
(Part 2 similar) ■

Special case: if $\mathbf{x}(t)$ is white noise, then $R_{xx}(t_1,t_2) = q(t_1)\delta(t_1 - t_2)$

It follows that $E\{|\mathbf{y}(t)|^2\} = q(t) * |h(t)|^2$

Example 4: Poisson Impulses

$$\mathbf{z}(t) = \sum_i \delta(t - \mathbf{t}_i) = \frac{d}{dt}\mathbf{x}(t)$$



here $L[\ ] = \dfrac{d}{dt}$

So $R_{zz}(t_1,t_2) = L_1\{L_2[R_{xx}(t_1,t_2)]\}$

$$= \frac{\partial^2}{\partial t_1 \partial t_2} R_{xx}(t_1,t_2)$$

$$= \frac{\partial}{\partial t_1}\left[\lambda^2 t_1 + \lambda U(t_1 - t_2)\right]$$

$$= \lambda^2 + \lambda\delta(t_1 - t_2)$$

For $\tau = t_1 - t_2,\ R_{zz}(\tau) = \lambda^2 + \lambda\delta(\tau)$

$$S_{zz}(\omega) = 2\pi\lambda^2\delta(\omega) + \lambda$$

$$E\{\mathbf{z}(t)\} = E\left\{\frac{d}{dt}\mathbf{x}(t)\right\} = \frac{d}{dt}E\{\mathbf{x}(t)\} = \frac{d}{dt}\lambda t = \lambda$$

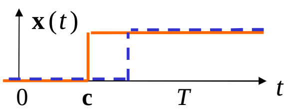
Hence $\mathbf{z}(t)$ at least WSS

Example 5: (non-stationary case)

$\mathbf{c}$ is uniform in $(0,T)$

Find autocorrelation if (i) $\mathbf{x}(t) = U(t - \mathbf{c})$, (ii) $\mathbf{y}(t) = \delta(t - \mathbf{c})$

(i)

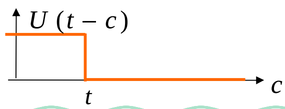


$$R_{xx}(t_1,t_2) = E\{U(t_1 - \mathbf{c})U(t_2 - \mathbf{c})\}$$

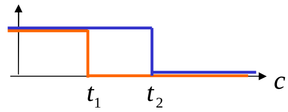
$$= \int U(t_1 - c)U(t_2 - c)f(c)dc \qquad f(c) = \text{density of } \mathbf{c}$$

$$= \int_0^T U(t_1 - c)U(t_2 - c)\frac{1}{T}dc$$

but $U(t-c) = \begin{cases} 0 & c > t \\ 1 & c \le t \end{cases}$



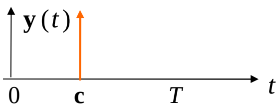
$$\Rightarrow U(t_1-c)U(t_2-c)=\begin{cases}0 & c>\min(t_1,t_2)\\ 1 & c\le\min(t_1,t_2)\end{cases}$$



$$R_{xx}(t_1,t_2)$$

$$=\begin{cases}0 & \min(t_1,t_2)<0\\ \displaystyle\int_0^{\min(t_1,t_2)}\frac{1}{T}dc & 0\le\min(t_1,t_2)<T\\ \displaystyle\int_0^{T}\frac{1}{T}dc & T\le\min(t_1,t_2)\end{cases}=\begin{cases}0 & \min(t_1,t_2)<0\\ t_1/T & 0\le\min(t_1,t_2)<T,t_1\le t_2\\ t_2/T & 0\le\min(t_1,t_2)<T,t_2<t_1\\ 1 & T\le\min(t_1,t_2)\end{cases}$$

(ii)



$$R_{yy}(t_1,t_2)=E\{\delta(t_1-\mathbf{c})\delta(t_2-\mathbf{c})\}$$

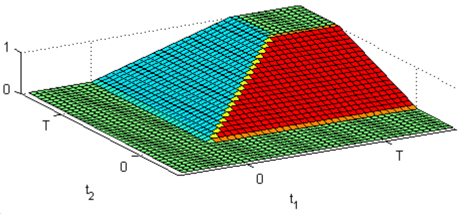
$$=\int_0^T\delta(t_1-c)\delta(t_2-c)\frac{1}{T}dc$$

$$\int\delta(t_1-c)\delta(t_2-c)dc=\delta(t_1-t_2)$$

$$=\begin{cases}\delta(t_1-t_2)/T & 0\le t_1,t_2<T\\ 0 & else\end{cases}$$

alternate approach:

$$R_{xx}(t_1,t_2)=$$



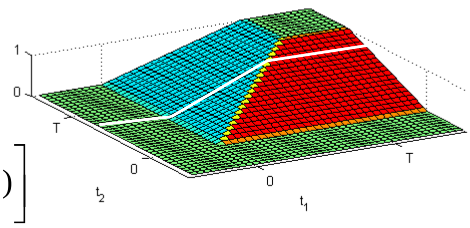
$$\frac{t_1}{T}\left[U(t_1)-U(t_1-T)\right]U(t_2-t_1)$$

$0 \le t_1 < T$ $\qquad$ $t_1 \le t_2$

$$+\frac{t_2}{T}\left[U(t_2)-U(t_2-T)\right]\left[1-U(t_2-t_1)\right] \qquad +U(t_1-T)U(t_2-T)$$

$0 \le t_2 < T$ $\qquad$ $t_2 < t_1$ $\qquad$ $T \le t_1, t_2$

$$\mathbf{y}(t)=\frac{d}{dt}\mathbf{x}(t) \qquad R_{yy}(t_1,t_2)=\frac{\partial^2}{\partial t_1 \partial t_2}R_{xx}(t_1,t_2)$$



$$\frac{\partial}{\partial t_1}R_{xx}(t_1,t_2)$$

$$=\frac{\partial}{\partial t_1}\left[\frac{t_1}{T}\left[U(t_1)-U(t_1-T)\right]U(t_2-t_1)\right]$$

$$+\frac{\partial}{\partial t_1}\left[\frac{t_2}{T}\left[U(t_2)-U(t_2-T)\right]\left[1-U(t_2-t_1)\right]\right]+\frac{\partial}{\partial t_1}U(t_1-T)U(t_2-T)$$

$$=\frac{1}{T}\left[U(t_1)-U(t_1-T)\right]U(t_2-t_1)+\frac{t_1}{T}\left[\delta(t_1)-\delta(t_1-T)\right]U(t_2-t_1)$$

$$-\frac{t_1}{T}\left[U(t_1)-U(t_1-T)\right]\delta(t_2-t_1)$$

cancels each other since $t_1 = t_2$

$$+\frac{t_2}{T}\left[U(t_2)-U(t_2-T)\right]\delta(t_2-t_1)+\delta(t_1-T)U(t_2-T)$$

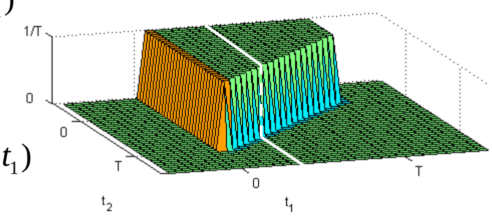
$$= \frac{1}{T}[U(t_1) - U(t_1 - T)]U(t_2 - t_1) + \frac{t_1}{T}\delta(t_1)U(t_2 - t_1)$$

zero since $t_1 = 0$

$$-\frac{t_1}{T}\delta(t_1 - T)U(t_2 - t_1) + \delta(t_1 - T)U(t_2 - T)$$

cancels each other since $t_1 = T$

$$= \frac{1}{T}[U(t_1) - U(t_1 - T)]U(t_2 - t_1)$$



$$\frac{\partial}{\partial t_2}\frac{1}{T}[U(t_1) - U(t_1 - T)]U(t_2 - t_1)$$

$$= \frac{1}{T}[U(t_1) - U(t_1 - T)]\delta(t_2 - t_1)$$ which is same as before

The relationship between cross-correlations may be used to express moments of the output in terms of the moments of the input.

For example, to find $R_{yyy}(t_1,t_2,t_3)$: (real case)

$$E\left[\mathbf{x}(t_1)\mathbf{x}(t_2)\mathbf{y}(t_3)\right] = L_3\left[E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\mathbf{x}(t_3)\}\right] = L_3\left[R_{xxx}(t_1,t_2,t_3)\right]$$

$$E\left[\mathbf{x}(t_1)\mathbf{y}(t_2)\mathbf{y}(t_3)\right] = L_2\left[E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\mathbf{y}(t_3)\}\right] = L_2\left[R_{xxy}(t_1,t_2,t_3)\right]$$

$$R_{yyy}(t_1,t_2,t_3) = E\left\{\mathbf{y}(t_1)\mathbf{y}(t_2)\mathbf{y}(t_3)\right\} = L_1\left[E\{\mathbf{x}(t_1)\mathbf{y}(t_2)\mathbf{y}(t_3)\}\right]$$

$$= L_1\left[R_{xyy}(t_1,t_2,t_3)\right] = L_1\left[L_2\left[L_3\left[R_{xxx}(t_1,t_2,t_3)\right]\right]\right]$$

Power Spectrum (Power Spectral Density, psd)

Fourier transform of the autocorrelation of a WSS stochastic process

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau \quad \text{where} \quad R_{xx}(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{x}^*(t)\}$$

$R_{xx}(\tau)$ conjugate symmetric $\Rightarrow$ $S_{xx}(\omega)$ real

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} \frac{d\omega}{2\pi}$$

If $\mathbf{x}(t)$ is real $\Rightarrow R_{xx}(\tau)$ real symmetric $\Rightarrow S_{xx}(\omega)$ real symmetric

Similarly, cross-power spectrum $S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$

Existence theorem:

If $\mathbf{x}(t) = ae^{j\boldsymbol{\omega}t+j\boldsymbol{\varphi}}$, $\boldsymbol{\omega}$ has density $f_{\boldsymbol{\omega}}(\omega)$ and $\boldsymbol{\varphi}$ is uniform in $-\pi,\pi$

then $R_{xx}(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{x}^*(t)\} = a^2 E\{e^{j\boldsymbol{\omega}t+j\boldsymbol{\omega}\tau+j\boldsymbol{\varphi}} e^{-j\boldsymbol{\omega}t-j\boldsymbol{\varphi}}\}$

$$= a^2 \int_{-\infty}^{\infty} e^{j\omega\tau} f_{\boldsymbol{\omega}}(\omega) d\omega = \int_{-\infty}^{\infty} \left[2\pi a^2 f_{\boldsymbol{\omega}}(\omega)\right] e^{j\omega\tau} \frac{d\omega}{2\pi}$$

$$\Rightarrow S_{xx}(\omega) = 2\pi a^2 f_{\boldsymbol{\omega}}(\omega) \quad \text{where} \quad a^2 = R_{xx}(0)$$

Thus, choosing $f_{\boldsymbol{\omega}}(\omega) = \dfrac{S(\omega)}{2\pi R_{xx}(0)}$ for some $S(\omega) \geq 0, \forall \omega$
and $\mathbf{x}(t) = ae^{j\boldsymbol{\omega}t+j\boldsymbol{\varphi}}$ would make $S_{xx}(\omega) = S(\omega)$

For real processes, choosing $f_{\boldsymbol{\omega}}(\omega) = S(\omega) / \pi R_{xx}(0)$ and
$\mathbf{x}(t) = a\cos(\boldsymbol{\omega}t + \boldsymbol{\varphi})$ would make $S_{xx}(\omega) = S(\omega)$

If a WSS process $\mathbf{x}(t)$, input to a linear system with impulse response $h(t)$, gives $\mathbf{y}(t)$ as a WSS output process, then

$$\mathbf{x}(t+\tau)[\mathbf{y}(t)]^* = \mathbf{x}(t+\tau)\left[\int_{-\infty}^{\infty}\mathbf{x}(t-\alpha)h(\alpha)d\alpha\right]^*$$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau+\alpha)h^*(\alpha)d\alpha$$

$$= R_{xx}(\tau) * h^*(-\tau)$$

Therefore $S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega)$ where $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$

Similarly $R_{yy}(\tau) = R_{xy}(\tau) * h(\tau) \Rightarrow S_{yy}(\omega) = S_{xy}(\omega)H(\omega)$

$$= S_{xx}(\omega)\left|H(\omega)\right|^2$$

Pictorial representation: (2nd edition, fig.10-5, p.273)

$$R_{xx}(\tau) \longrightarrow \boxed{h^*(-\tau)} \longrightarrow R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau) \longrightarrow \boxed{h(\tau)} \longrightarrow R_{yy}(\tau) = R_{xy}(\tau) * h(\tau)$$

$$S_{xx}(\omega) \longrightarrow \boxed{H^*(\omega)} \longrightarrow S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow S_{yy}(\omega) = S_{xy}(\omega)H(\omega)$$

If $\mathbf{x}(t)$ is white, then $R_{xx}(\tau) = q\delta(\tau)$ impulse

$\Rightarrow S_{xx}(\omega) = q$ flat

$$\Rightarrow S_{yy}(\omega) = q\left|H(\omega)\right|^2$$

Example 6: Hilbert Transform

Quadrature filter $H(\omega) = -j\,\mathrm{sgn}\,\omega = \begin{cases} -j & \omega > 0 \\ j & \omega < 0 \end{cases}$ allpass
$-90°$ phase shift

impulse response $h(t) = \dfrac{1}{\pi t}$

$$\mathbf{x}(t) \longrightarrow \boxed{H(\omega)} \longrightarrow \breve{\mathbf{x}}(t)$$

Hilbert transform

$\mathbf{x}(t)$ real process

$$\breve{\mathbf{x}}(t) = \mathbf{x}(t) * \frac{1}{\pi t}$$

<u>Example 6a:</u> $\mathbf{x}(t) = \mathbf{a}\cos\omega_0 t + \mathbf{b}\sin\omega_0 t$ (2nd edition, ex.10-16, p.284)

$$\Rightarrow \breve{\mathbf{x}}(t) = \mathbf{a}\cos(\omega_0 t - 90^\circ) + \mathbf{b}\sin(\omega_0 t - 90^\circ)$$
$$= \mathbf{a}\sin\omega_0 t - \mathbf{b}\cos\omega_0 t$$

$$S_{x\breve{x}}(\omega) = j\,\mathrm{sgn}\,\omega \cdot S_{xx}(\omega)$$

$$S_{\breve{x}\breve{x}}(\omega) = -j\,\mathrm{sgn}\,\omega \cdot j\,\mathrm{sgn}\,\omega \cdot S_{xx}(\omega) = S_{xx}(\omega) \qquad \text{since} \quad \mathrm{sgn}^2\,\omega = 1$$

Analytic signal (complex process) $\mathbf{z}(t) = \mathbf{x}(t) + j\breve{\mathbf{x}}(t)$

<u>Example 6a (contd.):</u> (2nd edition, ex.10-16, p.284)

$$\mathbf{x}(t) = \mathbf{a}\cos\omega_0 t + \mathbf{b}\sin\omega_0 t$$
$$\Rightarrow \mathbf{z}(t) = (\mathbf{a} - j\mathbf{b})e^{j\omega_0 t}$$

$$\mathbf{x}(t) \longrightarrow \boxed{2U(\omega)} \longrightarrow \mathbf{z}(t)$$

Frequency response $= 1 + j(-j\,\mathrm{sgn}\,\omega) = 2U(\omega)$

$$S_{zz}(\omega) = 4S_{xx}(\omega)U(\omega) \quad \text{since} \;\; U^2(\omega) = U(\omega)$$

$$= 2S_{xx}(\omega) + 2jS_{\breve{x}x}(\omega)$$

$$\text{since} \;\; S_{\breve{x}x}(\omega) = S^*_{x\breve{x}}(\omega) = -j\,\mathrm{sgn}\,\omega \cdot S_{xx}(\omega)$$

$$\Rightarrow R_{zz}(\tau) = 2R_{xx}(\tau) + 2jR_{\breve{x}x}(\tau)$$

<u>Wiener-Khinchin theorem:</u>

$$E\{|\mathbf{x}(t)|^2\} = R_{xx}(0) = \int_{-\infty}^{\infty} S_{xx}(\omega)\frac{d\omega}{2\pi} \geq 0$$

It follows that $S_{xx}(\omega) \geq 0$

<u>Property of correlation:</u>

Now $\quad R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}\frac{d\omega}{2\pi}$

$$\Rightarrow |R_{xx}(\tau)| \leq \int_{-\infty}^{\infty} |S_{xx}(\omega)e^{j\omega\tau}|\frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} S_{xx}(\omega)\frac{d\omega}{2\pi} = R_{xx}(0)$$

Thus, $|R_{xx}(\tau)| \leq R_{xx}(0)$ or $R_{xx}(\tau)$ is maximum at the origin.

If a process has other maxima, $R_{xx}(\tau_1) = R_{xx}(0)$ for $\tau_1 \neq 0$, then

$$|R_{xx}(\tau+\tau_1) - R_{xx}(\tau)|^2 = \left|E\left\{[\mathbf{x}(t+\tau+\tau_1) - \mathbf{x}(t+\tau)]\mathbf{x}^*(t)\right\}\right|^2$$

$$\leq E\left\{|\mathbf{x}(t+\tau+\tau_1) - \mathbf{x}(t+\tau)|^2\right\}E\left\{|\mathbf{x}(t)|^2\right\}$$

$$= E\left\{|\mathbf{x}(t+\tau+\tau_1)|^2 - \mathbf{x}(t+\tau+\tau_1)\mathbf{x}^*(t+\tau) - \mathbf{x}^*(t+\tau+\tau_1)\mathbf{x}(t+\tau) + |\mathbf{x}(t+\tau)|^2\right\} E\left\{|\mathbf{x}(t)|^2\right\}$$

$$= [R_{xx}(0) - R_{xx}(\tau_1) - R_{xx}^*(\tau_1) + R_{xx}(0)]R_{xx}(0)$$

but $R_{xx}(0) = \text{real}$, so $R_{xx}^*(\tau_1) = R_{xx}(0)$

$$= 0$$

or $|R_{xx}(\tau+\tau_1) - R_{xx}(\tau)|^2 \le 0$

or $R_{xx}(\tau+\tau_1) = R_{xx}(\tau), \forall\tau$

or $R_{xx}(\tau)$ periodic with period τ_1

Further, $|R_{xy}(\tau)|^2 \le R_{xx}(0)R_{yy}(0)$

because $|R_{xy}(\tau)|^2 = |E\{\mathbf{x}(t+\tau)\mathbf{y}^*(t)\}|^2 \le E\{|\mathbf{x}(t+\tau)|^2\}E\{|\mathbf{y}(t)|^2\}$

$$= R_{xx}(0)R_{yy}(0)$$

Discrete-Time (Digital) Processes

$$\mathbf{x}[n]$$

mean $\eta[n] = E\{\mathbf{x}[n]\}$

autocorrelation $R_{xx}[n_1,n_2] = E\{\mathbf{x}[n_1]\mathbf{x}^*[n_2]\}$

m-th order statistics $f(x_1,\cdots,x_m;n_1,\cdots,n_m)$

SSS $f(x_1,\cdots,x_m;n_1,\cdots,n_m) = f(x_1,\cdots,x_m;n_1+M,\cdots,n_m+M), \quad \forall M$

WSS $\eta[n]=\eta,\ R_{xx}[n_1,n_2] = R_{xx}[n_1-n_2] = R_{xx}[m],\ m = n_1 - n_2$

$\mathbf{x}[n]$ is white noise if $\mathbf{x}[n_1]$ and $\mathbf{x}[n_2]$ are uncorrelated for any $n_1 \ne n_2$

Therefore $R_{xx}[n_1,n_2] = q[n_1]\delta[n_1-n_2]$, where $\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n \ne 0 \end{cases}$

If $\mathbf{x}[n]$ and $\mathbf{y}[n]$ are input and output to a linear system, then

$$\mathbf{y}[n] = \mathbf{x}[n] * h[n] = \sum_{k=-\infty}^{\infty} \mathbf{x}[n-k]h[k]$$

where $h[n]$ is the impulse response of the system, $h[n] = L\left[\delta[n]\right]$

All earlier results are valid for discrete-time cases.

Power spectrum is discrete-time Fourier transform of autocorrelation,

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} R_{xx}[m]e^{-jm\omega}$$

$S_{xx}(\omega)$ is periodic with period 2π, and $S_{xx}(\omega) \geq 0$

$$R_{xx}[m] = \int_{-\pi}^{\pi} S_{xx}(\omega)e^{jm\omega}\frac{d\omega}{2\pi}$$

Types of Power Spectrum

Continuous-time process: power spectrum $R_{xx}(\tau) \xrightarrow{\text{FT}} S_{xx}(\omega)$

cross power spectrum $R_{xy}(\tau) \xrightarrow{\text{FT}} S_{xy}(\omega)$

covariance spectrum $C_{xx}(\tau) \xrightarrow{\text{FT}} S_{xx}^{c}(\omega)$

Laplace transform version: $R_{xx}(\tau) \xrightarrow{\text{LT}} \mathbf{S}_{xx}(s)$

$S_{xx}(\omega)$ and $\mathbf{S}_{xx}(s)$ are different. On imaginary axis $s = j\omega$

$$\Rightarrow \mathbf{S}_{xx}(j\omega) = S_{xx}(\omega)$$

Linear system:

impulse resp. freq. resp.

$$h(t) \xrightarrow{\text{FT}} H(\omega)$$

$$S_{xy}(\omega) = S_{xx}(\omega)H^{*}(\omega)$$

$$S_{yy}(\omega) = S_{xx}(\omega)\,|\,H(\omega)\,|^{2}$$

causal transfer function

$$h(t) \xrightarrow{\text{LT}} \mathbf{H}(s), \quad \mathbf{H}(j\omega) = H(\omega)$$

$$\mathbf{S}_{xy}(s) = \mathbf{S}_{xx}(s)\mathbf{H}(-s)$$

$h(t)$ real

$$\mathbf{S}_{yy}(s) = \mathbf{S}_{xx}(s)\mathbf{H}(s)\mathbf{H}(-s)$$

Discrete-time process: power spectrum $R_{xx}[m] \xrightarrow{\text{DTFT}} S_{xx}(\omega)$

cross power spectrum $R_{xy}[m] \xrightarrow{\text{DTFT}} S_{xy}(\omega)$

covariance spectrum $C_{xx}[m] \xrightarrow{\text{DTFT}} S_{xx}^{c}(\omega)$

DTFT is a special case of z-transform (on the unit circle)

z-transform version: $R_{xx}[m] \xrightarrow{\text{zT}} \mathbf{S}_{xx}(z) = \sum_{m=-\infty}^{\infty} R_{xx}[m] z^{-m}$

$S_{xx}(\omega)$ and $\mathbf{S}_{xx}(z)$ are different. On unit circle $z = e^{j\omega}$

$\Rightarrow \mathbf{S}_{xx}(e^{j\omega}) = S_{xx}(\omega)$

Linear system:

impulse resp. freq. response

$$h[n] \xrightarrow{\text{DTFT}} H(\omega)$$

$$S_{xy}(\omega) = S_{xx}(\omega) H^{*}(\omega)$$

$$S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2$$

transfer function

$$h[n] \xrightarrow{\text{zT}} \mathbf{H}(z), \quad \mathbf{H}(e^{j\omega}) = H(\omega)$$

$$\mathbf{S}_{xy}(z) = \mathbf{S}_{xx}(z) \mathbf{H}(z^{-1})$$

$$\mathbf{S}_{yy}(z) = \mathbf{S}_{xx}(z) \mathbf{H}(z) \mathbf{H}(z^{-1})$$

$h[n]$ real

Example 7: AR(1) Process

real white noise $\mathbf{x}[n] \rightarrow$ h[n] $\rightarrow \mathbf{y}[n]$

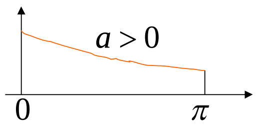
$$\mathbf{y}[n] = \mathbf{x}[n] + a\mathbf{y}[n-1]$$

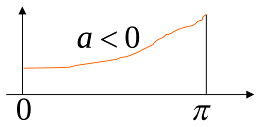
$$h[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \Rightarrow \mathbf{H}(z) = \frac{1}{1 - az^{-1}} \qquad |a| < 1 \text{ for stability}$$

Then $\mathbf{S}_{yy}(e^{j\omega}) = \mathbf{S}_{xx}(e^{j\omega}) \dfrac{1}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} = \dfrac{\mathbf{S}_{xx}(e^{j\omega})}{1 - 2a\cos\omega + a^2}$

Since the excitation $\mathbf{x}[n]$ is white noise, $\mathbf{S}_{xx}(e^{j\omega}) = q$ (WSS)

$$\mathbf{S}_{yy}(e^{j\omega}) = \frac{q}{1 - 2a\cos\omega + a^2}$$





or $$\mathbf{S}_{yy}(z) = \frac{q}{\left(1 - az^{-1}\right)\left(1 - az\right)} = \frac{q}{1 - a^2}\left[\frac{1}{1 - az^{-1}} + \frac{az}{1 - az}\right]$$

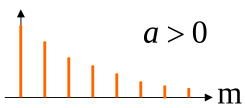
Now $$\frac{1}{1 - az^{-1}} = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} a^{|k|} z^{-k}$$

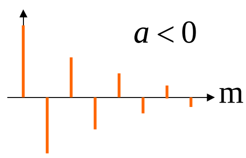
Also $$\frac{az}{1 - az} = \frac{1}{1 - az} - 1 = \sum_{l=0}^{\infty} a^l z^l - 1 = \sum_{l=1}^{\infty} a^l z^l = \sum_{k=-\infty}^{-1} a^{-k} z^{-k} = \sum_{k=-\infty}^{-1} a^{|k|} z^{-k}$$

Thus $$\mathbf{S}_{yy}(z) = \frac{q}{1 - a^2} \sum_{k=-\infty}^{\infty} a^{|k|} z^{-k}$$

But $$\mathbf{S}_{yy}(z) = \sum_{k=-\infty}^{\infty} R_{yy}[k] z^{-k}$$

Therefore $$R_{yy}[m] = \frac{q}{1 - a^2} a^{|m|}$$



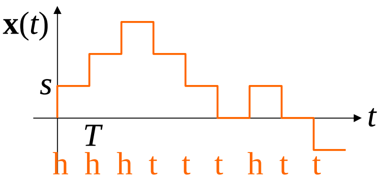


$$R_{yy}[0] = E\{\mathbf{y}^2[n]\} = \frac{q}{1 - a^2} \quad \text{where} \quad q = R_{xx}[0] = E\{\mathbf{x}^2[n]\}$$

Random Walk, Wiener Process

A fair coin is tossed once every T seconds, and a step of (real) length s is taken for heads, or length $-s$ for tails.

$\mathbf{x}(t)$ is a stochastic process called the random walk



Starting at 0, $\mathbf{x}(nT)$ will have value $ms = (k-(n-k))s$ if there are k heads and $n-k$ tails.

$$\Rightarrow P\{\mathbf{x}(nT)=ms\} = \binom{n}{k} 0.5^k 0.5^{n-k}$$

For large n and $k \approx np$, $\binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)^2/2npq}$

$$\Rightarrow P\{\mathbf{x}(nT)=ms\} \approx \frac{1}{\sqrt{2\pi(n/4)}} e^{-(m/2)^2/(2n/4)} \qquad \text{since } m = 2k-n$$

which is like a normal density in $m/2$ with mean 0, variance $n/4$

therefore $P\{\mathbf{x}(nT) \le ms\} \approx \int_{-\infty}^{m/2} N(0, n/4) = \int_{-\infty}^{(m/2)/\sqrt{n/4}} N(0,1)$

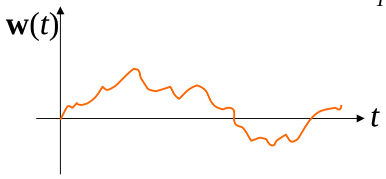
Each step is independent with mean 0 and variance s^2

$\mathbf{x}(nT)$ is the sum of n such steps

$\Rightarrow E\{\mathbf{x}(nT)\} = 0, \quad E\{\mathbf{x}^2(nT)\} = ns^2$

Let $T \to 0$ and $s^2 = \alpha T$

Then the discrete-state process $\mathbf{x}(t)$ becomes a continuous-state process called the Wiener process: $\mathbf{w}(t) = \lim_{T \to 0} \mathbf{x}(t)$



Substituting $w = ms$ and $t = nT$, or

$$\frac{m}{2\sqrt{n/4}} = \frac{m}{\sqrt{n}} = \frac{w/s}{\sqrt{\alpha t / s}} = \frac{w}{\sqrt{\alpha t}}$$

$$P\{\mathbf{w}(t) \le w\} = \int_{-\infty}^{w/\sqrt{\alpha t}} N(0,1)$$

Or, the first-order density of $\mathbf{w}(t)$ is $f(w,t) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-w^2/2\alpha t}$
(normal with mean 0, variance αt)

autocorrelation $R_{ww}(t_1,t_2) = E\{\mathbf{w}(t_1)\mathbf{w}(t_2)\}$

$$= E\{\mathbf{w}(t_1)[\mathbf{w}(t_1) + \mathbf{w}(t_2) - \mathbf{w}(t_1)]\}$$

$$= E\{\mathbf{w}^2(t_1)\} + E\{\mathbf{w}(t_1)[\mathbf{w}(t_2) - \mathbf{w}(t_1)]\}$$

Let $t_1 < t_2$, then $\mathbf{w}(t_1)$ and $\mathbf{w}(t_2) - \mathbf{w}(t_1)$ are independent

$$\Rightarrow E\{\mathbf{w}(t_1)[\mathbf{w}(t_2) - \mathbf{w}(t_1)]\} = E\{\mathbf{w}(t_1)\}E\{\mathbf{w}(t_2) - \mathbf{w}(t_1)\}$$

which is 0 since $E\{\mathbf{w}(t)\} = 0$

Therefore in this case $R_{ww}(t_1, t_2) = E\{\mathbf{w}^2(t_1)\} = \dfrac{t_1 s^2}{T} = \alpha t_1$

Similarly, it may be shown that for $t_1 > t_2$, $R_{ww}(t_1, t_2) = \alpha t_2$

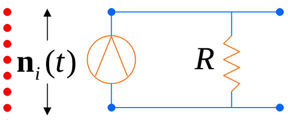
Therefore, $R_{ww}(t_1, t_2) = \alpha \min(t_1, t_2)$

Thermal Noise

Noise due to thermal agitation of atoms above 0°K

Reactive elements assumed noiseless

Resistors replaced by noiseless resistor in parallel with a current source



$\mathbf{n}_i(t)$ is a normal process, zero-mean, white

$$S_{n_i n_i}(\omega) = \frac{2kT}{R}$$

k Boltzmann constant

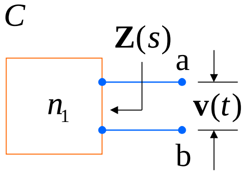
T Temperature in °Kelvin

Noise source of one resistor independent of that of another resistor

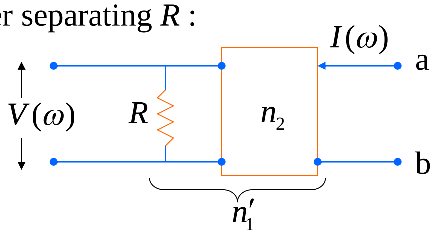
Let there be a passive network of R, L and C

$\mathbf{v}(t)$ = voltage across a and b due to $\mathbf{n}_i(t)$

$\mathbf{Z}(s)$ = impedance between a and b



Then consider separating R :



n_2 is the network without R, consisting only of reactive components.

n_1' is a linear system with frequency response $H(\omega)$.

$$H(\omega) = \frac{V(\omega)}{I(\omega)} = \frac{\text{output voltage}}{\text{input current}} \quad \text{in reverse mode}$$

If $I(\omega)$ is applied between a and b, then input power

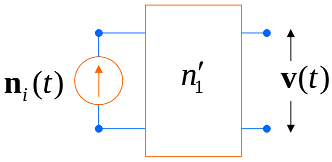
$= |I(\omega)|^2 \operatorname{Re}\{\mathbf{Z}(j\omega)\}$, since $\mathbf{Z}(s)$ is the impedance between a and b.

Since n_2 does not dissipate any power, power dissipated is across R, which is equal to $|V(\omega)|^2 / R$

Therefore $$|I(\omega)|^2 \operatorname{Re}\{\mathbf{Z}(j\omega)\} = \frac{|V(\omega)|^2}{R}$$

or $$|H(\omega)|^2 = R \cdot \operatorname{Re}\{\mathbf{Z}(j\omega)\}$$

Now, for network n_1', input process (current) has psd $S_{n_i n_i}(\omega) = \dfrac{2kT}{R}$



Output process (voltage) has psd

$$S_{vv}(\omega) = S_{n_i n_i}(\omega)\,|H(\omega)|^2 = 2kT\,\mathrm{Re}\{\mathbf{Z}(j\omega)\}$$

$$= kT\left[\mathbf{Z}(j\omega) + \mathbf{Z}^*(j\omega)\right]$$

but $\mathbf{Z}^*(j\omega) = \mathbf{Z}(-j\omega)$

$$= kT\left[\mathbf{Z}(j\omega) + \mathbf{Z}(-j\omega)\right]$$

Therefore $R_{vv}(\tau) = kT \cdot z(\tau) + kT \cdot z(-\tau)$

since $z(t)$ is inverse transform of $\mathbf{Z}(s)$,
then $z(-t)$ is inverse transform of $\mathbf{Z}(-s)$.

But, $z(t)$ valid for $t > 0$, $z(-t) = 0$ for $t > 0$

So $R_{vv}(\tau) = kT \cdot z(\tau), \quad \tau > 0$

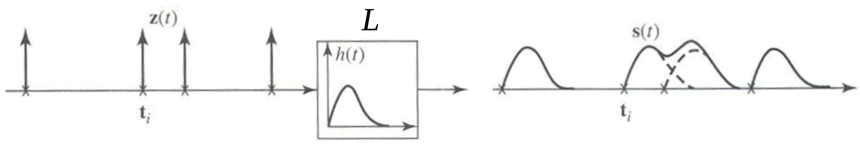
Shot Noise

$\mathbf{s}(t) = \sum_i h(t - \mathbf{t}_i)$ where $h(t)$ is a real function, $\mathbf{t}_i$ are a set of Poisson points with average density λ

$\mathbf{s}(t)$ is shot noise.

$\mathbf{s}(t)$ is output of a linear system driven by a train of Poisson impulses,

$\mathbf{s}(t) = L\left[\sum_i \delta\left(t - \mathbf{t}_i\right)\right]$, where L has impulse response $h(t)$.



Since Poisson impulses are SSS, $\mathbf{s}(t)$ is also SSS.

Shot noise is observed as the output of a system activated by a random sequence of impulses such as particle emissions.

$$E\{\mathbf{s}(t)\} = L\left[E\{\mathbf{z}(t)\}\right] = L[\lambda]$$

$$= \lambda \int_{-\infty}^{\infty} h(t)dt = \lambda H(0)$$

$S_{zz}(\omega) = 2\pi\lambda^2\delta(\omega) + \lambda$ (earlier result, Poisson impulses)

Therefore $S_{ss}(\omega) = S_{zz}(\omega)\left|H(\omega)\right|^2$

$$= 2\pi\lambda^2\left|H(\omega)\right|^2\delta(\omega) + \lambda\left|H(\omega)\right|^2$$

$$= 2\pi\lambda^2 H^2(0)\delta(\omega) + \lambda\left|H(\omega)\right|^2$$

Neglecting dc term, $S_{ss}(\omega)$ has the shape of $\left|H(\omega)\right|^2$.

Let $\rho(\tau)$ be inverse transform of $\left|H(\omega)\right|^2$

Since $\left|H(\omega)\right|^2 = H(\omega)H^*(\omega)$, and inverse transform of $H(\omega)$ is $h(t)$, inverse transform of $H^*(\omega)$ is $h^*(-t)$,

$\rho(\tau) = h(\tau) * h^*(-\tau) = h(\tau) * h(-\tau)$ since $h(t)$ is real

Then $R_{ss}(\tau) = \lambda^2 H^2(0) + \lambda\rho(\tau)$

variance $\sigma_s^2 = R_{ss}(0) - \eta_s^2 = \lambda^2 H^2(0) + \lambda\rho(0) - \lambda^2 H^2(0) = \lambda\rho(0)$

but $\rho(0) = \int_{-\infty}^{\infty} h(t)h^*\left[0-(-t)\right]dt = \int_{-\infty}^{\infty} h^2(t)dt$

so $\sigma_s^2 = \lambda\int_{-\infty}^{\infty} h^2(t)dt$

Modulation

Modulating process: $\mathbf{w}(t) = \mathbf{r}(t)e^{j\boldsymbol{\varphi}(t)}$ $\quad \begin{cases} \mathrm{Re}\left[\mathbf{w}(t)\right] = \mathbf{a}(t) \\ \mathrm{Im}\left[\mathbf{w}(t)\right] = \mathbf{b}(t) \end{cases}$

Modulated process: $\mathbf{z}(t) = \mathbf{w}(t)e^{j\omega_0 t}$

where $e^{j\omega_0 t}$ = (complex) sinusoid (carrier)

Demodulation: $\mathbf{w}(t) = \mathbf{z}(t)e^{-j\omega_0 t}$

Note:

$$\mathrm{Re}\left[\mathbf{z}(t)\right] = \mathbf{x}(t) = \mathbf{a}(t)\cos(\omega_0 t) - \mathbf{b}(t)\sin(\omega_0 t) = \mathbf{r}(t)\cos\left[\omega_0 t + \boldsymbol{\varphi}(t)\right]$$
$$\mathrm{Im}\left[\mathbf{z}(t)\right] = \mathbf{y}(t) = \mathbf{b}(t)\cos(\omega_0 t) + \mathbf{a}(t)\sin(\omega_0 t) = \mathbf{r}(t)\sin\left[\omega_0 t + \boldsymbol{\varphi}(t)\right]$$

Amplitude modulation by $\mathbf{r}(t)$, phase modulation by $\boldsymbol{\varphi}(t)$

Consider $\mathbf{x}(t)$, $E\{\mathbf{x}(t)\} = E\{\mathbf{a}(t)\}\cos(\omega_0 t) - E\{\mathbf{b}(t)\}\sin(\omega_0 t)$

$= 0$ since $\mathbf{a}(t), \mathbf{b}(t)$ are zero-mean, jointly WSS

$$E\{\mathbf{x}(t+\tau)\mathbf{x}(t)\} = E\Big\{\Big[\mathbf{a}(t+\tau)\cos\big(\omega_0(t+\tau)\big) - \mathbf{b}(t+\tau)\sin\big(\omega_0(t+\tau)\big)\Big]$$
$$\times\left[\mathbf{a}(t)\cos(\omega_0 t) - \mathbf{b}(t)\sin(\omega_0 t)\right]\Big\}$$
$$= E\left\{\mathbf{a}(t+\tau)\mathbf{a}(t)\left[\frac{\cos\big(\omega_0(2t+\tau)\big) + \cos(\omega_0\tau)}{2}\right]\right.$$
$$-\mathbf{a}(t+\tau)\mathbf{b}(t)\left[\frac{\sin\big(\omega_0(2t+\tau)\big) + \sin(-\omega_0\tau)}{2}\right]$$
$$-\mathbf{a}(t)\mathbf{b}(t+\tau)\left[\frac{\sin\big(\omega_0(2t+\tau)\big) + \sin(\omega_0\tau)}{2}\right]$$
$$\left.+\mathbf{b}(t)\mathbf{b}(t+\tau)\left[\frac{-\cos\big(\omega_0(2t+\tau)\big) + \cos(\omega_0\tau)}{2}\right]\right\}$$

$$= \frac{1}{2}\left[R_{aa}(\tau) + R_{bb}(\tau)\right]\cos(\omega_0\tau) + \frac{1}{2}\left[R_{ab}(\tau) - R_{ba}(\tau)\right]\sin(\omega_0\tau)$$

$$+\frac{1}{2}\left[R_{aa}(\tau) - R_{bb}(\tau)\right]\cos\left\{\omega_0(2t+\tau)\right\} + \frac{1}{2}\left[-R_{ab}(\tau) - R_{ba}(\tau)\right]\sin\left\{\omega_0(2t+\tau)\right\}$$

$\mathbf{x}(t)$ is WSS if $\quad R_{xx}(t_1, t_2) = R_{xx}(\tau)$

$$\Updownarrow$$

$$R_{aa}(\tau) = R_{bb}(\tau) \text{ and } R_{ab}(\tau) = -R_{ba}(\tau)$$

$\mathbf{x}(t)$ WSS $\Rightarrow$ $\mathbf{y}(t)$ WSS

$$R_{xx}(\tau) = R_{yy}(\tau) = R_{aa}(\tau)\cos(\omega_0\tau) + R_{ab}(\tau)\sin(\omega_0\tau)$$

$$R_{xy}(\tau) = -R_{yx}(\tau) = R_{ab}(\tau)\cos(\omega_0\tau) - R_{aa}(\tau)\sin(\omega_0\tau)$$

$$R_{ww}(\tau) = E\left\{\left[\mathbf{a}(t+\tau) + j\mathbf{b}(t+\tau)\right]\left[\mathbf{a}(t) - j\mathbf{b}(t)\right]\right\}$$

$$= R_{aa}(\tau) + R_{bb}(\tau) - jR_{ab}(\tau) + jR_{ba}(\tau)$$

$$= 2R_{aa}(\tau) - 2jR_{ab}(\tau)$$

$$R_{zz}(\tau) = E\{\mathbf{w}(t+\tau)e^{j\omega_0(t+\tau)}\mathbf{w}^*(t)e^{-j\omega_0 t}\}$$

$$= R_{ww}(\tau)e^{j\omega_0\tau}$$

For psd, $S_{ww}(\omega) = 2S_{aa}(\omega) - 2jS_{ab}(\omega)$ and $S_{zz}(\omega) = S_{ww}(\omega - \omega_0)$

Bandlimited Process

$\mathbf{x}(t)$ is bandlimited if $R_{xx}(0) < \infty$ (finite power)

and $S_{xx}(\omega) = 0$ for $|\omega| > \sigma$ (limited spectral width)

1) A bandlimited process may be sampled:

$$\mathbf{x}(t+\tau) = L[\mathbf{x}(t)] \text{ where } H(\omega) = e^{j\omega\tau}$$

But $e^{j\omega\tau}$ is a continuous function, hence may be expanded using Taylor series,

$$H(\omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial\omega^n} e^{j\omega\tau}\Big|_{\omega=0} \cdot \omega^n = \sum_{n=0}^{\infty} \frac{1}{n!} (j\tau)^n \omega^n = \sum_{n=0}^{\infty} (j\omega)^n \frac{\tau^n}{n!}$$

If $x(t) \leftrightarrow X(\omega)$ then $\frac{d}{dt} x(t) \leftrightarrow j\omega X(\omega)$

Therefore $L[\mathbf{x}(t)] = \sum_{n=0}^{\infty} \mathbf{x}^{(n)}(t) \frac{\tau^n}{n!}$, since $\mathbf{x}^{(1)}(t) = L_1[\mathbf{x}(t)]$ with $H_1(\omega) = j\omega$, differentiator

Thus $\mathbf{x}(t+\tau) = \sum_{n=0}^{\infty} \mathbf{x}^{(n)}(t) \frac{\tau^n}{n!}$

$\mathbf{x}^{(n)}(t)$ exists for all n

$R_{xx}(\tau) = \int_{-\sigma}^{\sigma} S_{xx}(\omega) e^{j\omega\tau} \frac{d\omega}{2\pi}$ since bandlimited
also $\int S_{xx}(\omega) d\omega$ is limited since $R_{xx}(0)$ is.

So, $R_{xx}^{(n)}(\tau) = \int_{-\sigma}^{\sigma} \frac{\partial^n}{\partial\tau^n} \left[S_{xx}(\omega) e^{j\omega\tau}\right] \frac{d\omega}{2\pi} = \int_{-\sigma}^{\sigma} (j\omega)^n S_{xx}(\omega) e^{j\omega\tau} \frac{d\omega}{2\pi}$

Thus, all derivatives of $R_{xx}(\tau)$ exist $\Rightarrow$ all derivatives of $\mathbf{x}(t)$ exist.

Therefore, knowing $\mathbf{x}^{(n)}(t)$ for, say $t = 0$, for all n is sufficient to construct $\mathbf{x}(t)$ for all t. Thus, a countably infinite values completely specify the bandlimited process.

2) A bandlimited process is continuous and smooth:

$$E\left\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\right\} = R_{xx}(0) - R_{xx}(\tau) - R_{xx}(-\tau) + R_{xx}(0)$$

$$= \int_{-\sigma}^{\sigma} S_{xx}(\omega)[2 - e^{j\omega\tau} - e^{-j\omega\tau}]\frac{d\omega}{2\pi} \qquad = 2\int_{-\sigma}^{\sigma} S_{xx}(\omega)[1 - \cos(\omega\tau)]\frac{d\omega}{2\pi}$$

$$= 2\int_{-\sigma}^{\sigma} S_{xx}(\omega)\cdot 2\sin^2\left(\frac{\omega\tau}{2}\right)\frac{d\omega}{2\pi}$$

$$\leq 4\int_{-\sigma}^{\sigma} S_{xx}(\omega)\left(\frac{\omega^2\tau^2}{4}\right)\frac{d\omega}{2\pi} \quad \text{since } \left|\sin\left(\frac{\omega\tau}{2}\right)\right| \leq \left|\frac{\omega\tau}{2}\right| \text{ or } \sin^2\left(\frac{\omega\tau}{2}\right) \leq \frac{\omega^2\tau^2}{4}$$

$$\text{and } S_{xx}(\omega) \geq 0$$

$$\leq \int_{-\sigma}^{\sigma} S_{xx}(\omega)\sigma^2\tau^2\frac{d\omega}{2\pi} = \sigma^2\tau^2 R_{xx}(0)$$

Thus $E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\} \leq \sigma^2\tau^2 R_{xx}(0)$, or signal does not change much for small τ

Sampling Expansion

$$\mathbf{x}(t+\tau) = \mathbf{x}(t) * F^{-1}\left[e^{j\omega\tau}\right]$$

Expand $e^{j\omega\tau}$ in a Fourier series in the interval $-\sigma \leq \omega \leq \sigma$

Then $e^{j\omega\tau} = \sum_{n=-\infty}^{\infty} a_n e^{jnT\omega}$ where $T = \frac{2\pi}{2\sigma}$ is fundamental period.

$$a_n = \frac{1}{2\sigma}\int_{-\sigma}^{\sigma} e^{j\omega\tau}e^{-jnT\omega}d\omega = \frac{\sin\left[\sigma(\tau - nT)\right]}{\sigma(\tau - nT)}$$

Since $\mathbf{x}(t)$ is bandlimited to σ

$$\mathbf{x}(t+\tau) = \mathbf{x}(t) * F^{-1}\left[\sum_{n=-\infty}^{\infty} \frac{\sin\left[\sigma(\tau - nT)\right]}{\sigma(\tau - nT)} e^{jnT\omega}\right]$$

$$= \sum_{n=-\infty}^{\infty} \left[\mathbf{x}(t) * F^{-1}[e^{jnT\omega}] \right] \frac{\sin\left[\sigma(\tau - nT)\right]}{\sigma(\tau - nT)}$$

$$= \sum_{n=-\infty}^{\infty} \mathbf{x}(t + nT) \frac{\sin\left[\sigma(\tau - nT)\right]}{\sigma(\tau - nT)}$$

Putting $t = 0$, $\quad \mathbf{x}(\tau) = \sum_{n=-\infty}^{\infty} \mathbf{x}(nT). \frac{\sin\left[\sigma(\tau - nT)\right]}{\sigma(\tau - nT)}$

continuous process — sampled (discrete) process — ideal low-pass filter impulse response

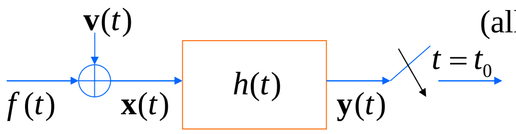
For any $T < \frac{2\pi}{2\sigma}$, it is still valid, since then $e^{j\omega\tau}$ is expanded for $(-\sigma', \sigma')$ for some $\sigma' > \sigma$.

Matched Filter (Detecting Signal in Noise)

Received signal $\mathbf{x}(t) = f(t) + \mathbf{v}(t)$, $f(t)$ = (shifted, scaled) known signal

$\mathbf{v}(t)$ = noise process, WSS, psd $S_{vv}(\omega)$

(all real processes)



output $\quad \mathbf{y}(t) = y_f(t) + \mathbf{y}_v(t)$

$$y_f(t) = \int_{-\infty}^{\infty} f(t-\alpha)h(\alpha)d\alpha, \quad Y_f(\omega) = F(\omega)H(\omega)$$

$$= \int_{-\infty}^{\infty} F(\omega)H(\omega)e^{j\omega t} \frac{d\omega}{2\pi}$$

Similarly, $\quad \mathbf{y}_v(t) = \mathbf{v}(t) * h(t)$

Output sampled at $t = t_0$, when SNR is $r^2 = \dfrac{|y_f(t_0)|^2}{E\{\mathbf{y}_v^2(t_0)\}}$

Find $h(t)$ to maximize r^2

Colored noise:

$$r^2 = \frac{\left|\int_{-\infty}^{\infty} F(\omega)H(\omega)e^{j\omega t_0}\frac{d\omega}{2\pi}\right|^2}{E\{\mathbf{y}_v^2(t_0)\}} = \frac{\left|\int_{-\infty}^{\infty}\left[\frac{F(\omega)}{\sqrt{S_{vv}(\omega)}}e^{j\omega t_0}\right]\left[\sqrt{S_{vv}(\omega)}H(\omega)\right]\frac{d\omega}{2\pi}\right|^2}{E\{\mathbf{y}_v^2(t_0)\}}$$

$$\leq \frac{\int_{-\infty}^{\infty}\frac{\left|F(\omega)e^{j\omega t_0}\right|^2}{S_{vv}(\omega)}\frac{d\omega}{2\pi}\cdot\int_{-\infty}^{\infty}S_{vv}(\omega)\,|H(\omega)|^2\frac{d\omega}{2\pi}}{E\{\mathbf{y}_v^2(t_0)\}}$$

but $E\{\mathbf{y}_v^2(t_0)\} = R_{y_v y_v}(0) = \int_{-\infty}^{\infty} S_{y_v y_v}(\omega)\frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} S_{vv}(\omega)\,|H(\omega)|^2\frac{d\omega}{2\pi}$

So $r^2 \leq \int_{-\infty}^{\infty}\frac{|F(\omega)|^2}{S_{vv}(\omega)}\frac{d\omega}{2\pi}$, equality if $k\left[\frac{F(\omega)}{\sqrt{S_{vv}(\omega)}}e^{j\omega t_0}\right]^* = \sqrt{S_{vv}(\omega)}H(\omega)$

for some constant k, or $H(\omega) = k\dfrac{F^*(\omega)}{S_{vv}(\omega)}e^{-j\omega t_0}$

White noise:

If $\mathbf{v}(t)$ is white noise, $S_{vv}(\omega) = S_0$ and $H(\omega) = k'F^*(\omega)e^{-j\omega t_0}$

or $h(t) = k'f(t_0 - t)$

Thus, optimum filter impulse response is (scaled, shifted) time-reversed signal $\Rightarrow$ hence, matched filter.

Colored noise from white noise (using innovations):

(2nd edition, chap.10-5, p.300)

$$\mathbf{x}(t) \longrightarrow \boxed{\Gamma_v(s)} \longrightarrow \mathbf{x}_1(t) \longrightarrow \boxed{H_1(\omega)} \longrightarrow \mathbf{y}(t)$$

$\mathbf{x}(t) = f(t) + \mathbf{v}(t)$ with $\mathbf{v}(t)$ colored noise having psd $S_{vv}(\omega)$

Use a whitening filter $\Gamma_v(s)$ such that $S_{vv}(\omega) = \dfrac{1}{|\Gamma_v(j\omega)|^2}$

$\Rightarrow \mathbf{x}_1(t) = f_1(t) + \mathbf{i}_v(t)$ where $\mathbf{i}_v(t)$ is the innovation of $\mathbf{v}(t)$

$S_{i_v i_v}(\omega) = 1$

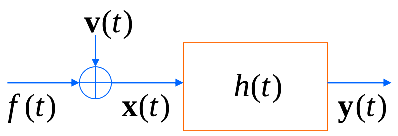
Maximizing the SNR for input $\mathbf{x}_1(t) \Rightarrow$ maximizing the SNR for input $\mathbf{x}(t)$

Optimum $H_1(\omega) = kF_1^*(\omega)e^{-j\omega t_0}$ where $F_1^*(\omega) = F^*(\omega)\Gamma_v^*(j\omega)$

Cascading with $\Gamma_v(s)$, optimum filter is

$$H(\omega) = H_1(\omega)\Gamma_v(j\omega) = kF^*(\omega)\Gamma_v^*(j\omega)\Gamma_v(j\omega)e^{-j\omega t_0}$$
$$= k\frac{F^*(\omega)}{S_{vv}(\omega)}e^{-j\omega t_0}$$

Smoothing (Estimating Signal in Noise)



$f(t)$ = unknown signal
$\mathbf{y}(t)$ = estimation of $f(t)$
(all real processes)

$\mathbf{x}(t)$ itself is an estimate of $f(t)$. Since $\mathbf{v}(t)$ is zero mean, $\mathbf{x}(t)$ is an unbiased estimate. However, the variance is large (equals $E\{\mathbf{v}^2(t)\}$).

Since $\mathbf{v}(t)$ is white, SNR may be improved by (weighted) averaging:

$$\mathbf{y}(t) = \int_{-T}^{T} \mathbf{x}(t+\tau)h(\tau)d\tau = \mathbf{x}(t) * h(t)$$

for some weighting function (window)

$h(t) > 0, \quad -T \le t \le T$

$h(t) = 0, \quad$ otherwise

$h(-t) = h(t), \quad$ symmetric

Now, estimator is biased with bias

$$b = E\{\mathbf{y}(t) - f(t)\} = E\{y_f(t) + \mathbf{y}_v(t) - f(t)\}$$

$$= y_f(t) - f(t) + E\{\mathbf{y}_v(t)\} = f(t) * h(t) - f(t)$$

And estimator variance is $\sigma^2 = E\{\mathbf{y}^2(t)\} - E^2\{\mathbf{y}(t)\} = E\{\mathbf{y}_v^2(t)\}$

since zero mean noise

$$= \int_{-\infty}^{\infty} q(t-\tau) h^2(\tau) d\tau$$

from slide 34

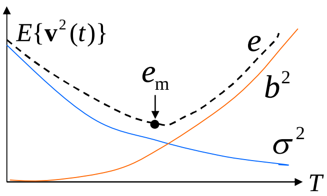
Now, mean square estimation error

$$e = E\left\{\left[\mathbf{y}(t) - f(t)\right]^2\right\} = E\left\{\left[y_f(t) - f(t) + \mathbf{y}_v(t)\right]^2\right\}$$

$$= \left[y_f(t) - f(t)\right]^2 + E\{\mathbf{y}_v^2(t)\} = b^2 + \sigma^2$$

For large T, σ^2 is small but b is large.

Typical behavior



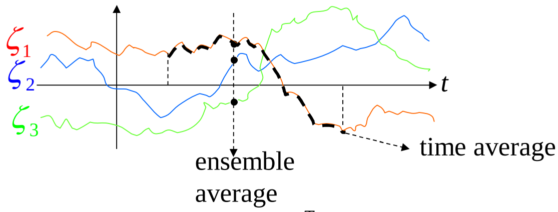
If $f(t)$ is known to be bandlimited, then $H(\omega) = 0$ outside the band does not introduce a bias but reduces σ^2. Thus, only in-band noise matters and out-of-band noise may be rejected by a filter.

Papoulis shows that for slow-varying $q(t)$ and quadratic function $f(t)$, e_m is achieved when $\sigma = 2b$, and for a parabolic window $h(t)$ which depends on $f(t)$.

Ergodicity

$\mathbf{x}(t)$ is a real stationary process, mean $\eta = E\{\mathbf{x}(t)\}$ ensemble average

Time average: $\boldsymbol{\eta}_T = \frac{1}{2T}\int_{-T}^{T} \mathbf{x}(t)dt$



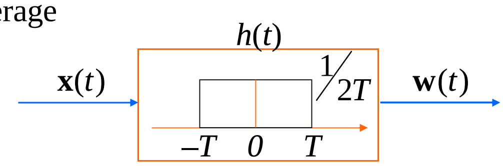
$\boldsymbol{\eta}_T$ is a random variable, $E\{\boldsymbol{\eta}_T\} = \frac{1}{2T}\int_{-T}^{T} E\{\mathbf{x}(t)\}dt = \eta$

Therefore, $\boldsymbol{\eta}_T$ is an estimate of η.

$\mathbf{x}(t)$ is a mean-ergodic process if $\boldsymbol{\eta}_T \to \eta$ as $T \to \infty$.

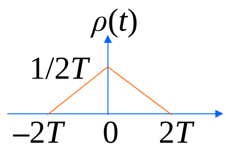
This is true if $\sigma_T \to 0$ as $T \to \infty$, σ_T^2 = variance of $\boldsymbol{\eta}_T$

Thus, mean-ergodicity $\Rightarrow$ ensemble average may be replaced by time average



$$\mathbf{w}(t) = \frac{1}{2T}\int_{t-T}^{t+T} \mathbf{x}(\alpha)d\alpha = L[\mathbf{x}(t)] = \mathbf{x}(t) * h(t), \quad \boldsymbol{\eta}_T = \mathbf{w}(0)$$

$$h(t) * h^*(-t) = \rho(t) = \begin{cases} \frac{1}{2T}\left(1 - \frac{|t|}{2T}\right) & |t| \le 2T \\ 0 & else \end{cases}$$



$$C_{ww}(\tau)=L\big[L[C_{xx}(\tau)]\big]=C_{xx}(\tau)*\rho(\tau)$$

where $C_{xx}(\tau)$ is autocovariance of $\mathbf{x}(t)$

$$=\frac{1}{2T}\int_{-2T}^{2T}C_{xx}(\tau-\alpha)\left[1-\frac{|\alpha|}{2T}\right]d\alpha$$

$$\sigma_T^2=C_{ww}(0)=\frac{1}{2T}\int_{-2T}^{2T}C_{xx}(-\alpha)\left[1-\frac{|\alpha|}{2T}\right]d\alpha=\frac{2}{2T}\int_{0}^{2T}C_{xx}(\alpha)\left[1-\frac{\alpha}{2T}\right]d\alpha$$

Thus, $\mathbf{x}(t)$ is mean-ergodic iff $\frac{1}{T}\int_{0}^{2T}C_{xx}(\alpha)\left[1-\frac{\alpha}{2T}\right]d\alpha\to 0$ as $T\to\infty$

Slutsky's Theorem: mean-ergodic $\Leftrightarrow \frac{1}{T}\int_{0}^{T}C_{xx}(\tau)d\tau\to 0$ as $T\to\infty$

Proof: ($\Rightarrow$) mean-ergodic means $\sigma_T\to 0$ as $T\to\infty$

Consider random variables $\boldsymbol{\eta}_T$ and $\mathbf{x}(0)$. They both have mean η.

So $E\{(\boldsymbol{\eta}_T-\eta)(\mathbf{x}(0)-\eta)\}=E\left\{\frac{1}{2T}\int_{-T}^{T}(\mathbf{x}(t)-\eta)(\mathbf{x}(0)-\eta)\,dt\right\}$

$$=\frac{1}{2T}\int_{-T}^{T}E\{(\mathbf{x}(t)-\eta)(\mathbf{x}(0)-\eta)\}dt=\frac{1}{2T}\int_{-T}^{T}C_{xx}(t)dt$$

but $E^2\{(\boldsymbol{\eta}_T-\eta)(\mathbf{x}(0)-\eta)\}\le E\{(\boldsymbol{\eta}_T-\eta)^2\}E\{(\mathbf{x}(0)-\eta)^2\}=\sigma_T^2C_{xx}(0)$

Therefore $\frac{1}{T}\int_{0}^{T}C_{xx}(\tau)d\tau\le\sigma_T\sqrt{C_{xx}(0)}\to 0$ as $T\to\infty$

$$(\Leftarrow)\quad \sigma_T^2 = \frac{1}{T}\int_0^{2T} C_{xx}(\tau)\left[1-\frac{\tau}{2T}\right]d\tau$$

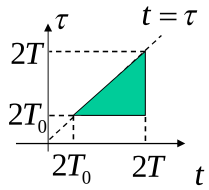
$$= \frac{1}{T}\int_0^{2T_0} C_{xx}(\tau)\left[1-\frac{\tau}{2T}\right]d\tau + \frac{1}{T}\int_{2T_0}^{2T} C_{xx}(\tau)\left[1-\frac{\tau}{2T}\right]d\tau = I_1 + I_2$$

$$I_1 \le \frac{1}{T}\int_0^{2T_0} C_{xx}(0)\left[1-\frac{\tau}{2T}\right]d\tau \quad \text{since } |C_{xx}(\tau)| \le C_{xx}(0)$$

$$\le \frac{1}{T}\int_0^{2T_0} C_{xx}(0)\times 1 \times d\tau = \frac{2T_0}{T}C_{xx}(0) \to 0 \text{ as } T \to \infty$$

$$I_2 = \frac{1}{2T^2}\int_{2T_0}^{2T} C_{xx}(\tau)(2T-\tau)d\tau = \frac{1}{2T^2}\int_{\tau=2T_0}^{2T} C_{xx}(\tau)\left[\int_{t=\tau}^{2T} dt\right]d\tau$$

$$= \frac{1}{2T^2}\int_{t=2T_0}^{2T}\left[\int_{\tau=2T_0}^{t} C_{xx}(\tau)d\tau\right]dt = \frac{1}{2T^2}\int_{2T_0}^{2T} I_3 dt$$



$$\text{now, given } \frac{1}{T}\int_0^{T} C_{xx}(\tau)d\tau \to 0 \text{ as } T \to \infty$$

$$\frac{1}{c_1}\int_0^{c_1} C_{xx}(\tau)d\tau \le \frac{1}{c_0}\int_0^{c_0} C_{xx}(\tau)d\tau \quad \text{for } c_1 > c_0$$

$$\text{or } \frac{1}{c_1}\int_{c_0}^{c_1} C_{xx}(\tau)d\tau \le \frac{1}{c_0}\int_0^{c_0} C_{xx}(\tau)d\tau$$

For some given $\varepsilon > 0$, we can always find a c_0 such that

$$\frac{1}{c_0}\int_0^{c_0} C_{xx}(\tau)d\tau < \varepsilon$$

then $\frac{1}{c_1}\int_{c_0}^{c_1} C_{xx}(\tau)d\tau < \varepsilon,$ or $\frac{1}{t}\int_c^t C_{xx}(\tau)d\tau < \varepsilon, \forall c > c_0, \forall t > c$

So, choosing $2T_0 = c_0, I_3 = \int_{2T_0}^{t} C_{xx}(\tau)d\tau < \varepsilon t$

then $I_2 < \frac{1}{2T^2}\int_{2T_0}^{2T} \varepsilon t dt = \frac{\varepsilon}{4T^2}(4T^2 - 4T_0^2) = \varepsilon\left(1 - \frac{T_0^2}{T^2}\right) \to \varepsilon$ as $T \to \infty$

Thus, $\sigma_T^2 = I_1 + I_2 < \varepsilon$ as $T \to \infty$.

Since ε is arbitrary, $\sigma_T \to 0$ as $T \to \infty$. ■

now $\sigma_T^2 = C_{ww}(0) = \int_{-\infty}^{\infty} S_{ww}^c(\omega)\frac{d\omega}{2\pi}$

where $S_{ww}^c(\omega) =$ covariance spectrum of $\mathbf{w}(t) = S_{xx}^c(\omega)\left[\frac{\sin(T\omega)}{T\omega}\right]^2$

so, $\sigma_T^2 = \int_{-\infty}^{\infty} S_{xx}^c(\omega)\frac{\sin^2(T\omega)}{T^2\omega^2}\frac{d\omega}{2\pi}$ As T large, $\frac{\sin^2(T\omega)}{T^2\omega^2} \approx 0$ for $\omega \neq 0$

So, $\sigma_T^2 \approx S_{xx}^c(0)\frac{1}{2T} \to 0$ as $T \to \infty$ if $S_{xx}^c(0)$ is finite, i.e., $S_{xx}^c(\omega)$ does not have an impulse at $\omega = 0$ [$S_{xx}^c(\omega)$ is continuous at origin].

Discrete-time Process:

$$\boldsymbol{\eta}_M = \frac{1}{2M+1}\sum_{n=-M}^{M}\mathbf{x}[n], \quad \sigma_M^2 = \frac{1}{2M+1}\sum_{m=-2M}^{2M} C_{xx}[m]\left(1 - \frac{|m|}{2M+1}\right)$$

$\mathbf{x}[n]$ is mean-ergodic iff $\frac{1}{M}\sum_{m=0}^{M} C_{xx}[m] \to 0$ as $M \to \infty$

Covariance-Ergodic Process:

Assume $\mathbf{x}(t)$ zero-mean, then time average estimate of $C_{xx}(\lambda)$ is

$$\mathbf{C}_T(\lambda)=\frac{1}{2T}\int_{-T}^{T}\mathbf{z}(t)dt \quad \text{where} \quad \mathbf{z}(t)=\mathbf{x}(t+\lambda)\mathbf{x}(t)$$

$\mathbf{x}(t)$ is covariance-ergodic iff $\frac{1}{T}\int_0^T C_{zz}(\tau)d\tau \to 0$ as $T\to\infty$

where $C_{zz}(\tau)=E\{\mathbf{x}(t+\lambda+\tau)\mathbf{x}(t+\tau)\mathbf{x}(t+\lambda)\mathbf{x}(t)\}-C_{xx}^2(\lambda)$

If we wish to estimate $C_{xx}(0)$, then

$$\mathbf{z}(t)=\mathbf{x}^2(t) \quad \text{and} \quad C_{zz}(\tau)=E\{\mathbf{x}^2(t+\tau)\mathbf{x}^2(t)\}-C_{xx}^2(0)$$

If $\mathbf{x}(t)$ is a normal process, then $C_{zz}(\tau)=2C_{xx}^2(\tau)$, so the condition becomes

$$\frac{1}{T}\int_0^T C_{xx}^2(\tau)d\tau \to 0 \quad \text{as} \quad T\to\infty$$

Distribution-Ergodic Process:

Let $\mathbf{y}(t)=U\left[x-\mathbf{x}(t)\right]=\begin{cases}1, & \mathbf{x}(t)\le x\\ 0, & \mathbf{x}(t)>x\end{cases}$

then $E\{\mathbf{y}(t)\}=P\{\mathbf{x}(t)\le x\}=F_1(x)$

Thus, $F_1(x)$ estimated by the time average of $\mathbf{y}(t)$:

$$\mathbf{F}_T(x)=\frac{1}{2T}\int_{-T}^{T}\mathbf{y}(t)dt$$

and $\mathbf{x}(t)$ is distribution-ergodic iff $\frac{1}{T}\int_0^T C_{yy}(\tau)d\tau \to 0$ as $T\to\infty$

$$C_{yy}(\tau)=E\{\mathbf{y}(t+\tau)\mathbf{y}(t)\}-E^2\{\mathbf{y}(t)\}=P\{\mathbf{x}(t+\tau)\le x,\mathbf{x}(t)\le x\}-F_1^2(x)$$
$$=F_2(x,x;\tau)-F_1^2(x)$$

where $F_2(x,x;\tau)$ is the second order distribution of $\mathbf{x}(t)$.

Measurement of Power Spectrum (Spectral Estimation)

In real life, a real process $\mathbf{x}(t)$ is available only from $-T$ to T

$$\mathbf{x}_T(t) = \begin{cases} \mathbf{x}(t) & |t| < T \\ 0 & |t| > T \end{cases}$$

$S_{xx}(\omega)$ can not be estimated directly since it is not an expectation.

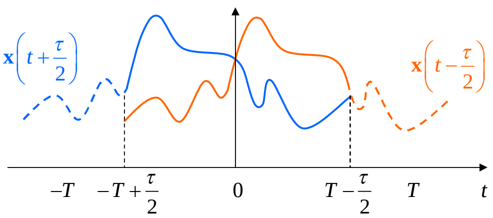
Autocorrelation estimate of power spectrum:

Determine $S_{xx}(\omega)$ from the estimate of autocorrelation

$$R_{xx}(\tau) = E\left\{\mathbf{x}\left(t+\frac{\tau}{2}\right)\mathbf{x}\left(t-\frac{\tau}{2}\right)\right\}$$

Assume covariance-ergodic

$$\Rightarrow R_{xx}(\tau) = \frac{1}{2T}\int_{-T}^{T}\mathbf{x}\left(t+\frac{\tau}{2}\right)\mathbf{x}\left(t-\frac{\tau}{2}\right)dt$$



Integrand available only in the interval $-T+\frac{\tau}{2} < t < T-\frac{\tau}{2}, \quad \tau > 0$

Option 1: $$\mathbf{R}^T(\tau) = \frac{1}{2T-\tau}\int_{-T+\tau/2}^{T-\tau/2}\mathbf{x}\left(t+\frac{\tau}{2}\right)\mathbf{x}\left(t-\frac{\tau}{2}\right)dt$$

Estimate is unbiased, but has large variance.

Option 2: $$\mathbf{R}_T(\tau) = \frac{1}{2T} \int_{-T+\tau/2}^{T-\tau/2} \mathbf{x}\left(t + \frac{\tau}{2}\right) \mathbf{x}\left(t - \frac{\tau}{2}\right) dt$$

smaller integration interval at large $|\tau|$

$\Rightarrow$ estimate is worse at large $|\tau|$

$\Rightarrow$ $R_{xx}(\tau)$ for large $|\tau|$ scaled down

Estimate is biased, but has smaller variance.

<u>Periodogram estimate:</u>

Take Fourier transform of the available signal,

$$\mathbf{X}_T(\omega) = \int_{-T}^{T} \mathbf{x}(t) e^{-j\omega t} dt$$

Then power spectrum is $\quad \mathbf{S}_T(\omega) = \dfrac{1}{2T} |\mathbf{X}_T(\omega)|^2$

In fact, this is the same as option 2 of autocorrelation estimate:

$$\begin{aligned}
\mathbf{R}_T(\tau) &= \frac{1}{2T} \int_{-T+\tau/2}^{T-\tau/2} \mathbf{x}\left(t + \frac{\tau}{2}\right) \mathbf{x}\left(t - \frac{\tau}{2}\right) dt \\
&= \frac{1}{2T} \int_{-\infty}^{\infty} \mathbf{x}_T\left(t + \frac{\tau}{2}\right) \mathbf{x}_T\left(t - \frac{\tau}{2}\right) dt \\
&= \frac{1}{2T} \int_{-\infty}^{\infty} \mathbf{x}_T(t') \mathbf{x}_T(t' - \tau) dt' \\
&= \frac{1}{2T} \mathbf{x}_T(\tau) * \mathbf{x}_T(-\tau)
\end{aligned}$$

Taking Fourier transform, $\quad \mathbf{S}_T(\omega) = \dfrac{1}{2T} \mathbf{X}_T(\omega) \mathbf{X}_T^*(\omega)$

$S_{xx}(\omega)$ is nearly constant in an interval of the order $1/T$ (large T)

$\Rightarrow$ asymptotically unbiased, $E\{\mathbf{S}_T(\omega)\} = S_{xx}(\omega)$

$S_{xx}(\omega)$ is not constant in an interval of the order $1/T$ (small T)

$\Rightarrow$ biased estimate

To reduce the bias, data window is used:

$$\mathbf{S}_c(\omega) = \frac{1}{2T}\left|\int_{-T}^{T} c(t)\mathbf{x}(t)e^{-j\omega t}dt\right|^2$$

$c(t)$ = data window, with Fourier transform $C(\omega)$

$$\Rightarrow \text{ bias } E\{\mathbf{S}_c(\omega)\} = \frac{1}{4\pi T} S_{xx}(\omega) * C^2(\omega)$$

Recall that, smaller integration interval at large $|\tau|$
$\Rightarrow$ estimate is worse at large $|\tau|$
$\Rightarrow$ variance of the estimator large as $\tau \to \infty$
$\Rightarrow \mathbf{S}_T(\omega)$ becomes noisy

Data window can not reduce the estimator variance

To reduce the variance, smoothed spectrum used:

$$\mathbf{S}_w(\omega) = \int_{-2T}^{2T} w(\tau)\mathbf{R}_T(\tau)e^{-j\omega\tau}d\tau$$

$w(t)$ = lag window $\longrightarrow$ $W(\omega)$ = spectral window

$$\Rightarrow \text{ bias } E\{\mathbf{S}_w(\omega)\} = \frac{1}{2\pi} S_{xx}(\omega) * W(\omega)$$

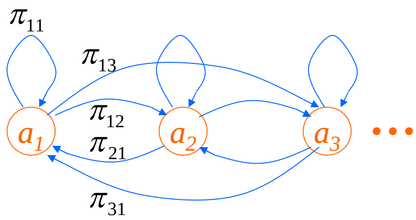
Variance is reduced when duration of $w(t)$ is small,
but bias is reduced when duration of $W(\omega)$ is small.

Markoff Chain and Markoff Process

1) Discrete-time Markoff Chains:

(discrete states a_1 , a_2 etc.)

Markoff process $\mathbf{x}_n$ with states a_i



state probabilities $p_i[n] = P\{\mathbf{x}_n = a_i\} \quad i = 1, 2, \ldots$

transition probabilities $\pi_{ij}[n_1, n_2] = P\{\mathbf{x}_{n_2} = a_j \mid \mathbf{x}_{n_1} = a_i\}$

All outgoing transitions $\sum_j \pi_{ij}[n_1, n_2] = 1$

All incoming transitions $\sum_i p_i[k]\pi_{ij}[k, n] = p_j[n]$

Chapman Kolmogoroff equation for any $n_1 < n_2 < n_3$

$$\pi_{ij}[n_1, n_3] = \sum_r \pi_{ir}[n_1, n_2]\pi_{rj}[n_2, n_3]$$

If transition probabilities are invariant to a shift,

$$\pi_{ij}[n_1, n_2] = \pi_{ij}[m] \quad \text{where} \quad m = n_2 - n_1$$

then the process is called homogeneous,
and the CK equation becomes $\pi_{ij}[n+k] = \sum_r \pi_{ir}[k]\pi_{rj}[n]$
(where $k = n_2 - n_1, n = n_3 - n_2$)

For a finite state Markoff chain transition matrix

$$\Pi[n]=\begin{bmatrix}\pi_{11}[n] & \cdots & \pi_{1N}[n]\\ \vdots & \ddots & \vdots \\ \pi_{N1}[n] & \cdots & \pi_{NN}[n]\end{bmatrix}$$

from CK equation, $\Pi[n+k]=\Pi[n]\Pi[k] \quad \Rightarrow \Pi[n]=\Pi^n[1]$

State probability vector $P[n]=\left[\, p_1[n] \quad \cdots \quad p_N[n] \right]$

$$=P[0]\Pi^n[1]$$

Stationary Markoff chain (invariant distribution), if

$$P[2]=P[1]=P \qquad \Rightarrow P[n]=P \quad \forall n$$

$$\Rightarrow P\Pi[1]=P$$

or P is an eigenvector of the transition matrix

Asymptotically stationary Markoff chain if $\Pi^n[1]$ tend to a limit as $n\to\infty$.

Otherwise, $P[n]$ depends on n, and the process is not stationary.

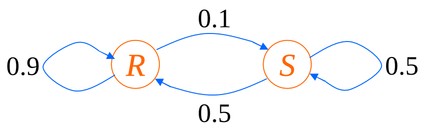
Example 8: Two relationship status: in a *R*elationship, *S*ingle

Probability that R breaks up next day is 0.1
Probability that S finds a boy/girlfriend next day is 0.5

Transition probabilities invariant with time $\Rightarrow$ homogeneous

Given $\pi_{12}[1]$=0.1, $\pi_{21}[1]$=0.5, we can find the transition matrix

$$\Pi[1] = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$



Find the probability that R remains R after 2 days:
From CK equation, transition matrix for 2 days is

$$\Pi[2] = \Pi^2[1] = \begin{bmatrix} 0.86 & 0.14 \\ 0.7 & 0.3 \end{bmatrix}$$

Given state probability vector $P[0] = \begin{bmatrix} 1 & 0 \end{bmatrix}$, the probability after 2 days is $P[2] = P[0]\Pi[2] = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix}$

$\Rightarrow$ R remains R after 2 days with probability 0.86

Find the probability that a student is in a relationship:
The steady state probability of this stationary Markoff chain is

$$P\Pi[1] = P$$

$$P\left(\Pi[1] - I\right) = 0$$

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} -0.1 & 0.1 \\ 0.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

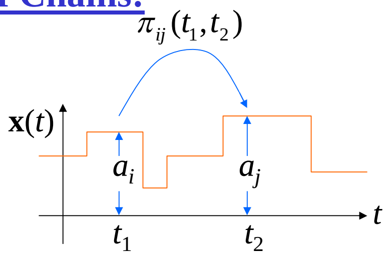
$$-0.1p_1 + 0.5(1 - p_1) = 0$$

$$p_1 = 0.83, p_2 = 0.17$$

$\Rightarrow$ A student is in a relationship with probability 0.83

2) Continuous-time Markoff Chains:

(discrete states a_1 , a_2 etc.)



$\mathbf{x}(t)$ = staircase function with discontinuities at random points

State probabilities $\quad p_i(t) = P\{\mathbf{x}(t) = a_i\}$

Transition probabilities $\quad \pi_{ij}(t_1, t_2) = P\{\mathbf{x}(t_2) = a_j \mid \mathbf{x}(t_1) = a_i\}$

All outgoing transitions $\quad \sum_j \pi_{ij}(t_1, t_2) = 1$

All incoming transitions $\quad \sum_i p_i(t_1)\pi_{ij}(t_1, t_2) = p_j(t_2)$

Chapman Kolmogoroff equation $\qquad$ for any $t_1 < t_2 < t_3$

$$\pi_{ij}(t_1, t_3) = \sum_r \pi_{ir}(t_1, t_2)\pi_{rj}(t_2, t_3)$$

Homogeneous process $\quad \pi_{ij}(t_1, t_2) = \pi_{ij}(\tau) \quad$ where $\quad \tau = t_2 - t_1$

CK equation becomes $\quad \pi_{ij}(\tau + \alpha) = \sum_r \pi_{ir}(\tau)\pi_{rj}(\alpha)$
(where $\alpha = t_3 - t_2$)

In vector form $\qquad \Pi(\tau + \alpha) = \Pi(\tau)\Pi(\alpha) \qquad \tau, \alpha \geq 0$

transition probability rates $\lambda_{ij} = \pi'_{ij}(0^+)$

Differentiating $\sum_j \pi_{ij}(\tau) = 1$ we obtain $\sum_j \lambda_{ij} = 0$

Also $\pi_{ij}(0) = \delta[i-j] = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \Rightarrow \lambda_{ij} \begin{cases} < 0 & i = j \\ > 0 & i \neq j \end{cases}$

define $\Pi'(0^+) = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{bmatrix}$

Differentiate CK equation with respect to α,

$$\Pi'(\tau + \alpha) = \Pi(\tau)\Pi'(\alpha)$$

Set α=0, $\Pi'(\tau) = \Pi(\tau)\Pi'(0)$

With initial condition $\Pi(0) = \mathbf{I}$,
the solution to these differential equations is $\Pi(\tau) = e^{\Pi'(0)\tau}$

Similarly, state probability vector $P(t) = \begin{bmatrix} p_1(t) & \cdots & p_N(t) \end{bmatrix}$

It follows from $\sum_i p_i(t_1)\pi_{ij}(t_1, t_2) = p_j(t_2)$ that

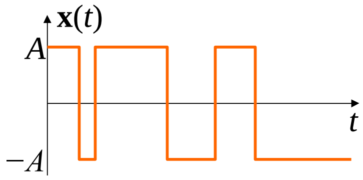
$$P(t+\tau) = P(t)\Pi(\tau)$$

Differentiate with respect to τ, $P'(t+\tau) = P(t)\Pi'(\tau)$

Set τ=0, $P'(t) = P(t)\Pi'(0)$

Its solution is $P(t) = P(0)e^{\Pi'(0)t}$

Example 9: Telegraph signal



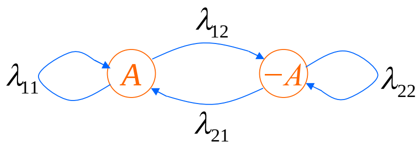
Two states $a_1 = A, \quad a_2 = -A$

Given

$$\pi_{11}(\Delta t) = P\{\mathbf{x}(t+\Delta t) = A \mid \mathbf{x}(t) = A\} = 1 - \mu_1 \Delta t$$

$$\pi_{22}(\Delta t) = P\{\mathbf{x}(t+\Delta t) = -A \mid \mathbf{x}(t) = -A\} = 1 - \mu_2 \Delta t$$

we can find $\pi_{12}(\Delta t), \pi_{21}(\Delta t)$ and hence $\lambda_{ij} = \pi'_{ij}$:



$$\Pi'(0) = \begin{bmatrix} -\mu_1 & \mu_1 \\ \mu_2 & -\mu_2 \end{bmatrix}$$

$$P'(t) = P(t)\Pi'(0) \quad \Rightarrow [\,p'_1(t) \quad p'_2(t)] = [\,p_1(t) \quad p_2(t)] \begin{bmatrix} -\mu_1 & \mu_1 \\ \mu_2 & -\mu_2 \end{bmatrix}$$

$$\begin{aligned} p'_1(t) &= -\mu_1 p_1(t) + \mu_2 p_2(t) \\ &= -(\mu_1+\mu_2) p_1(t) + \mu_2 \qquad \text{since } \; p_2(t) = 1 - p_1(t) \end{aligned}$$

Solution: $\quad p_1(t) = ce^{-(\mu_1+\mu_2)t} - \dfrac{\mu_2}{-(\mu_1+\mu_2)}$ for some c

Initial condition $\; p_1(t)|_{t=0} = p_1(0)$

$$\Rightarrow p_1(t) = \left[p_1(0) - \frac{\mu_2}{\mu_1+\mu_2} \right] e^{-(\mu_1+\mu_2)t} + \frac{\mu_2}{\mu_1+\mu_2}$$

This process is asymptotically stationary since

$$p_1(t) \underset{t\to\infty}{\rightarrow} \frac{\mu_2}{\mu_1+\mu_2} = p_1, \quad p_2(t) \underset{t\to\infty}{\rightarrow} \frac{\mu_1}{\mu_1+\mu_2} = p_2$$

Transition probabilities $\Pi'(\tau) = \Pi(\tau)\Pi'(0)$

$$= \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} -\mu_1 & \mu_1 \\ \mu_2 & -\mu_2 \end{bmatrix}$$

$\Rightarrow$ for example, $\pi'_{11}(\tau) = -\mu_1\pi_{11}(\tau) + \mu_2\pi_{12}(\tau)$

$= -(\mu_1 + \mu_2)\pi_{11}(\tau) + \mu_2$ since $\pi_{12}(\tau) = 1 - \pi_{11}(\tau)$

Solution: $\pi_{11}(\tau) = ce^{-(\mu_1+\mu_2)\tau} + \dfrac{\mu_2}{\mu_1 + \mu_2}$ for some c

Initial condition $\pi_{11}(0) = 1$

$$\Rightarrow \pi_{11}(\tau) = p_1 + p_2 e^{-(\mu_1+\mu_2)\tau}$$

$$\pi_{22}(\tau) = p_2 + p_1 e^{-(\mu_1+\mu_2)\tau} \quad \text{etc.}$$

Mean: $E\{\mathbf{x}(t)\} = p_1 \cdot A + p_2 \cdot (-A) = \dfrac{(\mu_2 - \mu_1)A}{\mu_1 + \mu_2}$

Autocorrelation: $P\{\mathbf{x}(t+\tau) = a_j, \mathbf{x}(t) = a_i\} = p_i \cdot \pi_{ij}(\tau)$

$$E\{\mathbf{x}(t+\tau)\mathbf{x}(t)\}$$

$$= A^2\left[p_1 \cdot \pi_{11}(\tau) + p_2 \cdot \pi_{22}(\tau)\right] - A^2\left[p_1 \cdot \pi_{12}(\tau) + p_2 \cdot \pi_{21}(\tau)\right]$$

$$= A^2\left[(p_1 - p_2)^2 + 4p_1p_2e^{-(\mu_1+\mu_2)\tau}\right]$$

$$= \eta^2 + 4A^2 p_1 p_2 e^{-(\mu_1+\mu_2)\tau}$$