Week 3

Monday, 28 August 2023 8:40 am

$\mathbb{P} \rightsquigarrow \mathbb{P}_X(A) = \mathbb{P}(X \in A)$ "pushforward" measure

$X : \Omega \to \mathbb{R}$ random variable legacy

function

$X(\omega) = x$

$\uparrow$ sample

discrete : $X : \Omega \to \{x_1, x_2, \ldots\}$

pmf $p_k(x_k) = \mathbb{P}(X = x_k)$, $k = 1, 2, \ldots$

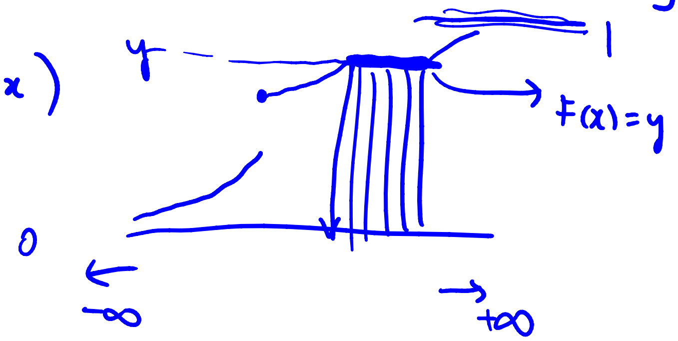
continuous : pdf $f_X(x)$

$\mathbb{P}(X \in A) = \int_A f_X(x)\,dx$

cumulative distribution function

cdf

$F(x) = \mathbb{P}(X \leq x)$



"distributed as"

$\downarrow$

$$\boxed{\begin{aligned} &X \sim F(x) \\ &F^{-1}(X) \sim \mathrm{Unif}[0,1] \end{aligned}}$$

$F^{-1}(y) = \inf\{x : F(x) \geq y\}$

sol^{n} to $g(x) = y$

pmf/pdf $X \sim f_X$ $\quad Y = g(X)$ $\quad f_Y(y) = \sum_k \frac{f_X(x_k)}{|g'(x_k)|}$

Discrete X, Y

Joint pmf $p_{X,Y}(x, y) := \mathbb{P}(\{X = x\} \cap \{Y = y\})$

$$\sum_x \sum_y p(x, y) = 1$$

$$\sum_y p(x, y) = \sum_y p(x) p(y \mid x)$$

$\mathbb{P}(Y = y \mid X = x)$

$$= p(x) \sum_y p(y \mid x)$$

$$= p(x) \underbrace{\sum_y p(y|x)}$$

$$= p(x) \quad \text{marginal pmf of } X$$

$$\sum_x p(x,y) = p(y) \quad \text{marginal pmf of } Y$$

$$p(x,y) = \mathbb{P}(\overset{A}{\{X=x\}} \cap \overset{B}{\{Y=y\}}) \overset{\text{“}\perp\!\!\!\perp\text{”}}{=} \mathbb{P}(A)\cdot\mathbb{P}(B)$$

$$= \mathbb{P}(X=x)\,\mathbb{P}(Y=y)$$

$$= p(x)\,p(y)$$

$X \perp\!\!\!\perp Y$

$$p(x)\,p(y|x)$$

$$p(y|x) = p(y)$$

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{p(x)\,p(y|x)}{\sum_{x'} p(x',y)}$$

$$= \frac{p(x)\,p(y|x)}{\sum_{x'} p(x')\,p(y|x')}$$

BSC

$$Z \sim \text{Bern}(\varepsilon) \perp\!\!\!\perp X.$$

$$X \longrightarrow \oplus \longrightarrow Y = X \oplus Z = (X+Z) \bmod 2$$

$X \sim \text{Bern}(p)$

$$p(x) = \begin{cases} p & \text{if } x=1 \\ 1-p & \text{if } x=0 \end{cases}$$

$$P_{X|Y}(x|y) = \frac{\boxed{p(y|x)}\,p(x)}{\boxed{p(y)}}$$

$$p(y|x) = \mathbb{P}(X \oplus Z = y \mid X = x)$$

$$= \mathbb{P}(x \oplus Z = y \mid X = x)$$

$$= \mathbb{P}(x \oplus Z = y \mid X = x)$$

$$= \mathbb{P}(x \oplus Z = y)$$

$$= \mathbb{P}(Z = x \oplus y)$$

$$= \begin{cases} \varepsilon & \text{if } x \oplus y = 1 \\ 1-\varepsilon & \text{if } x \oplus y = 0 \end{cases}$$

$0 \oplus 0 = 0$

$1 \oplus 0 = 1$

$0 \oplus 1 = 1$

$1 \oplus 1 = 0$

$$P_{X|Y}(0|1) = \frac{P_{Y|X}(1|0)\, P_X(0)}{P_Y(1)} = \frac{P_Z(1)\, P_X(0)}{P_Y(1)}$$

$$= \frac{\varepsilon(1-p)}{\underbrace{P_{Y|X}(1|0)}_{=P_Z(1)=\varepsilon}\underbrace{P_X(0)}_{=1-p} + \underbrace{P_{Y|X}(1|1)}_{=P_Z(0)=1-\varepsilon}\underbrace{P_X(1)}_{=p}}$$

$$= \frac{\varepsilon(1-p)}{\varepsilon(1-p) + (1-\varepsilon)p} = \cdots$$

$$P_Y(1) = \varepsilon(1-p) + (1-\varepsilon)p$$

$\varepsilon = \frac{1}{2}$ $\quad P_Y(1) = \frac{1}{2}(1-p) + \frac{1}{2}p = \frac{1}{2}$

$$P_Y(0) = 1 - P_Y(1) = \cdots$$

$$\mathbb{P}(X \neq Y) = P_{X,Y}(0,1) + P_{X,Y}(1,0)$$

$$= P_{Y|X}(1|0)P_X(0) + P_{Y|X}(0|1)P_X(1)$$

$$= P_Z(1)\, P_X(0) + P_Z(1)\, P_X(1)$$

$$= \cdots$$

$$P_Z(1)\, P_X(0) + \underline{P_Z(1)}\, P_X(1)$$

$$= P_Z(1)\,\big(\underbrace{P_X(0) + P_X(1)}\big)$$

$$= P_Z(1) \qquad Z \sim \mathrm{Bern}(\varepsilon) \qquad \mathbb{P}(Z=1) = \varepsilon$$

$$= \varepsilon \qquad 0 \leq \varepsilon \leq 1$$

$$\mathbb{P}(X = Y) = \underline{1-\varepsilon}$$

$\varepsilon = 1$: $X = 1 \to Y = 0 \to X = 1$; $X = 0 \to Y = 1 \to X = 0$

$$Y \in \{0, 1\} \qquad \mathbb{P}(\text{error}) = 0$$

Worst case error $= \boxed{\frac{1}{2}}$ when $Z \sim \mathrm{Bern}(\frac{1}{2})$

$$P_{Y|X}(y|x) = P_Z(x \oplus y) = \frac{1}{2} = P_Y(y)$$

$$\Rightarrow X \perp\!\!\!\perp Y.$$

$$P_{X|Y}(x|y) \underset{\text{lazy}}{\overset{\text{shorthand}}{=}} p(x|y)$$

$$\underset{\Phi}{P_Y(y)} = \underset{\Phi}{P_{X,Y}(0, y)} + \underset{\Phi}{P_{X,Y}(1, y)}$$

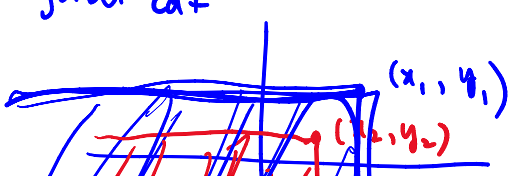
$$= P_{Y|X}(y|0)\, P_X(0) + P_{Y|X}(y|1)\, P_X(1)$$

$$P_Y(y) = \sum_x P_{Y|X}(y|x)\, P_X(x)$$

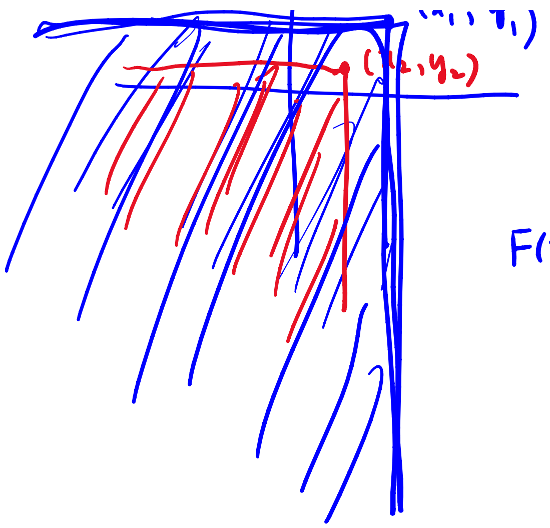
$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y) \quad \geq 0 \quad \leq 1$$

joint cdf

$$F(x_1, y_1) \geq F(x_2, y_2)$$



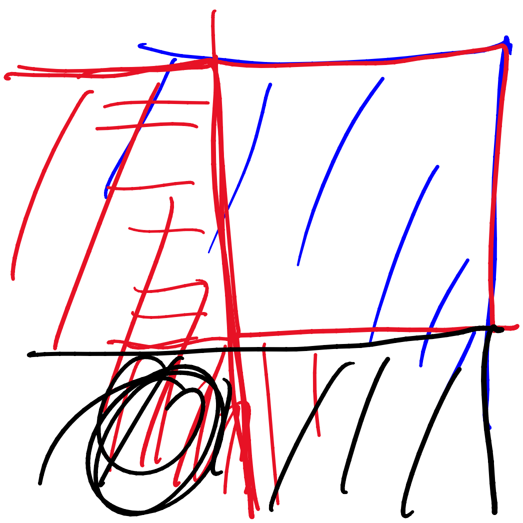
$$x_2 \leq x_1$$
$$y_2 \leq y_1$$



$x_2 \le x_1$

$y_2 \le y_1$

$$F(x, \infty) = \mathbb{P}(X \le x, Y \le \infty) = \mathbb{P}(X \le x) = F_X(x)$$



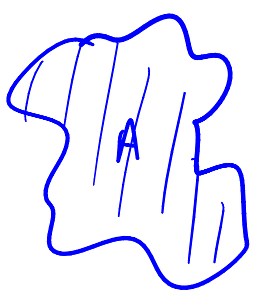
$$X \perp\!\!\!\perp Y$$

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y)$$

Continuous case: $\exists$ joint pdf $f_{X,Y}(x,y)$

s.t.

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\, du\, dv$$



$$\iint_A f_{X,Y}(x,y)\, dx\, dy = \mathbb{P}((X,Y) \in A)$$

marginal pdf

$$\boxed{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = f_X(x)}$$

cdf

$$F_X(x) = \mathbb{P}(X \le x, Y < \infty)$$

$$= \left(\int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv \right] du \right)$$

$\frac{\partial}{\partial x}$

Fundamental Thm of Calculus

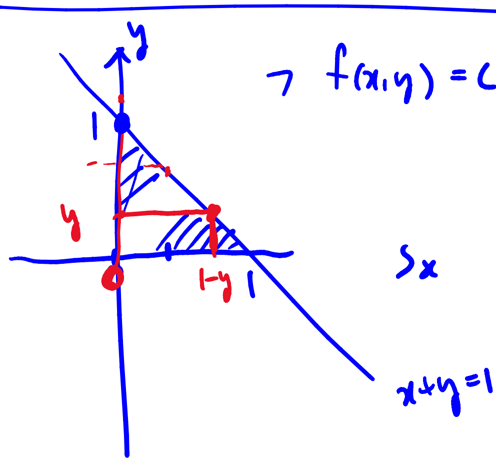
pdf

of calculus

pdf $f_X(x) = \frac{\partial}{\partial x}\Big(\qquad \Big)$ $\qquad \frac{d}{dx}\int_{-\infty}^{x} f(u)\,du = f(x)$

$$= \int_{-\infty}^{\infty} f_{X,Y}(x,v)\,dv$$

$X \perp\!\!\!\perp Y \quad f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$



$\rightarrow f(x,y) = c$

S_X

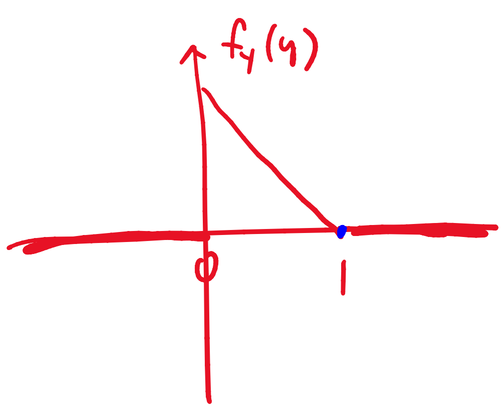
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\tfrac{1}{2}c = 1$$

$$c = 2$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{1-y} c\,dx = 2(1-y), \quad 0 \le y \le 1$$

$$= 0 \qquad y<0 \text{ or } y>1$$



If $X \perp\!\!\!\perp Y$, $\; f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$, $\forall\, x,y$

If $X \perp\!\!\!\perp Y$, $f_{X,Y}(x,y) = f_X(x) f_Y(y)$, $\forall x, y$

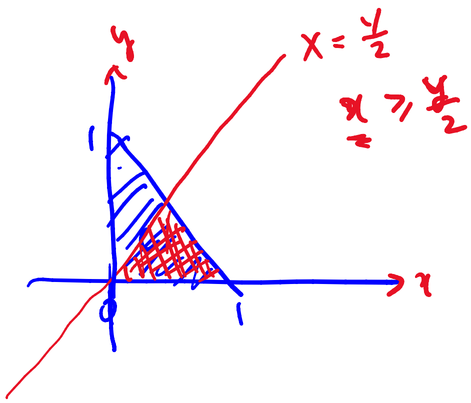
$$\boxed{f_{X,Y}(0,1) = 2}$$

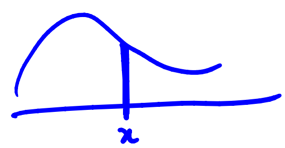
$$f_Y(1) = 2(1-1) = 0$$

$$\boxed{\underbrace{f_X(0)}_{\neq} \underbrace{f_Y(1)}_{=0} = 0}$$

not equal

$\therefore X \not\perp\!\!\!\perp Y$





$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$$

$$F_{Y|X}(y|x) = \mathbb{P}(Y \leq y \mid X = x)$$

$$= \lim_{\Delta x \to 0} \mathbb{P}(Y \leq y \mid x \leq X \leq x + \Delta x)$$

$$= \lim_{\Delta x \to 0} \frac{\mathbb{P}(Y \leq y,\ x \leq X \leq x + \Delta x)}{\underbrace{\mathbb{P}(x \leq X \leq x + \Delta x)}_{>0}}$$

$$\int_x^{x+\Delta x} \int_{-\infty}^{y} f_{X,Y}(x,u)\, du\, dx$$

$$\simeq \int_{-\infty}^{y} f_{X,Y}(x,u)\, du \cdot \Delta x$$

$$\simeq f_X(x) \Delta x$$

$$= \lim_{\Delta x \to 0} \frac{\int_{-\infty}^{y} f_{X,Y}(x,u)\, du}{f_X(x)} = \int_{-\infty}^{y} \frac{f_{X,Y}(x,u)}{f_X(x)}\, du$$

$$\boxed{F_{Y|X}(y|x) = \int_{-\infty}^{y} f_{Y|X}(u|x)\,du}$$

conditional cdf

conditional pdf

$$\begin{cases} 2, & x \geq 0, y \geq 0, x+y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

"$Y=y$" - fixed

$y \to$ constant

$$= \begin{cases} \frac{2}{2(1-y)} = \frac{1}{1-y}, & 0 \leq y < 1,\ 0 \leq x \leq 1-y \quad \cancel{x \geq 0,\ y \geq 0,\ x+y \leq 1} \quad \cancel{0 \leq y \leq 1} \\ 0, & \text{o.w.} \end{cases}$$

$$f_{X|Y}(x|y) = g_y(x)$$

variable

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\,f_X(x)}{f_Y(y)}$$

$$\to \int f_{X,Y}(x',y)\,dx'$$

$$\int f_{Y|X}(y|x')\,f_X(x')\,dx'$$

- $\Lambda \sim \text{Unif}[0,1]$

$X \mid \Lambda = \lambda \sim \text{Exp}(\lambda)$ $\quad f_{X|\Lambda}(x|\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$

$$X \mid \Lambda = \lambda \sim \text{Exp}(\lambda) \qquad f_{X|\Lambda}(x|\lambda) = \begin{cases} \ldots & , \\ 0 & ; \text{o.w.} \end{cases}$$

$$f_{\Lambda|X}(\lambda|3) = \frac{f_{X|\Lambda}(3|\lambda)\, f_\Lambda(\lambda)}{f_X(3)}$$

$$= \frac{\lambda e^{-3\lambda} \cdot 1}{\int_0^1 \lambda e^{-3\lambda}\, d\lambda} \qquad 0 \leq \lambda \leq 1$$

Θ - discrete

$Y|\Theta$ - continuous $\qquad f_{Y|\Theta}(y|\theta)$

likelihood $\qquad$ prior distribution

posterior pmf $\qquad \boxed{p_{\Theta|Y}(\theta|y)} = \dfrac{\boxed{f_{Y|\Theta}(y|\theta)}\,\boxed{p_\Theta(\theta)}}{f_Y(y)}$

"noise power"

$Z \sim N(0, N) \perp\!\!\!\perp \Theta$

$$\boxed{\Theta = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}} \longrightarrow \oplus \longrightarrow Y = \Theta + Z$$

$$p_{\Theta|Y}(\theta|y) = \frac{f_{Y|\Theta}(y|\theta)\, p_\Theta(\theta)}{f_{Y|\Theta}(y|+1)p_\Theta(1) + f_{Y|\Theta}(y|-1)p_\Theta(-1)}$$

$$Y = \Theta + Z$$

$$Y \mid \Theta = \theta \;=\; \theta + Z \sim N(\theta, N)$$

$$p_{\Theta|Y}(+1|y) = \frac{\frac{1}{\sqrt{2\pi N}} e^{-\frac{(y-1)^2}{2N}} \cdot p}{\ldots}$$

$$P_{\Theta|Y}(+1|y) = \frac{\frac{1}{\sqrt{2\pi N}} e^{-\frac{(y-1)^2}{2N}} \cdot p}{\frac{1}{\sqrt{2\pi N}} e^{-\frac{(y-1)^2}{2N}} \cdot p + \frac{1}{\sqrt{2\pi N}} e^{-\frac{(y+1)^2}{2N}} \cdot (1-p)}$$

min

$$P_e = \mathbb{P}(\hat{\Theta} \neq \Theta) \qquad \hat{\Theta} = d(Y) \longrightarrow \text{design } d(\cdot) \text{ s.t. } \min P_e .$$

$$= 1 - \mathbb{P}(\hat{\Theta} = \Theta)$$

$$= 1 - \underbrace{\int \mathbb{P}(d(Y) = \Theta \mid Y = y) f_Y(y)\, dy}_{\max .}$$

Given $Y = y$, choose $d(\cdot)$ to max $\mathbb{P}(d(y) = \Theta \mid Y = y)$ $\qquad \Theta \in \{\theta_0, \theta_1)$

$$d(y) = \begin{cases} \theta_0 & \text{if } P_{\Theta|Y}(\theta_0|y) \geq P_{\Theta|Y}(\theta_1|y) \\ \theta_1 & \text{o.w.} \end{cases}$$

$\downarrow$

maximum a posteriori prob.

(MAP)

$$P_{\Theta|Y}(\theta|y) = \frac{f_{Y|\Theta}(y|\theta) P_\Theta(\theta)}{f_Y(y)} \qquad = \frac{1}{2}$$

likelihood

$$\left.\begin{aligned} P_{\Theta|Y}(\theta_0|y) &\propto f_{Y|\Theta}(y|\theta_0) \\ P_{\Theta|Y}(\theta_1|y) &\propto f_{Y|\Theta}(y|\theta_1) \end{aligned}\right\} \text{ choose bigger}$$

$p = \frac{1}{2}$: maximum likelihood decoder

(ML)

"signal"

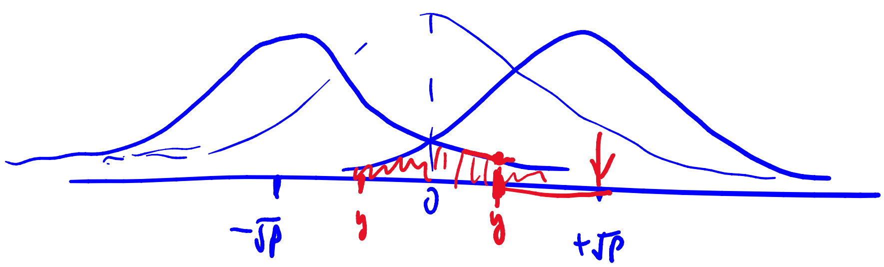
$p = \frac{1}{2}$: maximum likelihood decoder (ML)

"signal"

$$\Theta = \begin{cases} +\sqrt{P} \\ -\sqrt{P} \end{cases} \qquad Y = \Theta + \underbrace{Z}_{N(0,N)} \qquad \Theta \perp\!\!\!\perp Z$$

$$d(y) = \begin{cases} +\sqrt{P} & \dfrac{P_{\Theta|Y}(+\sqrt{P} \mid y)}{P_{\Theta|Y}(-\sqrt{P} \mid y)} > 1 \\ -\sqrt{P} & \text{o.w.} \end{cases}$$

$$Y \mid \Theta = \theta \sim N(\theta, N)$$



$$P_e = \mathbb{P}(d(Y) \neq \Theta)$$

$$= \mathbb{P}(\Theta = +\sqrt{P})\,\mathbb{P}(Y < 0 \mid \Theta = +\sqrt{P}) + \mathbb{P}(\Theta = -\sqrt{P})\,\mathbb{P}(Y > 0 \mid \Theta = -\sqrt{P})$$

$N(\sqrt{P}, N)$ $\qquad$ $N(-\sqrt{P}, N)$

$$= \frac{1}{2}\mathbb{P}(Z \leq -\sqrt{P}) + \frac{1}{2}\mathbb{P}(Z > \sqrt{P})$$

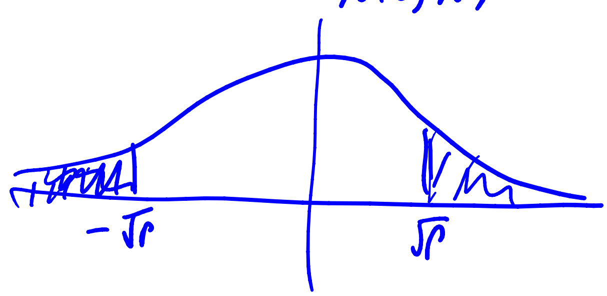
$$= \mathbb{P}\left(\frac{Z}{\sqrt{N}} > \sqrt{\frac{P}{N}}\right)$$

$$= Q\left(\sqrt{\frac{P}{N}}\right)$$

SNR $\uparrow$

$Q(\sqrt{SNR}) \downarrow$

$$\frac{P}{N} = SNR$$



$Z \sim N(0, N)$

$\frac{\cdot}{N} = \text{SNR}$

$Z \sim N(0, N)$

$\frac{Z}{\sqrt{N}} \sim N(0,1)$

$X, Y \qquad Z = g(X,Y)$

$X \perp\!\!\!\perp Y, \quad Z = X+Y$

$X \sim f_X(x)$

$Y \sim f_Y(y)$

$f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$F_Z(z) = \mathbb{P}(Z \leq z)$$

$$= \mathbb{P}(X+Y \leq z)$$

$$= \int_{\{(x,y):\, x+y \leq z\}} f_X(x) f_Y(y)\, dx\, dy$$

Fubini's Thm

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z-y} f_X(x)\, dx \right] f_Y(y)\, dy$$

$$\frac{\partial}{\partial z} F_Z(z) = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial z} \int_{-\infty}^{z-y} f_X(x)\, dx \right) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$

$$= f_X * f_Y\, (z)$$

$$f * g(x) = \int_{-\infty}^{\infty} f(x-\tau) g(\tau)\, d\tau$$

$X \perp\!\!\!\perp Y, \qquad U = \max\{X,Y\}$

$\qquad V = \min\{X,Y\}$

$U \leq u \Leftrightarrow X \leq u, Y \leq u$

$$F_U(u) = \mathbb{P}(U \leq u) = \mathbb{P}(X \leq u, Y \leq u) = \mathbb{P}(X \leq u)\, \mathbb{P}(Y \leq u)$$

$$= F_X(u)\, F_Y(u)$$

$$= F_X(u)\,F_Y(u)$$

$$\frac{\partial}{\partial u} F_U(u) = f_X(u) F_Y(u) + F_X(u) f_Y(u)$$

$X_1, X_2, \ldots, X_n$ independent, $X_i \sim F_X(\cdot)$ independent and identically distributed i.i.d.

$U = \max\{X_1, \ldots, X_n\}$

$$F_U(u) = F_X(u)^n \qquad f_U(u) = n\,F_X(u)^{n-1} f_X(u)$$

$V = \min\{X, Y\}$ $\quad V > v \Leftrightarrow X > v, Y > v$

$$F_V(v) = \mathbb{P}(V \le v) = 1 - \mathbb{P}(V > v)$$

$$= 1 - \mathbb{P}(X > v, Y > v)$$

$$= 1 - \underline{\mathbb{P}(X > v)}\,\mathbb{P}(Y > v)$$

$$= 1 - (1 - F_X(v))(1 - F_Y(v))$$

Ex. $X_1, \ldots, X_n$ i.i.d. $F_V(v)$?

$$\mathbb{P}(Y > 0 \mid \Theta = -\sqrt{P}) \sim N(-\sqrt{P}, N)$$