EE7401 Homework 2

Instructions:

● Please write your final solutions in the boxes shown below, then scan (if needed) and upload via NTULearn/Assignments/Homework2 by 23:59, 2 November 2023.
● Please do not copy from others, or let others copy your solution.
● Penalties may apply for late submission.
● Homework 2 carries 10% overall marks.

Q1) *Dilemma in lion city:* Two students living in *Singa*pore, Dil Singh and Emma Lyon, can't decide whether to take 179 or 179A to JP (Jurong Point). They know the following facts.

- 179A takes 5 minutes less than 179 to reach JP.
- 179A arrival is a Poisson arrival with arrival rate λ. In other words, $\mathbf{x}(t) = \mathbf{n}(0, t)$ is a Poisson process, where $\mathbf{n}(0, t)$ is the number of arrivals of 179A between time 0 and t minute. $\mathbf{n}(0, t)$ is k with probability $P\{\mathbf{n}(0, t) = k\} = e^{-\lambda t}(\lambda t)^k / k!$ where t is in minutes.

1.1) Dil, Emma are waiting at the bus stop, and a 179 arrives at time $t = 0$ minute. Dil prefers to take the 179. Emma, however, prefers to wait for the 179A.

(1.1.a) Find $\hat{t}$, the time of arrival of the 179A, such that 179A (Emma's preference) reaches JP at the same time as the 179 (Dil's preference).

$\hat{t} = \boxed{5}$

(2 marks)

(1.1.b) If the 179A arrives after $\hat{t}$, then Emma's preference reaches JP late. Find the probability P_{Emma}^{late} that Emma's preference reaches late, that is, the probability that no 179A arrives within time $\hat{t}$, or $P\{\mathbf{x}(\hat{t}) = 0\}$, in terms of λ.

$P_{Emma}^{late} = \boxed{5\lambda e^{-5\lambda}}$

(2 marks)

(1.1.c) Find the probability P_{Dil}^{late} that Dil's preference reaches late, in terms of λ.

$P_{Dil}^{late} = \boxed{e^{-\lambda}}$

(3 marks)

(1.1.d) Find a condition on the arrival rate λ such that Emma's preference is better, that is, $P_{Emma}^{late} < P_{Dil}^{late}$:

$\boxed{5\lambda e^{-5\lambda} < e^{-\lambda}}$

(3 marks)

1.2) Dil, Emma decides to meet at the bus stop and take a bus to JP. Dil comes to the bus stop at time $t = 0$ and waits for Emma. p 179A buses arrive between 0 and 10 minutes. Emma finally comes to the bus stop at time $t = 10$ and finds a visibly angry Dil waiting. Dil argues that the probability (say, P_1) of at least one 179A arriving in the next 5 minutes, given that he has already seen p 179A arriving in the past 10 minutes, is less than the usual probability of at least one 179A arriving in the next 5 minutes (say, P_2). Emma feels otherwise. Dil and Emma are comparing $P_1 =$ the conditional probability of greater than p arrivals in 0 to 15 minutes given there were p arrivals in 0 to 10 minutes $= P\{\mathbf{x}(15) > p \mid \mathbf{x}(10) = p\}$, with $P_2 =$ the probability of at least 1 arrival in 10 to 15 minutes $= P\{\mathbf{x}(15) - \mathbf{x}(10) \geq 1\}$.

(1.2.a) Compare $P\{\mathbf{x}(15) > p \mid \mathbf{x}(10) = p\}$ and $P\{\mathbf{x}(15) - \mathbf{x}(10) \geq 1\}$ to see if Dil is correct ($P_1 < P_2$), or if Emma is correct (P_1 not less than P_2). Show your steps.

Calculation of P_1:
$P_1 = P\{x(15) > p|x(10) = p\}$
$P_1 = P\{x(15) - x(10) > 0\}$
using the complement rule, this is equivalent to
$P_1 = 1 - P\{x(15) - x(10) \leq 1\}$
$P_1 = 1 - (P\{x(15) - x(10) = 1\} + P\{x(15) - x(10) = 1\})$
Next we calculate probabilities for 0 and 1 event and subtract from 1:
$P\{x(15) - x(10) = 0\} = \frac{(5\lambda)^0 e^{-5\lambda}}{0!} = e^{-5\lambda}$
$P\{x(15) - x(10) = 0\} = \frac{(5\lambda)^1 e^{-5\lambda}}{1!} = 5\lambda e^{-5\lambda}$
Now, add those two probabilities and subtract from 1:
$P_1 = 1 - (e^{-5\lambda} + 5\lambda e^{-5\lambda})$

Calculation of P_2:
$P_2 = P\{x(15) - x(10) \geq 1\}$
using the complement rule, this is equivalent to
$P_2 = 1 - P\{x(15) - x(10) < 1\}$
$P_2 = 1 - P\{x(15) - x(10) = 0\}$
$P_2 = 1 - 5\lambda e^{-5\lambda}$
Compare P_1 and P_2, it is clear to see that $P_1 < P_2$.

To conclude, $P_1 < P_2$ and Dil is correct.

(10 marks)

(1.2.b) Evaluate $P_2 = P\{\mathbf{x}(15) - \mathbf{x}(10) \geq 1\}$ for $\lambda = 0.2$ arrival/minute.

$P\{\mathbf{x}(15) - \mathbf{x}(10) \geq 1\} = \boxed{1 - e^{-1} = 0.6321}$

(5 marks)

Q2) *Fussy investor:* Mr. I. N. Vestor invests in the stock market. The unpredictable nature of the stock market means it is prudent to model the value of Mr. Vestor's investment at any given time by a wide sense stationary real process $\mathbf{x}(t)$. Let $R_{xx}(\tau)$ be the autocorrelation of $\mathbf{x}(t)$. Let $S_{xx}(\omega)$ be the power spectrum of $\mathbf{x}(t)$.

For obvious reasons, Vestor is always monitoring the rate of change of this value using an i-phone app. This rate of change $\mathbf{x}'(t)$ is essentially the output process when $\mathbf{x}(t)$ is passed through a differentiator. Recall that since differentiation in time domain is equivalent to multiplying by $j\omega$ in frequency domain, a differentiator is a linear system with frequency response $H(\omega) = j\omega$.

(2.a) Once a bull run (rate of change is positive, or the value increases) starts, Vestor asks you for how long will his good time continue. In effect, he is asking how the rate of change is correlated over time. Determine $R_{x'x'}(\tau)$, the autocorrelation of $\mathbf{x}'(t)$, in terms of $R_{xx}(\tau)$:

$$R_{x'x'}(\tau) = \boxed{-2\frac{d^2}{d\tau^2}R_{xx}(\tau)}$$

(8 marks)

(2.b) Find $S_{x'x'}(\omega)$, the power spectrum of $\mathbf{x}'(t)$, in terms of $S_{xx}(\omega)$:

$$S_{x'x'}(\omega) = \boxed{(\omega^2)\cdot S_{xx}(\omega) = (2\pi f)\cdot S_{xx}(\omega)}$$

(7 marks)

(2.c) After several years of bull run, I. N. Vestor accumulated a sizable wealth but was far from happy. When you asked him why, he said a larger value of $\mathbf{x}(t)$ would imply a negative rate of change. In order to convince him that this is not so, show that for a given t_1, the random variables $\mathbf{x}(t_1)$ and $\mathbf{x}'(t_1)$ are orthogonal.

To show that $x(t_1)$ and $x'(t_1)$ are orthogonal, we will show that $E\{x(t_1)x'(t_1)\} = 0$

1. If $x(t)$ and $x'(t_1)$ has a constant mean: if $\mu(t) = E\{x(t)\}$ is a constant, then $\mu'(t) = 0$
2. If $x(t)$ and $x'(t)$ are uncorrelated: as shown previously, if the covariance between $x(t_1)$ and $x'(t_1)$ is zero, then $E\{x(t_1)x'(t_1)\} = E\{x(t_1)\}E\{x'(t_1)\}$

Combining these two points, we get:
$E\{x(t_1)\}E\{x'(t_1)\} = \mu(t_1)\cdot 0 = 0$

That would imply that $x(t_1)$ and $x'(t_1)$ are orthogonal.

(5 marks)

(2.d) Unfortunately Mr. Vestor never took EE7401 or any similar course, and failed to appreciate orthogonality. However, he understands what correlated means (recall that you already explained correlation to him in part 1.a). To convince him, show that for a given t_1, the random variables $\mathbf{x}(t_1)$ and $\mathbf{x}'(t_1)$ are uncorrelated.

In general, it is not true that for any stochastic process $x(t_1)$ the random variables $x(t_1)$ and $x'(t_1)$ are uncorrelated. The correlation between these two random variables depends on the specific properties of the process.

To show uncorrelatedness, we would need to show that:

$Cov\{x(t_1)x'(t_1)\} = E\{x(t_1)x'(t_1)\} = E\{x(t_1)\}E\{x'(t_1)\}$

However, this equality does not hold for all stochastic processes. For example, consider a deterministic linear function $x(t_1) = mt + b$. In this case, $x(t_1)$ and $x'(t_1)$ are not uncorrelated because $x'(t_1) = m$ for all t_1, and thus $x(t_1)$ and $x'(t_1)$ are perfectly correlated.

For some specific types of stochastic processes, such as a Brownian motion (or Wiener process), the increments are independent, and thus the value of the process at a specific time and the derivative (or increment) at that time would be uncorrelated. However, this property does not extend to all random processes.

To make a general statement about the correlation between $x(t_1)$ and $x'(t_1)$, we would need additional information about the statistical properties of the process $x(t_1)$.

(5 marks)