NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2018-2019

EE7401 – PROBABILITY AND RANDOM PROCESSES

November/December 2018 Time Allowed: 3 hours

INSTRUCTIONS

1. This paper contains 5 questions and comprises 4 pages.
2. Answer all 5 questions.
3. All questions carry equal marks.
4. This is a closed-book examination.

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1. A binary random signal $X \sim$ Bern (p), $0 \le p \le 1$ is inputted to a random system. The system output is given by $Y = 6X + 2Z$, where Z is a random variable independent to X and has a pdf

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

(a) Derive the conditional pdf $f_{Y|X}(y \mid x)$.

(10 Marks)

(b) Estimate the value of the input X based on the system output Y that minimizes the probability of the wrong estimation.

(10 Marks)

2. A random variable has uniform pdf, $X \sim$ U[0, 3]. Given $X = x$, a random variable Y has a pdf

$$f_{Y|X}(y \mid x) = \begin{cases} cxe^{-xy} & y \ge 0 \\ 0 & \text{otherwise} \end{cases},$$

where c is a constant.

Note: Question No. 2 continues on page 3

(a) Find the marginal pdf $f_Y(y)$.

(10 Marks)

(b) Given $Y = 1/3$, estimate the value of X so that the mean square error of the estimation is minimized.

(10 Marks)

3. (a) You have S$3 in your pocket consisting of 1 coin of 1 dollar, 2 coins of 50 cents, 3 coins of 20 cents and 4 coins of 10 cents. Two coins are randomly picked out from your pocket and 8 coins remaining in your pocket. Suppose that every coin has the same chance to be picked out.

(i) Compute the probability that you picked out 40 cents.

(5 Marks)

(ii) Compute the probability that you picked out two coins of the same value.

(5 Marks)

(b) In a fair coin experiment, define the process $\mathbf{x}(t)$ as $\mathbf{x}(t) = g(t)$ if the head shows, and $\mathbf{x}(t) = -g(t)$ if the tail shows, where $g(t)$ is some given function of t.

(i) What is $\eta_x(t) = E\{\mathbf{x}(t)\}$, the mean of $\mathbf{x}(t)$?

(ii) Is the process $\mathbf{x}(t)$ WSS (wide-sense stationary)?

(5 Marks)

(c) Consider a complex random PPS (polynomial phase signal) process $\mathbf{s}(t) = \alpha.\exp\left(j\left(\mathbf{p}_0 + \mathbf{p}_1 t + \mathbf{p}_2 t^2\right)\right)$, where $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ are independent non-zero mean real random variables, and α is a real constant. If it is known that $\mathbf{s}(t)$ is WSS, show that $\mathbf{p}_2 = 0$.

[Hint: Find the condition such that $R_{ss}(t_1, t_2) = R_{ss}(\tau)$.]

(5 Marks)

4. (a) *McDonald's:* President Mcdonald Trump gets his food from McDonald's every day. He orders different items each day and gets 50% discount on the total price (after all, he is the president). He further pays a tip equal to 50% of his previous day's expenses. Let the total price of president Trump's order on n-th day be modeled by a discrete-time real stationary white noise process $\mathbf{v}[n]$ with autocorrelation $R_{vv}[m]=3\delta[m]$. Let the president's expenses at McDonald's on n-th day be another process $\mathbf{y}[n]$. Since the president pays $0.5\mathbf{v}[n]$ after discount, and pays a tip of $0.5\mathbf{y}[n-1]$, his expenses on n-th day is $\mathbf{y}[n]=0.5\mathbf{v}[n]+0.5\mathbf{y}[n-1]$. Therefore, $\mathbf{y}[n]$ is the output of a linear system with transfer function $H(z)=\dfrac{1}{2-z^{-1}}$ when $\mathbf{v}[n]$ is the input. In the following, help president Mcdonald Trump to find out the autocorrelation of $\mathbf{y}[n]$.

(i) Find $S_{vv}(\omega)$, the power spectrum of the price.

(ii) Find $S_{yy}(\omega)$, the power spectrum of the expenses.

(iii) Find $R_{yy}[m]$, the autocorrelation of the expenses. You may use either one of the following hints.
[Hint: frequency domain: Express $S_{yy}(\omega)$ as a sum of two parts of the form $\dfrac{\alpha}{1-ce^{-j\omega}}+\dfrac{\beta e^{j\omega}}{1-ce^{j\omega}}$ for appropriate constants α,β. Inverse Fourier transform of the first part is $\alpha c^{n}u[n]$, while inverse Fourier transform of the second part is $\beta c^{-n-1}u[-n-1]$. Now add both parts.]
[Hint: time domain: Rewrite $\mathbf{y}[n]=0.5\mathbf{v}[n]+0.5\mathbf{y}[n-1]$ as $\mathbf{y}[n]$ equals to the sum of only present and past inputs (no past output terms like $\mathbf{y}[n-k]$ should remain). Using this expression, find $E\{\mathbf{y}[n]\mathbf{y}[n+m]\}$ for $m\geq 0$. The $m<0$ case may be found from symmetry.]

(12 Marks)

(b) *Modulation:* Let $\mathbf{w}(t)=\mathbf{a}(t)+j\mathbf{b}(t)$ be a complex modulating process, where $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are real jointly WSS processes with auto/cross correlations $R_{aa}(\tau),R_{bb}(\tau),R_{ab}(\tau)$ and $R_{ba}(\tau)$. Let $\mathbf{x}(t)=\mathbf{a}(t)\cos(\omega_0 t)-\mathbf{b}(t)\sin(\omega_0 t)$ be the real part of the modulated process.

(i) Find $R_{ww}(t+\tau,t)$, the autocorrelation of $\mathbf{w}(t)$. Is $\mathbf{w}(t)$ WSS?

(ii) Find $R_{xx}(t+\tau,t)$, the autocorrelation of $\mathbf{x}(t)$, and hence find the condition(s) on $R_{aa}(\tau),R_{bb}(\tau),R_{ab}(\tau)$ and $R_{ba}(\tau)$ such that $\mathbf{x}(t)$ is WSS.

(8 Marks)

5. (a) *Dementia:* Prof. Dame N. Seah is suffering from dementia and keeps forgetting to bring her spectacle from her home to her office, or to bring it back from her office to her home. In the morning, Prof. Seah is fresh; if the spectacle is at her home, she remembers to take it to the office with a probability of 0.8. In the evening, Prof. Seah is tired; if the spectacle is at the office, she remembers to take it back home with a probability of only 0.4.

(i) Formulate the above problem as a Markoff chain, where the states represent the spectacle is at the home or at the office. Draw the Markoff chain and find the transition matrix.

(ii) Find the steady state probability p_{office} that the spectacle is at the office.

(iii) While the Professor could manage at her home without the spectacle, it is rather inconvenient when she does not have the spectacle at her office. Will p_{office} (the steady state probability that the spectacle is at the office) increase if the probability 0.4 (as above, probability to take the spectacle back home from office) is increased?

(11 Marks)

(b) *Chilling Winter:* Moving to Beijing, Qi Ling finds the winter to be very chilling. The temperature moves around zero degrees. Qi Ling modelled the temperature as a real WSS order 1 Markoff process $\mathbf{s}(t)$ where t is measured in hour, and found that its autocorrelation is $R_{ss}(\tau) = 5e^{-|\tau|/2}$. To better prepare against the chilling winter, Qi Ling wants to predict the temperature after 2 hours $\mathbf{s}(t+2)$ from the current temperature $\mathbf{s}(t)$.

(i) Determine the optimal predictor in the LMS (least mean square) error sense, that predicts $\mathbf{s}(t+2)$ using $\mathbf{s}(t)$.

(ii) Find the LMS error P for your predictor.

(iii) Qi Ling is not happy with the LMS error P, and wants to reduce it further. So Qi Ling designs another predictor that predicts $\mathbf{s}(t+2)$ from both the current temperature $\mathbf{s}(t)$ and the temperature an hour ago $\mathbf{s}(t-1)$. Comment on how much P may be reduced using this second predictor.

(9 Marks)

END OF PAPER

EE7401 PROBABILITY & RANDOM PROCESSES

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.