EE7401 Probability and Random Processes

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1 The probability of the events while its cdf is given

The cdf of a random variable X is given by:

$$F_X(x) = \begin{cases} \frac{1}{3} + \frac{2}{3}(x+1)^2, & -1 \leq x \leq 0, \\ 0, & x < -1. \end{cases}$$

Find the probability of the events $\{X > 0\}$ and $\{|X| \geq 1\}$

Answer

For the event $\{X > 0\}$:

Since the given domain for $F_X(x)$ is $-1 \leq x \leq 0$, this means that X cannot take any value outside of this interval. Hence, the probability that $X > 0$ is 0, because X can not be greater than 0 based on the domain provided.

Then, $P(X > 0) = 0$

For the event $\{|X| \geq 1\}$:

We know that $\{|X| > 1\}$ equals to $\{X < -1 \text{ or } X > 1\}$. Based on the domain provided, X does not take any values for $X > 1$ and $X < -1$. But note that for $X = -1$, $F_X(x) = \frac{1}{3}$ so that the probability for this events is:

$P(|X| \geq 1) = \frac{1}{3}$

$\therefore$ To conclude:

$P(X > 0) = 0$
$P(|X| \geq 1) = \frac{1}{3}$

2 Determine the Joint Density from Conditional Densities

Consider the joint probability density function $f_{X,Y}(x,y)$ for two random variables X and Y.

Show that $f_{X,Y}(x,y) = \dfrac{f_{Y|X}(y|x)}{\int \frac{f_{Y|X}(z|x)}{f_{X|Y}(x|z)} dz},$ ......(1)

i.e., the conditional densities determine the joint density.

Answer

In many references, the joint probability density function (PDF) is often symbolized by $f_{X,Y}(x,y)$. So, in this paper, we will use $f_{X,Y}(x,y)$ instead of $p_{X,Y}(x,y)$.

We know that:

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) \quad .....(2)$$

$$f_{Y|X}(z|x) = f(Y = z|X = x) = \frac{f_{X,Y}(x,z)}{f_X(x)} \quad .....(3)$$

$$f_{X|Y}(x|z) = f(X = x|Y = z) = \frac{f_{X,Y}(x,z)}{f_Y(z)} \quad .....(4)$$

From equation (3) and (4), we get:

$$\frac{f_{Y|X}(z|x)}{f_{X|Y}(x|z)} = \frac{\frac{\cancel{f_{X,Y}(x,z)}}{f_X(x)}}{\frac{\cancel{f_{X,Y}(x,z)}}{f_Y(z)}} = \frac{f_Y(z)}{f_X(x)} \quad .....(5)$$

Substitute equation (2) and (5) to (1), we will get:

$$\cancel{f_{Y|X}(y|x)} f_X(x) = \frac{\cancel{f_{Y|X}(y|x)}}{\int \frac{f_{Y|X}(z|x)}{f_{X|Y}(x|z)} dz}$$

$$\frac{1}{f_X(x)} = \int \frac{f_{Y|X}(z|x)}{f_{X|Y}(x|z)} dz = \int \frac{f_Y(z)}{f_X(x)} dz = \frac{1}{f_X(x)} \int f_Y(z) dz$$

Thus, $\displaystyle\int_{-\infty}^{\infty} f_Y(z) dz = 1$, which is consistent with the properties of probability density function.

3 Mean of X

Suppose the random variable θ has pmf $\mathbb{P}(\theta = 3) = 0.4, \mathbb{P}(\theta = 8) = 0.6$, and $Y_1, Y_2, ...$ are i.i.d random variables, independent of θ with mean 10. What is the mean of $X = Y_1 + Y_2 + ... + Y_\theta$?

Answer

We need to find $E[X]$ where $X = Y_1 + Y_2 + ... + Y_\theta$.

Given θ, the mean of X is:

$E[X|\theta = k] = E[Y_1 + Y_2 + ... + Y_\theta]$

Since all Y_i are identically distributed and have a mean of 10,

$E[X|\theta = k] = k \times E[Y_1] = 10k$

For the two given values of θ:

For $\theta = 3$:
$E[X|\theta = 3] = 3 \times 10 = 30$

For $\theta = 8$:
$E[X|\theta = 8] = 8 \times 10 = 80$

Using the law of total expectation:

$E[X] = P(\theta = 3)E[X|\theta = 3] + P(\theta = 8)E[X|\theta = 8]$

$E[X] = 0.4(30) + 0.6(80)$

$E[X] = 12 + 48 = 60$

$\therefore$ The answer is 60.