EE7401 Probability and Random Processes
RA 1 Solutions

Note that there can be many possible solutions. This is just one possible approach.

1.

(a) A sample point is of the form (Box with gold bar, Box chosen by contestant, Box opened by Monty Hall). The sample space Ω consists of sample points of the form:

$$(x, y, z), \quad \text{where } x \in \{A, B, C\},\ y = x,\ z \in \{A, B, C\} \backslash \{x\} \quad \text{(6 samples)}$$
$$(x, y, z), \quad \text{where } x \in \{A, B, C\},\ y \neq x,\ z \in \{A, B, C\} \backslash \{x, y\} \quad \text{(6 samples).}$$

For example, the sample points with $x = A$ are (A, A, B), (A, A, C), (A, B, C), (A, C, B).

(b) To specify the probability measure, we need to find the probability of each sample point as Ω is a discrete space. Consider first sample points of the form (x, x, z). We have

$$\begin{aligned}
&\mathbb{P}((x, x, z)) \\
&= \mathbb{P}(\text{Box containing gold bar is } x, \text{ Contestant chooses } x, \text{ Monty Hall opens } z) \\
&= \mathbb{P}(\text{Box containing gold bar is } x)\mathbb{P}(\text{Contestant chooses } x \,|\, \text{Box containing gold bar is } x) \\
&\quad \cdot \mathbb{P}(\text{Monty Hall opens } z \,|\, \text{Box containing gold bar is } x, \text{ Contestant chooses } x) \\
&= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \\
&= \frac{1}{18}.
\end{aligned}$$

For sample points of the form (x, y, z), $y \neq x$, we have

$$\begin{aligned}
&\mathbb{P}((x, y, z)) \\
&= \mathbb{P}(\text{Box containing gold bar is } x, \text{ Contestant chooses } y, \text{ Monty Hall opens } z) \\
&= \mathbb{P}(\text{Box containing gold bar is } x)\mathbb{P}(\text{Contestant chooses } y \,|\, \text{Box containing gold bar is } x) \\
&\quad \cdot \mathbb{P}(\text{Monty Hall opens } z \,|\, \text{Box containing gold bar is } x, \text{ Contestant chooses } y) \\
&= \frac{1}{3} \cdot \frac{1}{3} \cdot 1 \\
&= \frac{1}{9}.
\end{aligned}$$

The event E that the contestant wins by deciding to switch consists of the sample points of the form (x, y, z), $y \neq x$. Therefore,

$$\mathbb{P}(E) = \frac{1}{9} \cdot 6 = \frac{2}{3}.$$

Note that the probability of the contestant winning by staying with her first choice is only $\frac{1}{18} \cdot 6 = \frac{1}{3}$, so the contestant has a higher chance of winning by switching.

(c) Let w be the box the contestant chooses after switching (set $w = \text{null}$ if contestant does not switch). Reusing the same notations as before but with $z = (z_1, \ldots, z_p)$, a sample point is of the form (x, y, z, w). For $y \neq x$ and $w = x$, we have

$$\mathbb{P}((x, y, z, w)) = \frac{1}{n}\frac{1}{n}\frac{1}{\binom{n-2}{p}}\frac{1}{n-p-1}.$$

There are $n(n-1)\binom{n-2}{p}$ such sample points. Therefore,

$$\begin{aligned}
\mathbb{P}(\text{Contestant wins after switching}) &= \mathbb{P}(\{(x, y, z, w) \,:\, y \neq x,\ w = x\}) \\
&= n(n-1)\binom{n-2}{p}\frac{1}{n}\frac{1}{n}\frac{1}{\binom{n-2}{p}}\frac{1}{n-p-1} \\
&= \frac{n-1}{n(n-p-1)} > \frac{1}{n}.
\end{aligned}$$

This calculation shows that it is always better to switch.

2. We are given $\mathbb{P}(A \,|\, B) \geq \mathbb{P}(A)$. Since $\mathbb{P}(A) = \mathbb{P}(B)\mathbb{P}(A \,|\, B) + \mathbb{P}(B^{\mathsf{c}})\mathbb{P}(A \,|\, B^{\mathsf{c}})$, we have

$$\begin{aligned}
\mathbb{P}(A \,|\, B^{\mathsf{c}}) &= \frac{\mathbb{P}(A) - \mathbb{P}(B)\mathbb{P}(A \,|\, B)}{\mathbb{P}(B^{\mathsf{c}})} \\
&\leq \frac{\mathbb{P}(A) - \mathbb{P}(B)\mathbb{P}(A)}{\mathbb{P}(B^{\mathsf{c}})} \\
&= \frac{\mathbb{P}(A)(1 - \mathbb{P}(B))}{\mathbb{P}(B^{\mathsf{c}})} \\
&= \mathbb{P}(A).
\end{aligned}$$

Therefore, $\mathbb{P}(A \,|\, B^{\mathsf{c}}) \leq \mathbb{P}(A)$.