NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2022-2023

EE7401 – PROBABILITY AND RANDOM PROCESSES

November / December 2022 Time Allowed: 3 hours

INSTRUCTIONS

1. This paper contains 4 questions and comprises 5 pages.
2. Answer all 4 questions.
3. All questions carry equal marks.
4. This is a closed book examination.
5. Unless specifically stated, all symbols have their usual meanings.

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1. (a) Suppose X is selected uniformly at random from the interval $[-1, 1]$. Define the events

$$A = \{X < 0\},\ B = \{|X - 0.5| < 1\},\ \text{and } C = \{X > 0.75\}.$$

Find the probabilities of

(i) $A \cap B$,

(ii) $A \cap C$,

(iii) $A \cup B$, and

(iv) $A \cup C$.

(6 Marks)

(b) Let A be the event that a patient develops long COVID symptoms and B be the event that the patient is unvaccinated. A study has found that if a patient is unvaccinated, he or she is more likely to develop long COVID symptoms, i.e., $\mathbb{P}(A \mid B) \geq \mathbb{P}(A)$. Given a *vaccinated* patient, what can you say about the conditional probability of the patient developing long COVID symptoms? Justify your answer rigorously.

(7 Marks)

Note: Question No. 1 continues on page 2.

(c) Let $X \sim \text{Exp}(\lambda)$ be an exponential random variable with parameter λ and $Y = \lfloor X \rfloor$ be the integer part of X, i.e., $Y = k$ for $k \leq X < k+1$, $k = 0, 1, \ldots$.

(i) Find the pmf of Y.

(ii) Let $Z = X - Y$ be the quantization error. Find the pdf of Z.

(12 Marks)

2. (a) Consider the noisy channel shown in Figure 1, where X and Z are independent, and a and b are constants.

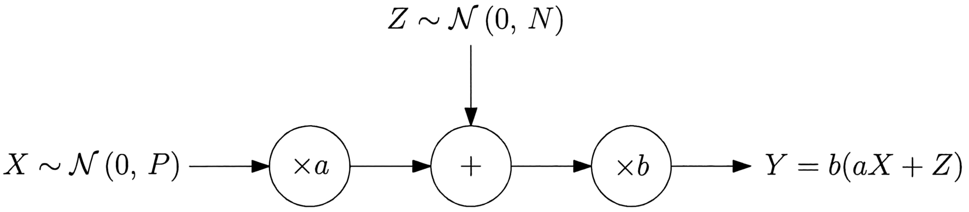

Figure 1

(i) Find the mean and variance of Y.

(ii) Find the covariance between X and Y.

(iii) Are the minimum mean square error (MMSE) estimate of X given Y and its MMSE linear estimate the same? Why or why not? Find the MMSE linear estimate of X given Y and its MSE.

(20 Marks)

(b) Suppose $X_1, X_2, \ldots$ are i.i.d. random variables, each with pdf $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Let $Y_n = \min\{X_1, X_2, \ldots, X_n\}$. Show that Y_n converges to 0 in probability as $n \to \infty$.

(5 Marks)

3. (a) *Channelling:* Entrusted with the task of channelling two real signals, modelled by real jointly wide sense stationary (WSS) random processes $\mathrm{x}(t)$ and $\mathrm{y}(t)$, researcher Chan Eh Ling realizes that at the receiver of one signal, the other signal acts as an interference. The auto-correlations $R_{xx}(\tau)$, $R_{yy}(\tau)$, and the cross-correlation $R_{xy}(\tau)$ are available to researcher Chan. Chan Eh Ling compares the magnitude of the cross-correlation of two signals, $|R_{xy}(\tau)|$, to the average power of both signals, $\frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$.
Without any further information, you should be able to compare $|R_{xy}(\tau)|$ and $\frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$. In particular, determine the most appropriate one among the following:

Note: Question No. 3 continues on page 3.

(i) $|R_{xy}(\tau)| < \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$
(ii) $|R_{xy}(\tau)| \leq \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$
(iii) $|R_{xy}(\tau)| = \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$
(iv) $|R_{xy}(\tau)| \geq \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$
(v) $|R_{xy}(\tau)| > \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$

(4 Marks)

(b) Continuing the same task, Chan Eh Ling now considers combining two signals into a complex signal, modelled by a complex WSS random process $\mathbf{w}(t)$ having auto-correlation $R_{ww}(\tau)$. Chan studies how much the signal $\mathbf{w}(t)$ changes during a time difference of τ. This change, in the mean square sense, is given by $E\{|\mathbf{w}(t+\tau) - \mathbf{w}(t)|^2\}$. Chan feels this change should be compared to the real part of the difference in auto-correlation values, $\mathrm{Re}[R_{ww}(0) - R_{ww}(\tau)]$.
Without any further information, you should be able to compare $E\{|\mathbf{w}(t+\tau) - \mathbf{w}(t)|^2\}$ and $2\mathrm{Re}[R_{ww}(0) - R_{ww}(\tau)]$. In particular, determine the most appropriate one among the following:

(i) $E\{|\mathbf{w}(t+\tau) - \mathbf{w}(t)|^2\} < 2\mathrm{Re}[R_{ww}(0) - R_{ww}(\tau)]$
(ii) $E\{|\mathbf{w}(t+\tau) - \mathbf{w}(t)|^2\} \leq 2\mathrm{Re}[R_{ww}(0) - R_{ww}(\tau)]$
(iii) $E\{|\mathbf{w}(t+\tau) - \mathbf{w}(t)|^2\} = 2\mathrm{Re}[R_{ww}(0) - R_{ww}(\tau)]$
(iv) $E\{|\mathbf{w}(t+\tau) - \mathbf{w}(t)|^2\} \geq 2\mathrm{Re}[R_{ww}(0) - R_{ww}(\tau)]$
(v) $E\{|\mathbf{w}(t+\tau) - \mathbf{w}(t)|^2\} > 2\mathrm{Re}[R_{ww}(0) - R_{ww}(\tau)]$

(3 Marks)

(c) *Haze Law:* With the deteriorating haze situation in Singapore, Ms. Hazel Aw introduces a law against haze. The Pollutant Standards Index (PSI) is modelled as a real stationary white noise process $\mathbf{w}[n]$ with auto-correlation $R_{ww}[m] = \delta[m]$. Ms. Aw measures the haze using $\mathrm{s}[n]$, which is a random process that is obtained by passing $\mathbf{w}[n]$ through a linear system with transfer function $\mathbf{H}(z) = 1 + \frac{1}{2}z^{-1}$ (Figure 2). Since Hazel Aw uses the auto-correlation of $\mathrm{s}[n]$ in the proposed law, you'll help her find the auto-correlation.

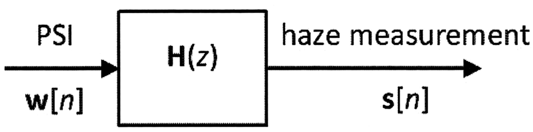

Figure 2

(i) Find $S_{ss}(\omega)$, the power spectrum of $\mathrm{s}[n]$. Simplify so that $S_{ss}(\omega)$ is a real function of ω.

Note: Question No. 3 continues on page 4.

(ii) Find $\mathrm{S}_{ss}(z)$, the z-transform domain power spectrum of $\mathrm{s}[n]$. [Hint: z-transform of $y[n]$ is defined as $Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n}$. The z-transform of $\delta[n-n_0]$, an impulse at n_0, is therefore z^{-n_0}.]

(iii) Find $R_{ss}[m]$, the auto-correlation of $\mathrm{s}[n]$. [Hint: You may use any of the following approaches: the time-domain approach, the inverse Fourier transform approach, or the inverse z-transform approach.]

(9 Marks)

(d) Ms. Aw, however, finds that the experimental values of $\mathrm{s}[n]$ are noisy. The experimental values of the measured haze are given by $\mathrm{x}[n] = \mathrm{s}[n] + \mathrm{v}[n]$, where $\mathrm{v}[n]$ is a real stationary white noise (see Figure 3), which is uncorrelated to $\mathrm{s}[n]$, and has the auto-correlation $R_{vv}[m] = \frac{7}{8}\delta[m]$. Ms. Aw decides to use a whitening filter to reduce the experimental noise, and needs your help.

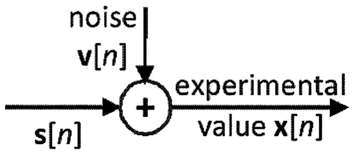

Figure 3

(i) Find $\mathrm{S}_{xx}(z)$, the z-transform domain power spectrum of $\mathrm{x}[n]$.

(ii) The transfer function of a discrete-time whitening filter $\mathbf{\Gamma}_x(z)$ is such that $\mathrm{S}_{xx}(z) = \frac{1}{\mathbf{\Gamma}_x(z)\mathbf{\Gamma}_x(z^{-1})}$. Find the whitening filter $\mathbf{\Gamma}_x(z)$ by factorizing the z-transform domain power spectrum $\mathrm{S}_{xx}(z)$.

(9 Marks)

4. (a) *Addiction:* Mr. A. Dickson has lots of money, and has an addiction to gambling. He decides to play non-stop until he hits the jackpot of one million dollars. Denote the time the jackpot is won by a random variable $\mathbf{c}$. From the published odds, Mr. A. Dickson figures that winning of the jackpot is equally likely to happen anytime during the next 500 days. Therefore $\mathbf{c}$ has uniform density in $0 \leq t \leq 500$ where time t is measured in days. Let a process $\mathrm{x}(t) = 10^6\delta(t-\mathbf{c})$ represents Mr. A. Dickson's gain in dollars when he hits the jackpot. However, playing for the jackpot costs him \$3000 a day. Let another process $\mathrm{y}(t) = -3000\{U(t) - U(t-\mathbf{c})\}$ represents his cost in dollars, where $U(t)$ denotes the unit step function, and the negative amplitude denotes loss of money.

(i) In this part, find $R_{xx}(t_1, t_2)$, the auto-correlation of the winning process. [Hint: $\mathrm{x}(t)$ is a non-stationary process. $\int_p^q \delta(t_1-c)\delta(t_2-c)dc = \delta(t_1-t_2)$ if both t_1 and t_2 are inside the interval $[p,q]$, but = 0 otherwise.]

Note: Question No. 4 continues on page 5.

(ii) In this part, we find the expectation of $\mathbf{x}(t) + \mathbf{y}(t)$, the expected gain or loss of Mr. A. Dickson. First, show that the expectation of the winning process $\mathbf{x}(t)$ is given by

$$E\{\mathbf{x}(t)\} = \begin{cases} 2000 & 0 \leq t \leq 500 \\ 0 & \text{otherwise} \end{cases}$$

which confirms that the expected winning does not change over time since **c** is uniform. [Hint: $\int_p^q \delta(t-c)dc = 1$ if t is inside the interval $[p, q]$, but = 0 if t is outside this interval.]

(iii) Second, find $E\{\mathbf{y}(t)\}$, the expectation of the cost process $\mathbf{y}(t)$. [Hint: $\int_p^q U(t-c)dc = \int_p^q dc$ if $q < t$, $= \int_p^t dc$ if t is inside the interval $[p, q]$, and = 0 if $t < p$. The expected cost $E\{\mathbf{y}(t)\}$ should be negative since cost is negative. It should be a function of time, and it should approach 0 over time, since Mr. A. Dickson stops playing as soon as he wins, and his win becomes more and more likely as time passes.]

(14 Marks)

(b) *Reign of a Queen:* In a faraway country called United Queendom (UQ), a Queen reigns for as long as she lives. Let a_i denote the i-th year of the reign of the Queen. For simplicity, assume that $1 \leq i \leq \infty$.

- With probability p, the Queen does not survive the i-th year, and a new Queen is coronated such that the reign of the (new) Queen restarts at a_1.
- Otherwise, the Queen lives through the i-th year, and her reign then extends to a_{i+1}.

(i) Formulate the above problem as a discrete-time Markoff chain with infinite number of states. Draw the Markoff chain, showing a few states and transition probabilities. Find the state transition matrix.

(ii) Find the steady state probabilities. What is the probability that any Queen of UQ reigns for more than 70 years?

(11 Marks)

END OF PAPER

EE7401 PROBABILITY & RANDOM PROCESSES

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.