Week 1

Monday, 14 August 2023 6:47 pm

<1600 Greece, Egypt games of chance

1600s Fermat, Pascal, Huygens "classical"

1700s Bernoulli, Fourier

↳ strong law of large numbers binary r.v.

1800s Gauss, Laplace

$\boxed{1933}$ Andrei Kolmogorov. measure-theoretic foundations

Fisher, Neyman

Ito calculus

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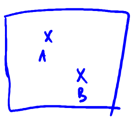
lightning



"discrete"

A: "chance of lightning strike at A" 0.4

B: " " " " " B" 0.2

N: "chance of no lightning " $\boxed{0.4}$

"outcomes"

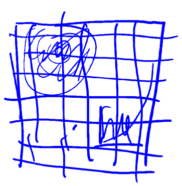
"samples"

sample space $\Omega = \{A, B, N\}$

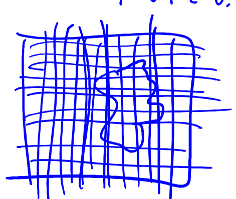
$\{A \text{ or } B\} = \{\text{having lightning}\}$

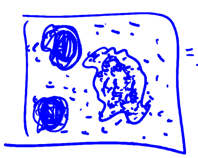
$\mathbb{P}(\cdot) = 0.4 + 0.2 = 0.6$

$= 1 - 0.4 = 0.6$



"continuous"

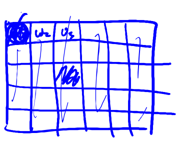




$= \Omega$

collection of events that we want prob. for

$(\Omega, \boxed{\mathcal{F}}, \mathbb{P})$



$\mathcal{F} = \{ \omega_1, \omega_2, \omega_3, \ldots$

$\omega_1 \cup \omega_2, \quad \omega_1 \cup \omega_3, \ldots$

$\omega_1^c, \omega_2^c, \ldots$

$\vdots$

$\}$

Discrete

$\Omega = \{H, T\} \xrightarrow{n \text{ times}}$

$n = 4$ $\qquad 2^4 = 16$

HHTT HTHT ...

HHHH ...

Discrete

$\Omega = \{H, T\} \xrightarrow{n \text{ times}}$ $n = 4$ $2^4 = 16$

HHTT HTHT ...

HHHH ....

$\Omega = \{1, 2, \cdots, 6\}$

$\Omega = \{ \text{HHTT}, \text{HTHT}, \ldots \}$

countably infinite: 0, 1, 2, 3, ...

uncountably infinite: $\mathbb{R}$

$\Omega = \{1, 2, 3\}$

$\mathcal{F} = \{\phi, \{1\}, \{2\}, \{3\},$ ← events that you can define the probability

$\{1,2\}, \{1,3\}, \{2,3\},$ — e.g. A

$\{1,2,3\} \}$

$0 \le \cdot \le 1$

$\mathbb{P}(A) = \mathbb{P}(\{2\}) + \mathbb{P}(\{3\})$

$A \in \mathcal{F}$ $\quad \boxed{\mathbb{P}(A)} = \sum_{a \in A} \boxed{\mathbb{P}(\{a\})}$

$\mathbb{P}(\Omega) = \mathbb{P}(\{1,2,3\}) = 1$

odd number

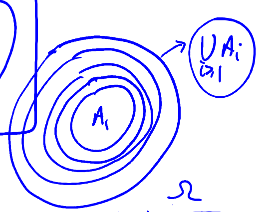
$\mathbb{P}(\{1,3,5\}) = \mathbb{P}(\{1\}) + \mathbb{P}(\{3\}) + \mathbb{P}(\{5\})$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

$(\Omega, \mathcal{F}, \mathbb{P})$ $\quad \boxed{\mathbb{P} : \mathcal{F} \to [0, 1]}$

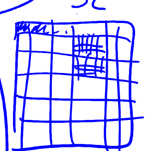
$\mathcal{F}$ = collection of events → σ-algebra, σ-field

- $\phi \in \mathcal{F}$
- If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
- $\underbrace{A_1, A_2, \cdots}_{\text{countable}} \in \mathcal{F} \Rightarrow \boxed{\bigcup_{i \ge 1} A_i \in \mathcal{F}}$



→ Hold naturally for discrete Ω

$\mathbb{P} : \mathcal{F} \to [0, 1]$ Axioms



- $\mathbb{P}(\Omega) = 1$
- $A_1, A_2, \cdots \in \mathcal{F}$, $\quad A_i \cap A_j = \phi, \; i \ne j$

$\mathbb{P}\left(\bigcup_{i \ge 1} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ Countably additive

$A_i \cap A_j = \phi$

$\mathbb{P}(\bigcup A_i) = \sum \mathbb{P}(A_i)$?

$A_i \cap A_j = \phi$

$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$?

$A_1, A_2, \dots, A_n, \overset{A_{n+1}}{\phi}, \overset{A_{n+2}}{\phi}, \dots$

$\mathbb{P}\left(\bigcup_{i\geq 1} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right)$

$\mathbb{P}(\phi) = 0$

$\sum_{i\geq 1} \mathbb{P}(A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i) + \boxed{\sum_{i>n} \mathbb{P}(A_i)}$

$\phi = \phi \cup \phi \cup \dots$ $\qquad \phi \cap \phi = \phi$

$\mathbb{P}(\phi)$ $\quad \mathbb{P}(\phi) = \underbrace{\mathbb{P}(\phi) + \mathbb{P}(\phi) + \dots}_{\text{countable}}$

If $\mathbb{P}(\phi) > 0$, $RHS = \infty$

$\therefore \mathbb{P}(\phi) = 0 = RHS$

$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$ $\qquad A_i \cap A_j = \phi$

$\mathbb{P}((a,b)) = F(b) - F(a)$

$-\infty$ $\quad a \quad b \quad \infty$

$\mathcal{F}$ "generated by open sets"

= Borel σ-algebra

- $1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c)$

$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

- $A \subset B$ $\qquad \mathbb{P}(A) \leq \mathbb{P}(B)$

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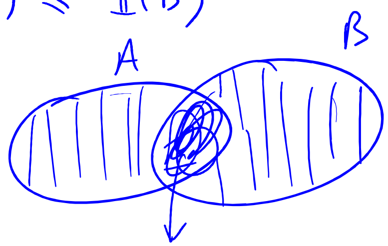

$A \cap B$

- $\mathbb{P}(A \cup B)$

$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$\leq \mathbb{P}(A) + \mathbb{P}(B)$$

(induction

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$$

union bound

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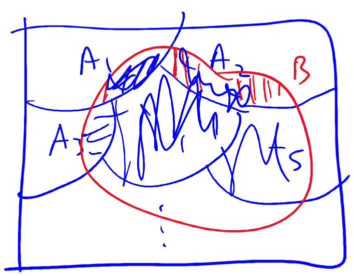


Ω

$$\Omega = \bigcup_{i \geq 1} A_i$$

$$B = B \cap \Omega = B \cap \left(\bigcup_{i \geq 1} A_i\right)$$

$$= \bigcup_{i \geq 1} (B \cap A_i)$$

$$\mathbb{P}(B) = \mathbb{P}\left(\bigcup_{i \geq 1} (B \cap A_i)\right)$$

$$= \sum_{i \geq 1} \mathbb{P}(B \cap A_i)$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\, \mathbb{P}(B)$$

Bayes' Thm.

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\,\mathbb{P}(B)$$

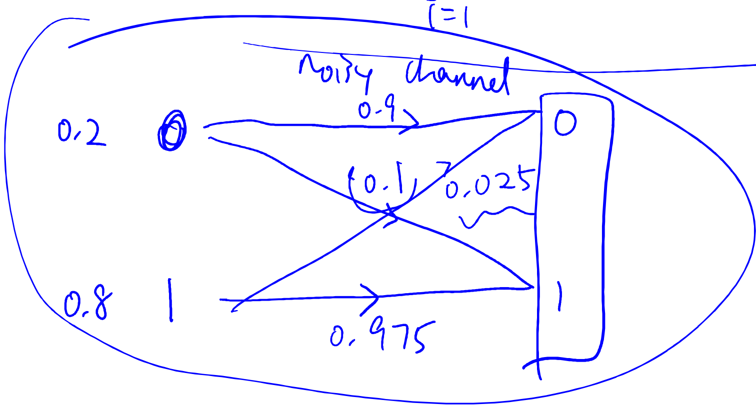
Bayes' Thm.

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$$

$$\Omega = A_1 \cup A_2 \cup A_3 \cdots \cup A_n \qquad A_i \cap A_j = \phi$$

$$\mathbb{P}(A_j \mid B) = \frac{\mathbb{P}(B \mid A_j)\,\mathbb{P}(A_j)}{\boxed{\mathbb{P}(B)}}$$

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(B \mid A_i)\,\mathbb{P}(A_i)$$



$$\mathbb{P}(B \mid A) = 0.9$$

$$\Omega = \{(\overset{tx}{0}, \overset{rx}{0}), (0,1), (1,0), (1,1)\}$$

$$A = \{0 \text{ is sent}\} = \{(0,0), (0,1)\}$$

$A = \{0 \text{ is sent}\} = \{(0,0),(0,1)\}$

$B = \{0 \text{ is received}\} = \{(0,0),(1,0)\}$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\,\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{0.9 \times 0.2}{\underbrace{\mathbb{P}(B|A)\,\mathbb{P}(A)}_{0.9 \times 0.2} + \underbrace{\mathbb{P}(B|A^c)\,\mathbb{P}(A^c)}_{0.025 \times 0.8}}$$

$$\mathbb{P}(A \cap B) = \boxed{\mathbb{P}(A|B)}\,\mathbb{P}(B)$$

$$= \boxed{\mathbb{P}(A)}\,\mathbb{P}(B)$$

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

A and B are "independent"

$E_1 = \{\text{first bit has error}\}$

$E_2 = \{\text{2nd bit has error}\}$

$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\,\mathbb{P}(E_2)$$

$$\mathbb{P}(E_2) = \mathbb{P}(E_1)$$

$$E_1 = \{\underbrace{(0,1)}_{A \cap B^c},\ \underbrace{(1,0)}_{A^c \cap B}\} = (A \cap B^c) \cup (A^c \cap B)$$

$$\begin{aligned}
\mathbb{P}(E_1) &= \mathbb{P}(A \cap B^c) + \mathbb{P}(A^c \cap B) \\
&= \mathbb{P}(B^c \mid A)\,\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\,\mathbb{P}(A^c) \\
&= 0.1 \times 0.2 + 0.025 \times 0.8 \\
&= 0.04
\end{aligned}$$

$$\mathbb{P}(E_1 \cap E_2) = \underline{\mathbb{P}(E_1)}\,\underline{\mathbb{P}(E_2)} = (0.04)^2$$

$A_1, A_2, \ldots, A_n$ — independent

$$\mathbb{P}(A_1 \cap A_5 \cap A_{10}) = \mathbb{P}(A_1)\,\mathbb{P}(A_5)\,\mathbb{P}(A_{10})$$

$A \perp\!\!\!\perp B$

$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cdots \cap A_{i_k}) = \prod_{i=1}^{k} \mathbb{P}(A_{i_i})$$ ✓

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2) \cdots \mathbb{P}(A_n)$$ ✓

$A_1 \quad A_2 \quad A_3$

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$$
$$\mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_3)$$
$$\mathbb{P}(A_2 \cap A_3) = \quad \vdots$$
$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \quad \vdots$$

$A = \{ \text{1st die} : 1, 2 \text{ or } 3 \}$ ($\frac{1}{6}$)

$B = \{ \text{1st die} : 2, 3 \text{ or } 6 \}$

$$\frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$$

$C = \{ \text{sum} = 9 \} = \{ \underline{(3,6)}, \underline{(4,5)}, \underline{(5,4)}, \underline{(6,3)} \}$

$$\mathbb{P}(C) = 4 \times \frac{1}{36} = \frac{1}{9}$$

$$\underline{\mathbb{P}(A \cap B \cap C)} = \mathbb{P}(\{(3,6)\}) = \frac{1}{36}$$

$\mathbb{P}(C) = 4 \times \frac{1}{36} = \frac{1}{9}$

$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\{(3,6)\}) = \frac{1}{36}$

$\parallel$

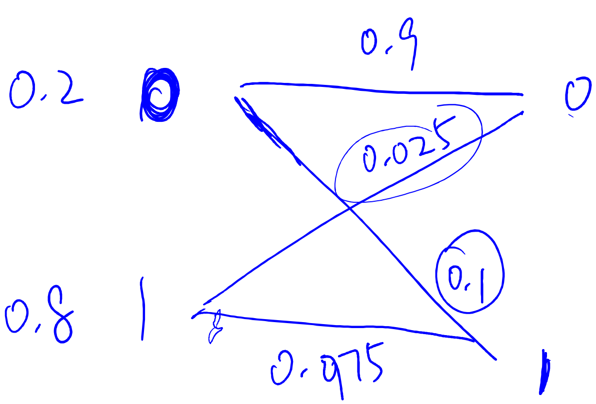
$\mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C) = \frac{1}{9} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{36}$

$\mathbb{P}(A) = \frac{3}{6} = \frac{1}{2}$

$\mathbb{P}(B) = \frac{3}{6} = \frac{1}{2}$

$\mathbb{P}(A \cap B) = \mathbb{P}(\text{1st die is 2 or 3}) = \frac{2}{6} = \frac{1}{3}$

$\mathbb{P}(A)\,\mathbb{P}(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$



$E_1 = \{\text{received bit is in error}\}$

$= \{(0,1), (1,0)\}$

$\mathbb{P}(E_1) = \boxed{\mathbb{P}((0,1))} + \boxed{\mathbb{P}((1,0))}$

$= \mathbb{P}(rx\ 1 \mid tx\ 0)\,\mathbb{P}(tx\ 0) \quad \mathbb{P}(rx\ 0 \mid tx\ 1)\,\mathbb{P}(tx\ 1)$

$= 0.1 \times 0.2$

$\mathbb{P}(+x1)$

$= 0.025 \times 0.8$

$\mathbb{P}(E_2) = \mathbb{P}(E_1)$ $\qquad E_1 \perp\!\!\!\perp E_2$

$E_1 \rightarrow$ 1st bit error

$E_2 \rightarrow$ 2nd bit error

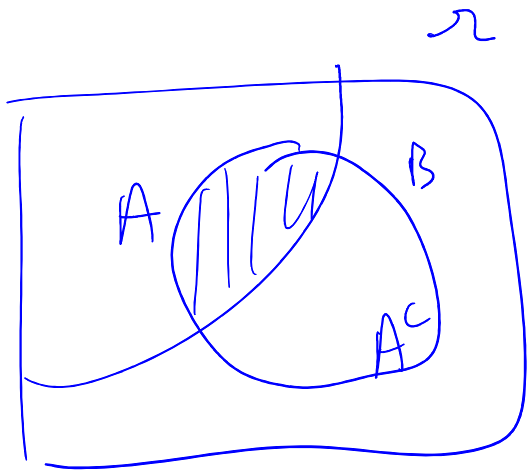
$\mathbb{P}(E_1) = \mathbb{P}(E_2)$

Power set – collection of all possible subsets

$\Omega = \{1, 2\}$ $\quad \mathcal{Y} = \{\phi, \{1\}, \{2\}, \{1,2\}\}$ = power set of Ω

$|\mathcal{Y}| = 2^{|\Omega|}$ $\qquad = \boxed{2^{\Omega}}$

$$\Omega = \{a, b, c\} \qquad 2^{\Omega} = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$



$$\mathbb{P}(B) \qquad \mathbb{P}(B|A) \qquad \mathbb{P}(B|A^c)$$

$$\mathbb{P}(B) = \mathbb{P}(B|A)\,\mathbb{P}(A) + \mathbb{P}(B|A^c)\,\mathbb{P}(A^c)$$

$$\Omega = \{0, 1, 2, 3, \ldots\}$$

$$\mathbb{P}(\#\text{ of packets is } k) = \frac{(\lambda T)^k}{k!} e^{-\lambda T}, \quad \lambda > 0.$$

$$k = 0, 1, 2, \ldots$$