Week 2

Monday, 21 August 2023 10:44 am

$(\underbrace{\Omega}_{\text{sample space}}, \underbrace{\mathcal{F}}_{\text{collection of events, "measurable" sets}}, \mathbb{P})$

$$\boxed{\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}}$$

Coin toss × 3 independent

$A \perp\!\!\!\perp B$: $\mathbb{P}(A\cap B) = \mathbb{P}(A)\mathbb{P}(B)$

$\mathbb{P}(A\cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$

$\mathbb{P}(A|B) = \mathbb{P}(A)$

X : no. of "T" in $\omega \in \Omega$

random variable

	Ω		$X(\omega)$
$\frac{1}{2^3}$	HHH	$\to 0$	$X(\omega) = X(HHH) = 0$
$\frac{1}{2^3}$	HHT	$\to 1$	$X(\omega) = 1$
	HTH	$\to 1$	$X(\omega) = 1$
	THH	$\to 1$	$X(\omega) = 1$
	HTT	$\to 2$	$X(\omega) = 2$
	THT	$\to 2$	$X(\omega) = 2$
	TTH	$\to 2$	$X(\omega) = 2$
	TTT	$\to 3$	$X(\omega) = 3$

$X(\omega)$ ~~"random"~~

$\mathbb{P}(X=0) = \frac{1}{2^3}$

$\mathbb{P}(X=1) = \mathbb{P}(HHT) + \mathbb{P}(HTH) + \mathbb{P}(THH) = \frac{3}{2^3}$

$\mathbb{P}(X=2) = \frac{3}{2^3}$

$\mathbb{P}(X=3) = \frac{1}{2^3}$

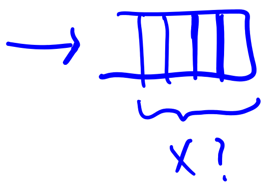
$(\underbrace{\{0,1,2,3\}}_{\Omega_X}, 2^{\Omega_X}, \mathbb{P}_X)$

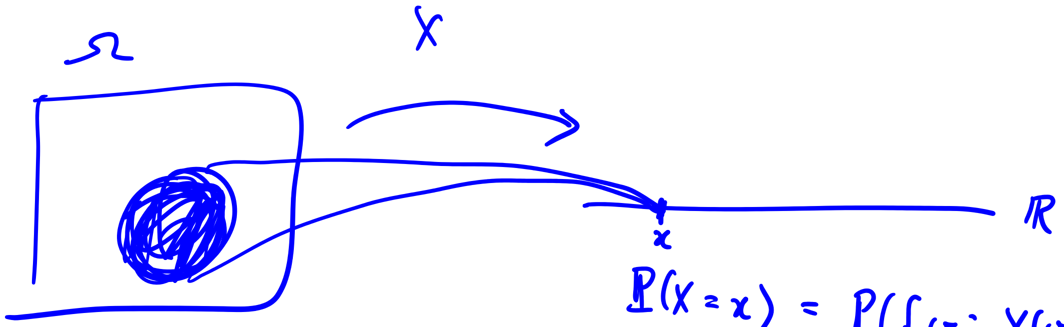
$\boxed{\mathbb{P}_X}(0) = \mathbb{P}(X=0) = \frac{1}{2^3}$

$\mathbb{P}_X(2) = \mathbb{P}(X=2) = \frac{3}{2^3}$

$X(\omega) = \#$ of "T" in ω

$Y(\omega) = \begin{cases} 0 & \text{if no "T" in } \omega \\ 1 & \text{o.w.} \end{cases}$

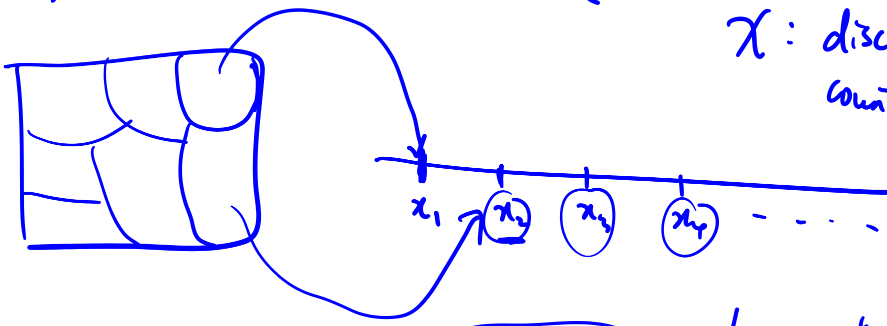




$$\mathbb{P}(X = x) = \mathbb{P}(\{\omega : X(\omega) = x\})$$

$X : \Omega \to \mathbb{R}$ discrete

$\mathcal{X}$: discrete set $\{x_1, x_2, \dots\}$
countable set



$\mathbb{P}$ $\quad P_X$

event

Distribution of X, $\boxed{\mathbb{P}_X(x_k)} = \mathbb{P}(\{X = x_k\})$

$\boxed{P_X(x_k)} = \mathbb{P}(X = x_k)$ r.v. realized value

pmf $\quad X \quad x$

$\boxed{p(x)} \quad \boxed{P_X(x)}$

$$\mathbb{P}(\{X=0\} \cup \{X=1\}) \qquad p(0,1) \quad X$$

$$\mathbb{P}(X \in \{0,1\}) = p(0) + p(1) \qquad p(0)$$

$$\mathbb{P}(X \in A) = \sum_{a \in A} P_X(a) \qquad P_X(1.5) = 0$$

$A = \{0, 1, 1.5, 2.5\}$ $\qquad \mathbb{P}(X \in A) = \sum_{a \in A \cap \mathcal{X}} P_X(a)$

$\mathcal{X}$ = set of values that $X(\omega)$ can take

$(\Omega \quad \mathcal{M} \quad \mathbb{P})$ $\qquad = \text{range}(X)$

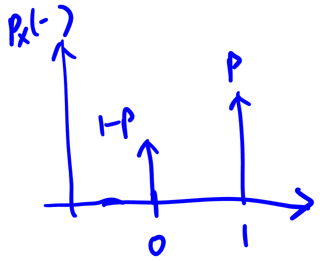
$\mathcal{X}$ = set of values that $X(\omega)$ can take = range(X)

$(\Omega, \boxed{\mathcal{F}}, \mathbb{P})$

Borel sets : events that are generated by open sets

Bernoulli $X \sim \text{Bern}(p)$

$$X(\omega) = \begin{cases} 0 & \text{w.p. } 1-p \\ 1 & \text{w.p. } p \end{cases}$$



independent tosses

H	p
TH	$(1-p)p$
TTH	$(1-p)^2 p$
$\underbrace{TT\cdots T}_{k-1} H$	$(1-p)^{k-1} p$

Geometric $X \sim \text{Geom}(p)$

Binomial $X \sim \text{Bin}(n, p)$

tossing n coins, independent, H w.p. p $\quad X$ = total no of heads

n Bernoulli r.v. Bern(p) $\quad X = \sum_{i=1}^{n} X_i$

$X_1, X_2, \ldots, X_n$

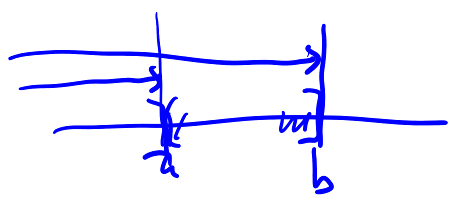
$X \sim \text{Poisson}(\lambda) \quad p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots$

$X_n \sim \text{Bin}(n, p) \quad p = \frac{\lambda}{n}$

$$p_{X_n}(k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \xrightarrow{n \to \infty} \frac{\lambda^k}{k!} e^{-\lambda}$$

Poisson

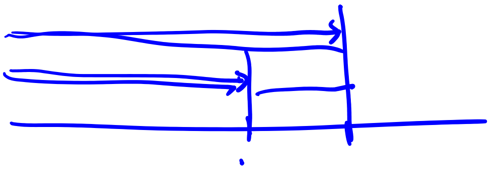
CDF

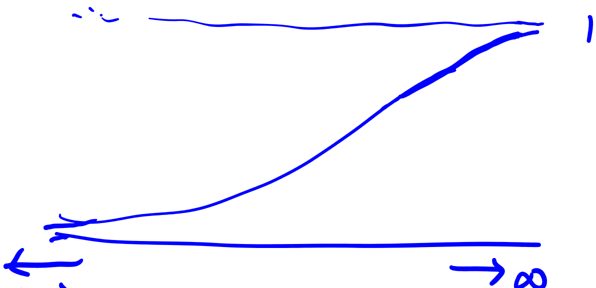


$F(x) \quad F_X(x) = \mathbb{P}(X \leq x)$

$$\mathbb{P}(X \in (a, b]) = \mathbb{P}(a < X \leq b) = \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a)$$

$$\mathbb{P}(X \in (a,b]) = \mathbb{P}(a < X \le b) = \mathbb{P}(X \le b) - \mathbb{P}(X \le a)$$
$$= F(b) - F(a)$$

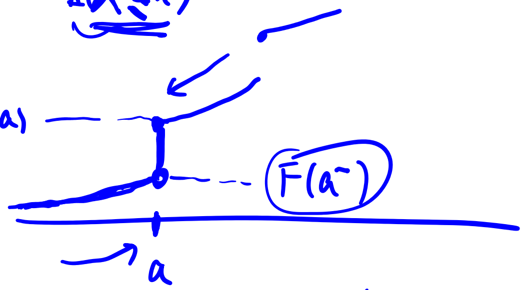




$x \to -\infty$ $\qquad$ $F(x) \to 0$

$F(x) \to 1$ $\qquad$ $x \to \infty$

$F(x) = \mathbb{P}(X \le x)$



$a = 10$

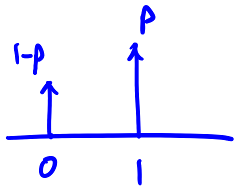
$9.9999999\ldots$

a^-

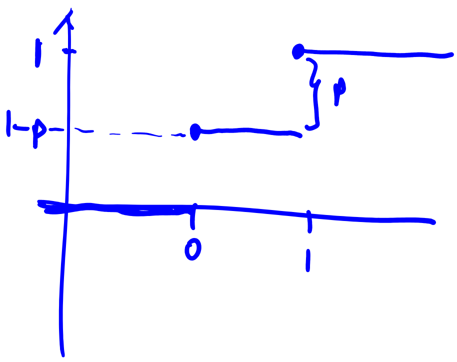
$$F(a) - F(a^-)$$

$$\lim_{x \to a^-} F(x)$$

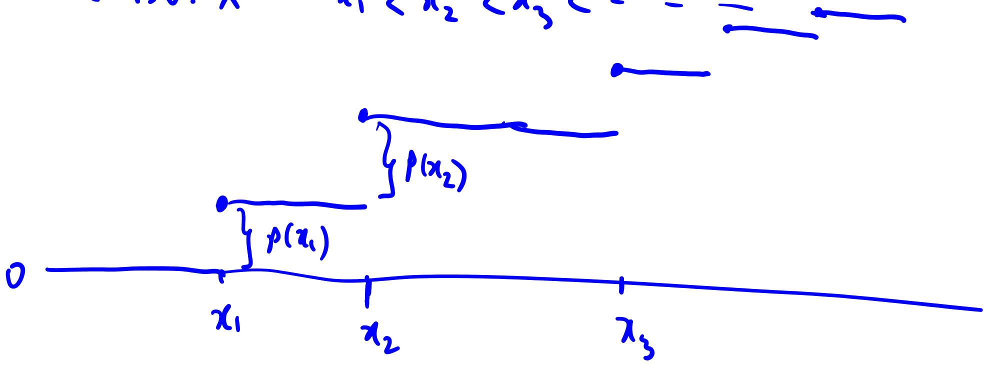
Bernoulli



$F(x) = \mathbb{P}(X \le x)$



Discrete r.v. X $\qquad x_1 < x_2 < x_3 < \cdots$



If $F(x)$ is differentiable (i.e., $F'(x) = \frac{d}{dx}F(x)$ exists)

If $F(x)$ is differentiable (i.e., $F'(x) = \frac{d}{dx}F(x)$ exists) at every x,

$$= \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

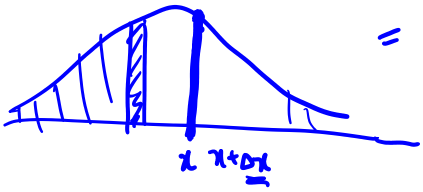
pdf $f(x) = f_X(x) = \frac{d}{dx}F(x)$, X is a continuous r.v.

$$\int_{-\infty}^{a} f(x)\,dx = F(a)$$

$$\int_a^b f(x)\,dx = F(b) - F(a) = \mathbb{P}(X \in (a,b])$$

$$\int_A f(x)\,dx = \mathbb{P}(X \in A)$$

$f(x)$

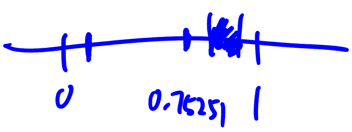


$\rightarrow \int_x^{x+\Delta x} f(u)\,du = \mathbb{P}(X \in [x, x+\Delta x])$

pdf $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

0.314526...

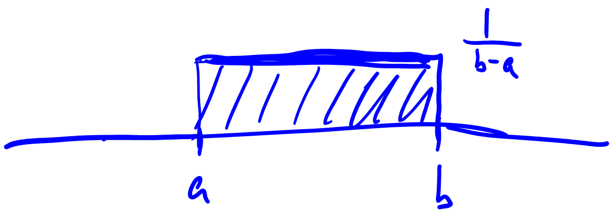


$\mathbb{P}(X = x) \;\cancel{=}\; f(x)$?

$\parallel$

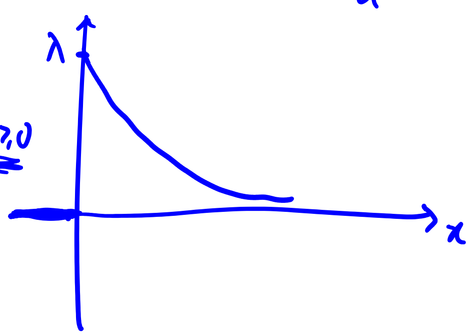
0

Uniform $X \sim \text{Unif}[a,b]$



Exp. $X \sim \text{Exp}(\lambda)$

$$= \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$



$\overbrace{\mathbb{P}(X \geq 20 \mid X \geq 10)}^{X\text{ - service}} = \mathbb{P}(\{X \geq 20\} \cap \{X \geq 10\})$

$$\boxed{\mathbb{P}(\underbrace{X \geq 20}_{A} \mid \underbrace{X \geq 10}_{B})} = \frac{\mathbb{P}(\{X \geq 20\} \cap \{X \geq 10\})}{\mathbb{P}(X \geq 10)}$$

$\lambda = 0.1$

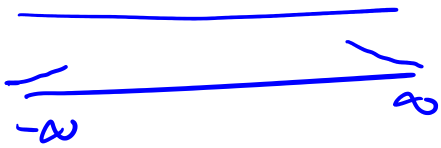
$$= \frac{\mathbb{P}(X \geq 20)}{\mathbb{P}(X \geq 10)} = \frac{e^{-\lambda \cdot 20}}{e^{-\lambda \cdot 10}} = \boxed{e^{-\lambda \cdot 10}}$$

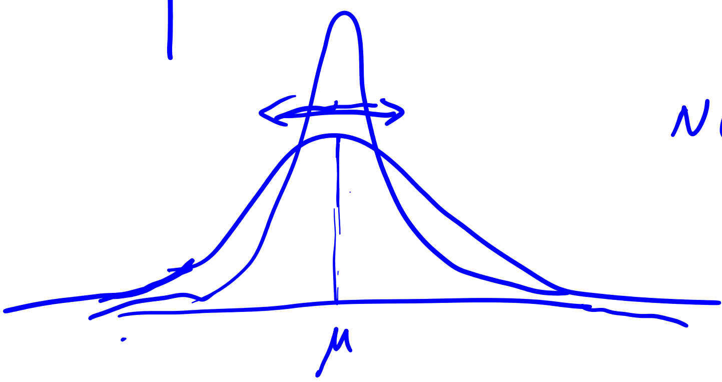
$$= e^{-1}$$

$$\mathbb{P}(X \geq x) = \int_x^\infty f(u)\,du = \int_x^\infty \lambda e^{-\lambda u}\,du \quad \boxed{= \mathbb{P}(X \geq 10)}$$

$$= -e^{-\lambda u}\Big|_x^\infty = e^{-\lambda x}$$

$$\mathbb{P}(X \geq a \mid X \geq b) = \mathbb{P}(X \geq a-b) = e^{-\lambda(a-b)}$$ memoryless.



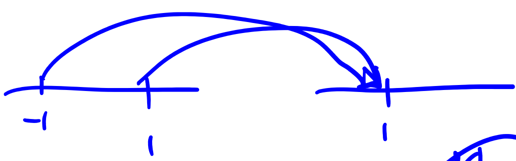


$N(\mu, \sigma^2)$

$X \qquad Y = g(X)$

$X \quad Y = X^2$

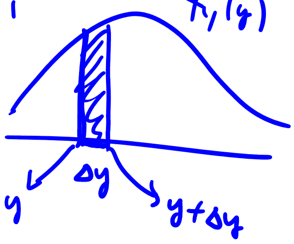
$X = -1, 1 \longrightarrow Y = X^2 = 1$



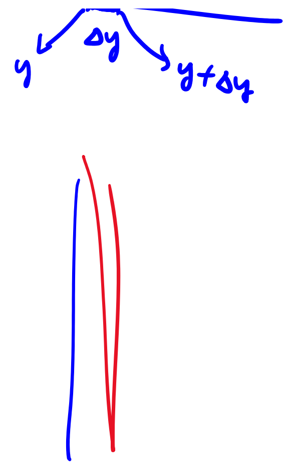
$Y = aX + b$

$a > 0$

$X \sim f_X(x)$, $f_Y(y)$?



$f_Y(u)\,\Delta u \sim \mathbb{P}(u < Y < u + \dots)$



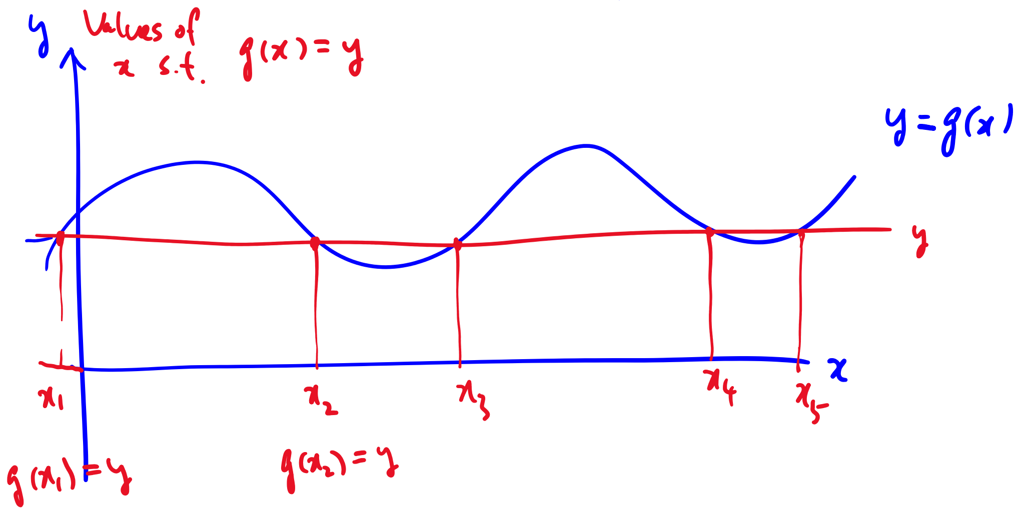
$$f_Y(y)\,\Delta y \simeq \mathbb{P}(y < Y < y+\Delta y)$$

$$= \mathbb{P}(y < aX+b < y+\Delta y)$$

$$= \mathbb{P}\left(\frac{y-b}{a} < X < \frac{y-b}{a} + \boxed{\frac{\Delta y}{a}}\right)$$

$$\simeq f_X\left(\frac{y-b}{a}\right)\frac{\Delta y}{a}$$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$



$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} + \dots + \frac{f_X(x_5)}{|g'(x_5)|}$$

$Y = X^2$ $\quad g(x) = x^2 \quad g'(x) = 2x$

$y = g(x)$

$\boxed{y = x^2}$

$x = \pm\sqrt{y}$ $\quad x_1 = -\sqrt{y}$

$x_2 = +\sqrt{y}$

$$f_Y(y) = \frac{f_X(-\sqrt{y})}{|2(-\sqrt{y})|} + \frac{f_X(+\sqrt{y})}{|2(+\sqrt{y})|}$$

$$= \frac{1}{2\sqrt{y}}\left(f_X(-\sqrt{y}) + f_X(\sqrt{y})\right)$$

$$= \frac{1}{2\sqrt{y}} \left(f_X(-\sqrt{y}) + f_X(\sqrt{y}) \right)$$

$X \sim N(\mu, \sigma^2) \qquad Y = aX + b$

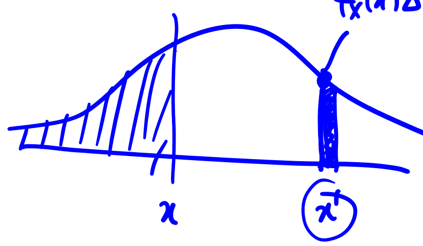
$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{|a|} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2} \right)$$

$$= \frac{1}{\sqrt{2\pi (a\sigma)^2}} \exp\left(- \frac{1}{2a^2\sigma^2} \left(y - (a\mu + b) \right)^2 \right)$$

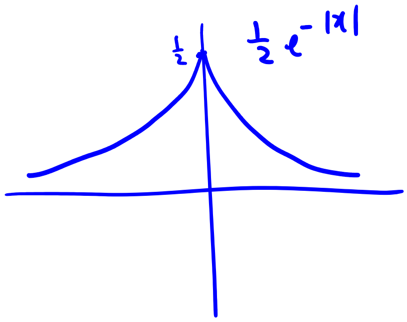
$$\sim N(a\mu + b, a^2\sigma^2)$$

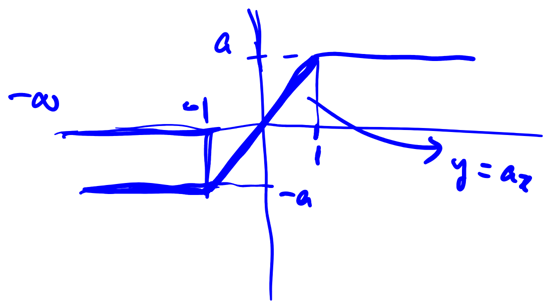
$F(x) = \mathbb{P}(X \leq x)$



$F_Y(y) = \mathbb{P}(Y \leq y)$

$= \mathbb{P}(g(X) \leq y) = \mathbb{P}(\{x : g(x) \leq y\})$





$y < -a$

$F_Y(y) = \mathbb{P}(Y \leq y) = 0$

$y = -a$

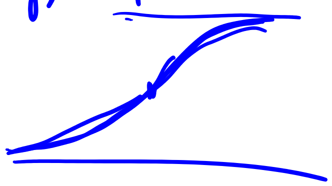
$$F_Y(-a) = \int_{-\infty}^{-1} \frac{1}{2} e^{x} \, dx = \frac{e^{-1}}{2}$$

$-a < y < a$:

$-a < y < a$:

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(aX \le y)$$
$$= \mathbb{P}\left(X \le \frac{y}{a}\right)$$
$$= \int_{-\infty}^{\frac{y}{a}} \frac{1}{2} e^{-|x|}\, dx = \dots$$

$y \geq a \quad F_Y(y) = \mathbb{P}(Y \le y) = \underline{1}$



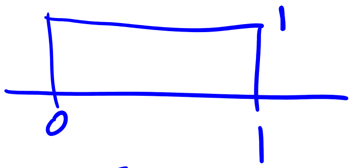
$X \sim F_X(x)$

$Y = F_X(X)$ $\qquad\qquad \underline{F_X(x)} = \underline{\mathbb{P}(X \le x)}$

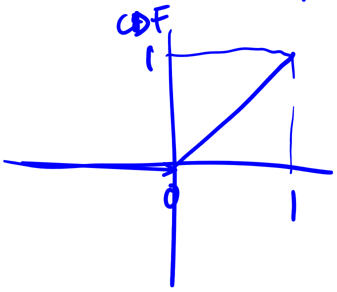
$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\underline{F_X(X)} \le y)$$
$$= \underline{\mathbb{P}(X \le F_X^{-1}(y))}$$
$$= F_X(F_X^{-1}(y))$$

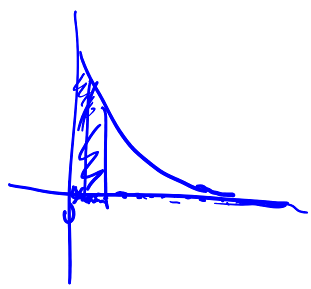
$\boxed{F_Y(y) = y}$ $\Rightarrow$ Y is Unif$[0,1]$ r.v.

$U \sim \text{Unif}[0,1]$



$$\underline{F_U(u)} = \mathbb{P}(U \le u) = \int_0^u 1\, dx = \underline{u}$$





$X \sim \underline{F_X(x)}$, $\underline{Y} = F_X(X) \sim \underline{\text{Unif}[0,1]}$

$U \sim \text{Unif}[0,1] \qquad F_X^{-1}(U) \to X$

$\boxed{U \sim \text{Unif}[0,1] \qquad F_X^{-1}(U) \to X}$

"pseudorandom"

$\mathbb{P}(U \leq u) = u$

$\mathbb{P}(\boxed{F_X^{-1}(u)} \leq x) \; \cdots\cdots = F_X(x)$

$= \mathbb{P}(U \leq \boxed{F_X(x)}) = F_X(x)$

$F_X(x) = \mathbb{P}(X \leq x) = \int_0^x \lambda e^{-\lambda y}\, dy = -e^{\lambda y}\Big|_{y=0}^{y=x}$

$= 1 - e^{-\lambda x}$

$u = 1 - e^{-\lambda x}$

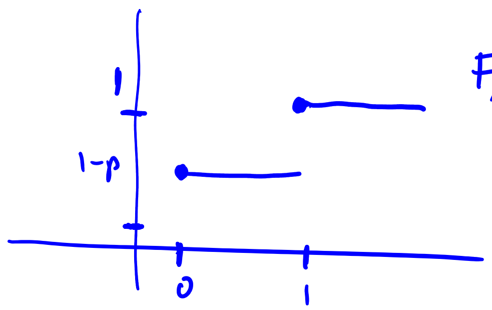
$e^{-\lambda x} = 1 - u$

$x = -\frac{1}{\lambda}\log(1-u) = F_X^{-1}(u)$

python.rand() $\quad 0.15 \quad 0.8 \quad 0.45$

$-\frac{1}{\lambda}\log(1-0.15) \qquad -\frac{1}{\lambda}\log(1-0.8) \quad - - -$

Bernoulli: $\quad X \sim \text{Bern}(p)$



$F_X(\cdot)$

$F_X(1) = 1$

$F_X(0) = 1-p$

$0 < u \leq 1-p \quad$ Find the smallest x s.t. $F_X(x) \geq u$

$X = F_X^{-1}(u) = 0$

$1-p < u \leq 1 \quad X = F_X^{-1}(u) = 1$

w.p. $(1-p)$

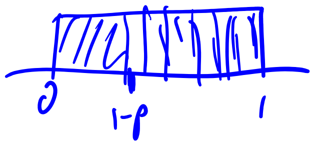
$X = \begin{cases} 0 & \text{if} \quad 0 < u \leq 1-p \end{cases}$

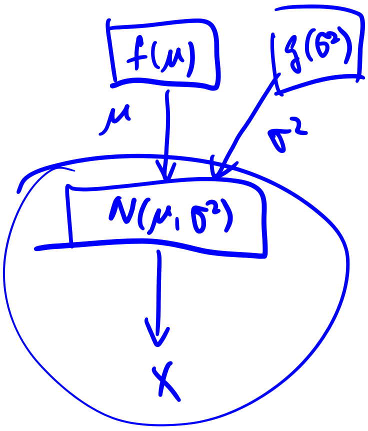
$u \sim \text{Unif}[0,1]$

$$X = \begin{cases} 0 & \text{if } \quad 0 < u \le 1-p \\ 1 & \text{if } \quad 1-p < u \le 1 \end{cases}$$

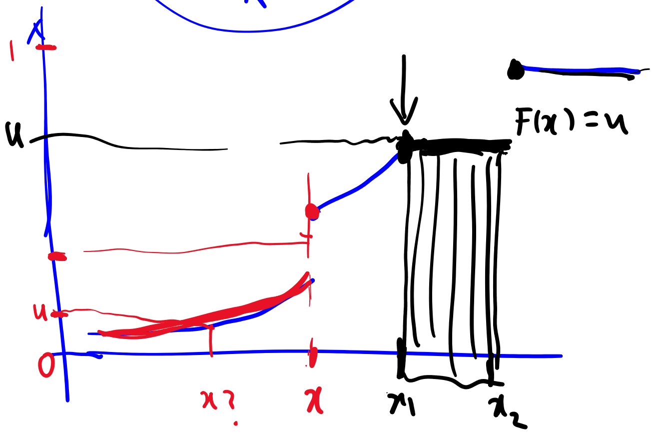
w.p. p

$u \sim \mathrm{Unif}[0,1]$





hierarchical distribution



pseudoinverse

$$F^{-1}(u) = \inf_x \{ x : F(x) \geq u \}$$

$$F(x_2) = F(x_1)$$

$$P(x_1 < X \leq x_2) = F(x_2) - F(x_1) = 0$$