EE7401 Probability and Random Processes Practice

1.

Two discrete random variables X and Y have possible values of 4, 5 and 6. Table 1 specifies one of their probability-mass functions (PMFs): $p_{X,Y}(x,y)$, $p_{Y|X}(y|x)$ or $p_{X|Y}(x|y)$.

$$\sum_{x,y} P(x,y) = 1$$

$$a, b, c \geq 0$$

Table 1

y \ x	4	5	6
4	***a***	**0.1**	**0.5**
5	**0.2**	***b***	**0.4**
6	**0.3**	**0.2**	***c***

Give the conditions (values of a, b, and c), under which the Table 1 specifies PMFs, $p_{Y|X}(y|x)$, $p_{X|Y}(x|y)$ and $p_{X,Y}(x,y)$, respectively.

$P_{Y|X}(y|x)$: $a=0.5$, $b=0.7$, $c=0.1$

$P_{X|Y}(x|y)$: $a=0.4$, $b=0.4$, $c=0.5$

$P_{X,Y}(x,y)$: not possible.

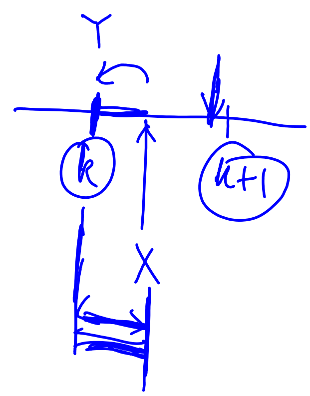
$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$

floor

2. Let $X \sim \text{Exp}(\lambda)$ be an exponential random variable with parameter λ and $Y = \lfloor X \rfloor$ be the integer part of X, i.e., $\boxed{Y = k}$ for $k \leq X < k+1$, $k = 0, 1, \ldots$.

(i) Find the pmf of Y.

(ii) Let $Z = X - Y$ be the quantization error. Find the pdf of Z.



(i) $p_Y(k) = \mathbb{P}(Y = k), \quad k = 0, 1, 2, \ldots$

$$= \mathbb{P}(k \leq X < k+1)$$

$$= \int_k^{k+1} f_X(x)\,dx$$

$$= \int_k^{k+1} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_k^{k+1}$$

total prob. law $\Omega = \bigcup_{k \geq 0} A_k$ $\quad = e^{-\lambda k} - e^{-\lambda(k+1)}$

$\mathbb{P}(B) = \sum_{k=0}^{\infty} \mathbb{P}(B \cap A_k) \quad = e^{-\lambda k}(1 - e^{-\lambda})$

(ii) $\boxed{\text{cdf} \longrightarrow \text{pdf}}$ $\quad f_Z(z) = \frac{d}{dz}\left(\frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}\right) = \begin{cases} \frac{\lambda e^{-\lambda z}}{1 - e^{-\lambda}}, & 0 \leq z < 1 \\ 0, & \text{o.w.} \end{cases}$

$\boxed{0 \leq z < 1}$

$\boxed{\mathbb{P}(Z \leq z)} = \boxed{\mathbb{P}(X - Y \leq z)}$

$0 \leq z < 1$

$\boxed{Y = \lfloor X \rfloor}$

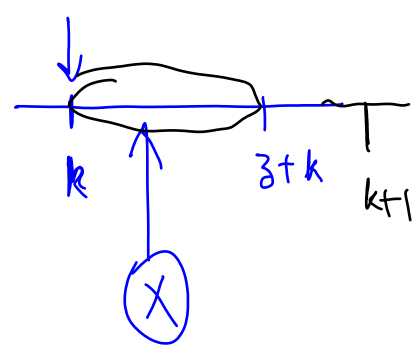
$$= \sum_{k=0}^{\infty} \mathbb{P}(\underbrace{X - Y \leq z}_{B}, \underbrace{Y = k}_{A_k})$$

$X - k \leq z$

$$= \sum_{k=0}^{\infty} \mathbb{P}(X \leq z + k, Y = k)$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(k \leq X \leq z + k)$$

$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ $\qquad \int_k^{z+k} \lambda e^{-\lambda x}\,dx = e^{-\lambda k}(1 - e^{-\lambda z})$



$$= \left(\sum_{k=0}^{\infty} e^{-\lambda k}\right)(1-e^{-\lambda z}) = \frac{1}{1-e^{-\lambda}}(1-e^{-\lambda z})$$

3.

Two continuous random variables X and Y have joint probability density function (pdf)

$$p_{X,Y}(x,y) = \lambda y^2 e^{-xy}, \quad x \ge 0,\ 1 \le y \le 5,$$

where λ is a constant.

given Y $\quad \mathbb{E}[O \mid Y]$

MMSE

(a) Express the minimal mean square error estimation of X as a function of the random variable Y. If we observed that Y=2, what is the estimate of X that has the minimal mean square error?

MMSE of Y $\quad \mathbb{E}[Y]$

(b) For unknown value of X, find the estimate of Y that has the minimal mean square error. What is the mean square error of the estimate?

$\mathbb{E}[(\hat{Y}-Y)^2]$ $\to$ minimizer $= \mathbb{E}[Y]$

$x \ge 0,\ 1 \le y \le 5$

(a) $\mathbb{E}[X \mid Y]$

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} = \frac{\lambda y^2 e^{-xy}}{\lambda y} = y e^{-xy} \quad [\text{Note: } \mathrm{Exp}(y)]$$

$1 \le y \le 5$

$$p_Y(y) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dx = \int_0^{\infty} \lambda y^2 e^{-xy}\,dx = \lambda y^2 \left[-\frac{e^{-xy}}{y}\right]_0^{\infty} = \lambda y$$

$$\mathbb{E}[X \mid Y=y] = \int_0^{\infty} x\, p_{X|Y}(x|y)\,dx = \int_0^{\infty} x y e^{-xy}\,dx = y \int_0^{\infty} x e^{-xy}\,dx$$

$$\left. -\frac{1}{y^2} e^{-xy} \right|_{x=0}^{x=\infty}$$

integration by parts

$$= y\left(\left.-\frac{x e^{-xy}}{y}\right|_{x=0}^{x=\infty} + \int_0^{\infty} \frac{e^{-xy}}{y}\,dx\right) = y \cdot \frac{1}{y^2} = \frac{1}{y}$$

$Y=2$

$$\hat{X}_{MMSE} = \mathbb{E}[X \mid Y=2] = \frac{1}{2}$$

$1 \le y \le 5$, $P_Y(y) = \lambda y$

$$\int_1^5 \lambda y \, dy = 1 \Rightarrow \lambda \frac{y^2}{2}\Big|_1^5 = 1$$

$$\lambda = \frac{1}{12}$$

$$\mathbb{E}[Y] = \int_1^5 y \cdot \lambda y \, dy = \lambda \left(\frac{y^3}{3} \Big|_1^5 \right) = \frac{1}{12} \cdot \bigcirc$$

$$= \frac{31}{9}$$

$$MSE = var(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$= \lambda \int_1^5 y^3 \, dy - \left(\frac{31}{9}\right)^2$$

$$\simeq 1.136$$

MSE : mean squared error $= \mathbb{E}[(\hat{X} - X)^2]$

MMSE : minimum mean squared error = minimum MSE

$\rightarrow$ when $\hat{X} = \mathbb{E}[X \mid Y]$

MMSE estimator

MMSE : minimum MSE estimate