1.a) Per bit error rate = probability of a bit being in error
$= 10^{-8}$

Now, for a binary channel, we define

$$X_i = \begin{cases} 1, & \text{if ith is in error} \\ 0, & \text{if it is not} \end{cases}$$

Note that for a chain of length $B = 1000$
number of total erroneous bits in the chain $= \sum_{i=1}^{1000} \{X_i = 1\}$

$$\therefore E[N] = E\left[\sum_{i=1}^{1000} X_i\right] = \sum_{i=1}^{1000} E(X_i)$$

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = 10^{-8}$$

$$\therefore E[N] = 1000 \times 10^{-8} = 10^{-5} = 0.00001$$

1b) The Markov Inequality states that for any non-negative random variable X and any $a > 0$,

$$P(X \geq a) \leq \frac{E(X)}{a}$$

In this case, we are looking for an upper bound on the probability that a block of 1000 bits has 10 or more erroneous bits. Therefore, X is the number of erroneous bits in a block of 1000 bits, and $a = 10$.

Plugging in the values

$$P(X \geq 10) \leq \frac{10^{-5}}{10}$$

$$\therefore P(X \geq 10) \leq 10^{-6}$$

So, using the Markov inequality, we find that the upper bound on the probability that a block of 1000 bits has 10 or more erroneus bits is 10^{-6}.

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2) We have a random variable X which takes values 1 and -1 with probability $\frac{1}{2}$ each, and noise Σ which is uniformly distributed in the range $[-2,2]$. The sum of X and Z is observed as another random variable Y, such that $Y = X + Z$

2 a) $\boxed{\text{Conditional PMF}}$ $P_{X|Y}(x|y)$

Using Bayes' theorem, we have

$$P_{X|Y}(x|y) = \frac{P_{Y|X}(y|x)\, P_X(x)}{P_Y(y)}$$

Given x, $Y = X + Z$, so $Z = Y - X$. Given a value for x, the distribution of Z is uniform over $[-2,2]$, so the conditional pdf $P_{Y|X}(y|x)$ is:

$P_{Y|X}(y|x) = \frac{1}{4}$ for $y-x \in [-2,2]$ and 0 otherwise

$P_X(x)$ is $\frac{1}{2}$ for $x=1$ or $x=-1$

for $P_Y(y) = \sum_x P_{Y|X}(y|x)\, P_X(x)$

$$P_Y(y) = \frac{1}{4}\left(1_{[-3,1]}(y) + 1_{[-1,3]}(y)\right)$$

$P_Y(y) = \frac{1}{4}$ for $y \in [-3,3]$ and 0 otherwise.

So $P_{X|Y}(x|y) = \dfrac{P_{Y|X}(y|x)\, P_X(x)}{P_Y(y)} = \dfrac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{1}{4}} = \frac{1}{2}$ for the appropri ranges of y given x.

$\boxed{\text{MMSE of X given Y and its MSE}}$

The MMSE estimator is given by:

$$\hat{X}_{MMSE} = E[X|Y=y] = \sum_x x \cdot P_{X|Y}(x|y) = 1\cdot\frac{1}{2} + (-1)\cdot\frac{1}{2} = 0$$

The MMSE is 0, which means we would always estimate X to given any Y. This is not a valid value for X, but it minimiz the mean square error

To find the MSE, we compute :

$$MSE_{MMSE} = E\left[(X-\hat{x}_{MMSE})^2\right] = E[X^2] = \left[\frac{1}{2}(1^2+(-0)^2)\right] = \frac{1}{2}$$

(2b) MAP Decoder and its Probability of Error

The MAP decision rule is : $\begin{cases} 1 & \text{if } P_{X|Y}(1|y) > P_{X|Y}(-1|y) \\ -1 & \text{otherwise} \end{cases}$

Since $P_{X|Y}(1|y)$ and $P_{X|Y}(-1|y)$ are both $\frac{1}{2}$, the MAP decoder can be based on whether y is closer to 1 or -1. If $y>0$, decide $X=1$ if $y<0$, decide $X=-1$

The probability of error is when $Y>0$ but $X=-1$, or when $y<0$ but $X=1$. Each of these happens with probability $\frac{1}{4}$ so the total probability of error is $\frac{1}{2}$.

Comparing MAP decoder's MSE to the minimum MSE

The MAP decoder's MSE is given by:

$$MSE_{MAP} = E\left[(X-\hat{X}_{MAP})^2\right]$$

Given our decision rule, we have $\hat{X}_{MAP} = \text{sign}(Y)$, so

$$MSE_{MAP} = E\left[(X-\text{sign}(Y))^2\right]$$

$$MSE_{MAP} = \frac{1}{2}\int_{-3}^{0}(1+1)^2\frac{1}{4}\,dy + \frac{1}{2}\int_{0}^{3}(1-1)^2\frac{1}{4}\,dy = \frac{1}{2}$$

The MSE of the MAP decoder is the same as the minimum MSE, which is $\frac{1}{2}$.