EE7401 Probability and Random Processes

Nurfitri Anbarsanti G2104045K

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1 Monty Hall game show

The gold bars are placed in three boxes A, B, or C. The participant is asked to choose a box. The host of the game, Monty Hall, opens an empty box that has not been selected. The participant has a choice to switch to the third box or not.

1.1 Monty Hall Problem's Sample Space

The sample space for this problem is $\Omega = \{(A, A), (A, B), (A, C), (B, B), (B, C), (C, C)\}$

The first element in each pair is the participant's initial choice and the second element is the actual location of the gold bar.

1.2 Assumption and Pre-Calculation

I assume that the Monty's Hall choice is random so that Monty Hall could choose the box containing the gold bar. There is no certainty that Monty Hall would choose the empty box.

In this case,

$P(\text{Gold in box i}) = \frac{1}{3}$ for every $i = A, B, C$

At first, the participant is assumed that he/she has chosen box A.

$P(\text{Gold in A} \cap \text{host choose B}) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$P(\text{Gold in A} \cap \text{host choose C}) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$P(\text{Gold in B} \cap \text{host choose B}) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$P(\text{Gold in B} \cap \text{host choose C}) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$P(\text{Gold in C} \cap \text{host choose B}) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$P(\text{Gold in C} \cap \text{host choose B}) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

1.3 The probability of winning the gold bar if the contestant decides to keep

At first, the participant is assumed that he/she has chosen box A. The host's choice is random so the host could choose the box containing the gold bar. There is no certainty that the host would choose the empty box.

$$\begin{aligned} P(\text{keep and win}) &= P(\text{Gold in A}|(\text{host choose B} \cup \text{host choose C})) \\ &= \tfrac{P(\text{Gold in A } \cap (\text{host choose B } \cup \text{ host choose C})}{P(\text{host choose B } \cup \text{ host choose C})} \ ... \ (1) \end{aligned}$$

Using Bayes' Theorem,

$$\begin{aligned} P(\text{host choose B}) &= P(\text{host choose B} \mid \text{gold in A})P(\text{gold in A}) \\ &\quad + P(\text{host choose B} \mid \text{gold in B})P(\text{gold in B}) \\ &\quad + P(\text{host choose B} \mid \text{gold in C})P(\text{gold in C}) \\ &= (\tfrac{1}{2} \times \tfrac{1}{3}) + (\tfrac{1}{2} \times \tfrac{1}{3}) + (\tfrac{1}{2} \times \tfrac{1}{3}) \\ &= \tfrac{1}{2} \end{aligned}$$

$$\begin{aligned} P(\text{host choose C}) &= P(\text{host choose C} \mid \text{gold in A})P(\text{gold in A}) \\ &\quad + P(\text{host choose C} \mid \text{gold in B})P(\text{gold in B}) \\ &\quad + P(\text{host choose C} \mid \text{gold in C})P(\text{gold in C}) \\ &= (\tfrac{1}{2} \times \tfrac{1}{3}) + (\tfrac{1}{2} \times \tfrac{1}{3}) + (\tfrac{1}{2} \times \tfrac{1}{3}) \\ &= \tfrac{1}{2} \end{aligned}$$

Thus,

$P(\text{host choose B} \cup \text{host choose C})) = \frac{1}{2} + \frac{1}{2} = 1$

Back to Equation (1),

$$\begin{aligned} P(\text{keep and win}) &= \tfrac{P(\text{Gold in A } \cap (\text{host choose B } \cup \text{ host choose C})}{P(\text{host choose B } \cup \text{ host choose C})} \ ... \ (1) \\ &= \tfrac{1/3 \times 1}{1} \\ &= 1/3 \end{aligned}$$

In conclusion,

$$\begin{aligned} \therefore \ P(\text{keep and win}) &= P(\text{Gold in A}|(\text{host choose B} \cup \text{host choose C})) \\ &= P(\text{Gold in A}) \\ &= 1/3. \end{aligned}$$

Besides, this indicates that Gold in A and the host's choice are independent.

1.4 The probability of winning the gold bar if the contestant decides to switch

At first, the participant is assumed that he/she has chosen box A. The host's choice is random so the host could choose the box containing the gold bar. There is no certainty that the host would choose the empty box.

$$
\begin{aligned}
P(\text{switch and win}) &= P(\text{Gold in B or C } |(\text{ host choose B or C}) \\
&= \frac{P((\text{Gold in B or C }) \cap (\text{ host choose B or C}))}{P(\text{host choose B or C})} \text{ ... } (2)
\end{aligned}
$$

From subsection 1.4,

$P(\text{host choose B or C}) = P(\text{host choose B} \cup \text{host choose C})) = \frac{1}{2} + \frac{1}{2} = 1$

$P(\text{Gold in B or C }) = 2/3$

Back to Equation (2),

$$
\begin{aligned}
P(\text{switch and win}) &= \frac{P((\text{Gold in B or C }) \cap (\text{ host choose B or C}))}{P(\text{host choose B or C})} \\
&= \frac{2/3 \times 1}{1} \\
&= 2/3
\end{aligned}
$$

In conclusion,

$$
\begin{aligned}
\therefore\ P(\text{switch and win}) &= P(\text{Gold in B or C}|(\text{host choose B} \cup \text{host choose C})) \\
&= P(\text{Gold in B or C}) \\
&= 2/3.
\end{aligned}
$$

Besides, this indicates that the location of the gold bar and the host's choice are independent.

1.5 The probability of winning the gold bar if the contestant decides to switch if there are n boxes in total and the host reveals ANY p BOXES

We assumed that there are n boxes, i.e. $\{b_1, b_2, b_3, ..., b_n\}$ and there are p boxes revealed by the host.

At first, the participant is assumed that he/she has chosen box b_1. The host's choice is random so the host could choose the box containing the gold bar. There is no certainty that the host would choose the empty box.

$$\begin{aligned} P(\text{switch and win}) &= P(\text{gold in box other than } b_1 \mid \text{host reveals p boxes}) \\ &= \frac{P(\text{gold in box other than } b_1 \cap \text{host reveals p boxes})}{P(\text{host reveals p boxes}} \text{ .... (3)} \end{aligned}$$

$P(\text{gold in } b_1) = 1/n$

$P(\text{gold in box other than } b_1) = 1/(n-1)$

$P(\text{host reveals p boxes}) = C_p^{n-1} = \frac{(n-1)!}{p!(n-p-1)!}$

$P(\text{gold in box other than } b_1 \ \cap \ \text{host reveals p boxes}) = \frac{1}{(n-1)} \times \frac{(n-1)!}{p!(n-p-1)!}$

Back to Equation (3),

$$\begin{aligned} P(\text{switch and win}) &= \frac{P(\text{gold in box other than } b_1 \cap \text{host reveals p boxes})}{P(\text{host reveals p boxes}} \\ &= \frac{\frac{1}{(n-1)} \times \frac{(n-1)!}{p!(n-p-1)!}}{\frac{(n-1)!}{p!(n-p-1)!}} \end{aligned}$$

$P(\text{switch and win}) = \frac{1}{(n-1)}$

$$\begin{aligned} \therefore \ P(\text{switch and win}) &= P(\text{gold in box other than } b_1 \ ) \\ &= \frac{1}{(n-1)} \end{aligned}$$

Besides, this indicates that the location of the gold bar and the host's choice are independent.

However, the results will be different if the host intentionally reveals **empty boxes** that will be calculated in the next subsection.

1.6 The probability of winning the gold bar if the contestant decides to switch if there are n boxes in total and the host reveals p EMPTY BOXES

We assumed that there are n boxes, i.e. $\{b_1, b_2, b_3, ..., b_n\}$ and there are p boxes revealed by the host.

At first, the participant is assumed that he/she has chosen box b_1. The host will choice the empty boxes so the host could not choose the box containing the gold bar. There is a certainty that the host would choose the empty box.

$P(\text{switch and win}) = P(\text{gold in box other than } b_1 \mid \text{host reveals p empty boxes})$

$$= \frac{P(\text{gold in box other than } b_1 \cap \text{host reveals p empty boxes})}{P(\text{host reveals p empty boxes}} \quad .... \ (4)$$

$P(\text{gold in } b_1) = 1/n$

$P(\text{gold in box other than } b_1) = 1/(n-1)$

$$P(\text{gold in box other than } b_1 \ \cap \ \text{host reveals p empty boxes}) = \frac{1}{(n-1)} \times C_p^{n-2}$$
$$= \frac{1}{(n-1)} \times \frac{(n-2)!}{p!(n-p-2)!}$$

Using Bayes' Theorem,

$P(\text{host reveals p empty boxes})$
$= P(\text{host reveals p empty boxes} \mid \text{gold in } b_1)P(\text{gold in } b_1)$
$\quad +P(\text{host reveals p empty boxes} \mid \text{gold in box other than } b_1)P(\text{gold in box other than } b_1)$
$= \{C_p^{n-1} \times \frac{1}{n}\} + \{C_p^{n-2} \times \frac{1}{(n-1)}\}$

$= \{\frac{(n-1)!}{p!(n-p-1)!} \times \frac{1}{n}\} + \{\frac{(n-2)!}{p!(n-p-2)!} \times \frac{1}{(n-1)}\}$

$= \{\frac{(n-1)(n-2)!}{p!(n-p-1)(n-p-2)!} \times \frac{1}{n}\} + \{\frac{(n-2)!}{p!(n-p-2)!} \times \frac{1}{(n-1)}\}$

$= \frac{(n-2)!}{p!(n-p-2)!}\{\frac{(n-1)}{n(n-p-1)} + \frac{1}{(n-1)}\}$

$= \frac{(n-2)!}{p!(n-p-2)!}\{\frac{(2n^2-n(p+2)-n+1)}{n(n-1)(n-p-1)}\}$

Back to Equation (4),

$$P(\text{switch and win}) = \frac{P(\text{gold in box other than } b_1 \cap \text{host reveals p empty boxes})}{P(\text{host reveals p empty boxes}}$$

$$= \frac{\frac{1}{(n-1)} \times \frac{(n-2)!}{p!(n-p-2)!}}{\frac{(n-2)!}{p!(n-p-2)!}\{\frac{(2n^2-n(p+2)-n+1)}{n(n-1)(n-p-1)}\}}$$

$$= \frac{n^2-pn-n}{2n^2-n(p+2)-n+1}$$

$\therefore\ P(\text{switch and win}) = \frac{n^2-pn-n}{2n^2-n(p+2)-n+1}$

$$\neq P(\text{gold in box other than } b_1\ )$$

Besides, this indicates that the location of the gold bar and the host's choice are not independent.

2 Covid 19 case

Let us define the events, i.e.

A = patient develops long COVID symptomps
B = patient is unvaccinated
B^C = patient is vaccinated

Known that $P(A|B) \geq P(A)$, what can we say about $P(A|B^C)$?

Using Bayes' Theorem,

$P(A) = P(A|B)P(B) + P(A|B^C)P(B^C)$

Thus,

$P(A|B) \geq P(A)$

$P(A|B) \geq P(A|B)P(B) + P(A|B^C)P(B^C)$

$P(A|B)(1 - P(B)) \geq P(A|B^C)P(B^C)$

$P(A|B)(P(B^C)) \geq P(A|B^C)P(B^C)$

Assume that $(P(B^C)) \neq 0$ then

$\therefore\ P(A|B) \geq P(A|B^C)$

We can conclude that the conditional probability of her developing long COVID symptoms given that *she is vaccinated* is **less than** the conditional probability of her developing long COVID symptoms given that *she is unvaccinated.*