NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2020-2021

EE7401 – PROBABILITY AND RANDOM PROCESSES

November / December 2020 Time Allowed: 2 hours

INSTRUCTIONS

1. This paper contains 4 questions and comprises 4 pages.
2. Answer all 4 questions.
3. All questions carry equal marks.
4. This is a closed book examination.
5. Unless specifically stated, all symbols have their usual meanings.

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1. A random variable X is specified by its probability density function (pdf)

$$f_X(x) = \begin{cases} a(1-0.1x), & 0 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}.$$

Five numbers drawn from X are x_1=0.3, x_2=1, x_3=2.5, x_4=4.5 and x_5=7.

(a) From the 5 random numbers x_k, generate 5 random numbers y_k that obey the uniform distribution from zero to one.

(12 Marks)

(b) From the 5 random numbers x_k, generate 5 random numbers z_k that obey the distribution specified by the pdf

$$f_Z(z) = \begin{cases} bz, & 0 \le z \le 10 \\ 0, & \text{otherwise} \end{cases}.$$

(13 Marks)

2. A random variable X has the probability mass function (pmf) $p_X(0) = 0.4$ and $p_X(2) = 0.6$. The conditional probability density function (pdf) $f_{Y|X}(y\,|\,x)$ of a random variable Y is given by

$$f_{Y|X}(y\,|\,0) = \begin{cases} b_1 y & 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y\,|\,2) = \begin{cases} b_2(4-y) & 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

It is observed that $y = 2.5$.

(a) Estimate the value of X to achieve the lowest probability of the wrong estimation.

(13 Marks)

(b) Estimate the value of X to have the minimal mean square error of the estimation.

(12 Marks)

3. *Spread of the pandemic:* Komal Vidyarthi, an NTU researcher who is lovingly called Ko. Vid. by her friends, uses the following exponential model to study the spread of a pandemic in a community. The percentage of infected people is a real random variable $\mathbf{x}$ having uniform density $f_x(x)$ from 0 to 10. The number of people who are at a risk of infection at time $t \ge 0$ is a real non-stationary random process $\mathbf{y}(t) = e^{\mathbf{x}t}$.

(a) Komal wishes to find the first order statistics of $\mathbf{y}(t)$. In the first step, find $\eta_y(t)$, the mean of $\mathbf{y}(t)$, for $t \ge 0$.

(6 Marks)

(b) In the second step, using $f_x(x)$, the density of $\mathbf{x}$, find $f_y(y, t)$, the first order density of $\mathbf{y}(t)$, for $t \ge 0$.

(10 Marks)

(c) Ko. Vid. found that $\mathbf{y}(t)$ has a strange property. The second order statistics of $\mathbf{y}(t)$ may be found from its first order statistics. Express $R_{yy}(t_1, t_2)$, the autocorrelation of $\mathbf{y}(t)$, in terms of $\eta_y(t)$, the mean of $\mathbf{y}(t)$, for $t_1, t_2 \ge 0$.

(5 Marks)

(d) Combining your results of parts (a) and (c), find $R_{yy}(t_1, t_2)$ for $t_1, t_2 \ge 0$.

(4 Marks)

4. (a) Let $\mathbf{x}[n]$ be a discrete time real stationary random process with autocorrelation

$$R_{xx}[m] = \begin{cases} 2 & m = 0 \\ 1 & m = -1,1 \\ 0 & else \end{cases}.$$

Let $\mathbf{y}[n] = 2\mathbf{x}[n] - \mathbf{x}[n-1]$ be another random process. Note that $\mathbf{y}[n]$ is the output process when the input process $\mathbf{x}[n]$ passes through a linear system with impulse response

$$h[n] = \begin{cases} 2 & n = 0 \\ -1 & n = 1 \\ 0 & else \end{cases}.$$

(i) Find $S_{xx}(\omega)$, the power spectrum of $\mathbf{x}[n]$. Also find $\mathbf{S}_{xx}(z)$, the z-transform-domain power spectrum of $\mathbf{x}[n]$.

(ii) Find $R_{yy}[m]$, the autocorrelation of $\mathbf{y}[n]$.

(iii) Find $S_{yy}(\omega)$, the power spectrum of $\mathbf{y}[n]$. Also find $\mathbf{S}_{yy}(z)$, the z-transform-domain power spectrum of $\mathbf{y}[n]$.

(14 Marks)

(b) *PCR (polymerase chain reaction) test:* Everyone in Jurong Technological University is PCR tested each week, with one of three possible outcomes: negative, (positive and) asymptomatic, or (positive and) symptomatic. The following facts are observed.

- An asymptomatic person cannot become symptomatic next week. Similarly, a symptomatic person cannot become asymptomatic next week.
- Without any medical treatment, recovery of an asymptomatic person is slow. The probability that an asymptomatic person becomes negative next week is observed to be M times higher than the probability that a negative person becomes asymptomatic next week. The value of M is observed to be 2.
- With medical treatment, recovery of a symptomatic person is fast. As a result, the probability that a symptomatic person becomes negative next week is N times higher than the probability that a negative person becomes symptomatic next week. The value of N is observed to be 4.

(i) Formulate the above problem as a discrete-time Markoff chain (with unit of time being week), where the states are negative, asymptomatic, and symptomatic. Draw the Markoff chain and find the transition matrix. If the actual transition probabilities are not specified above, you may use variables.

Note: Question No. 4 continues on page 4.

(ii) Find the steady state probabilities. If the population of the Jurong Technological University is 35000, how many people are symptomatic on an average in the steady state? [Hint: You should be able to evaluate the steady state probabilities in spite of any variable you may have used in part (b)(i).]

(iii) With improved medical treatment, it is possible to achieve faster recovery and, therefore, lesser number of symptomatic people. Find the condition on N such that the number of symptomatic people on an average does not exceed 3500 in the steady state.

(11 Marks)

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.