Week 4

Monday, 4 September 2023 5:02 pm

Observe $Y \rightsquigarrow X$?

$$X \longrightarrow \boxed{\text{noisy channel}} \longrightarrow Y \longrightarrow \hat{X}$$

(X, Y)

Joint cdf $F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$

- discrete : pmf $p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$

$$F_{X,Y}(x,y) = \sum_{\substack{x' \le x \\ y' \le y}} p_{X,Y}(x', y')$$

$$\mathbb{P}((X,Y) \in A) = \sum_{(x,y) \in A} p_{X,Y}(x,y)$$

marginal pmf $p_X(x) = \sum_y p_{X,Y}(x,y)$

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)\,\boxed{p_X(x)}}{p_Y(y)}$$

likelihood model

prior

Bayes' rule

posterior

- Continuous joint pdf $f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x)$

marginal pdf $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$$

- MAP decoder

"a posterior prob."

$$X \longrightarrow \boxed{\text{noisy channel}} \longrightarrow Y \longrightarrow \hat{X} = \arg\max_x \underline{p_{X|Y}(x|y)}$$

discrete

$f_{Y|X}(y|x)$

e.g. $Y = X + Z$, $Z \sim N(0,1)$

MAP decoder.

"$\int$" = "Σ"

Expectation $\boxed{X : \Omega \to \mathbb{R}}$

Assume pdf/pmf

discrete

$$\boxed{\mathbb{E}[X] = \int_{-\infty}^{\infty} x \underbrace{f_X(x)}_{} \, dx \qquad \sum_x x\, p_X(x)}$$

$\mathbb{E}$ - expectation operator

"mean"

constant

constant

Operator : $X \rightsquigarrow X'$

"a..."

"mean"

"average" constant constant Operator: $X \rightsquigarrow X'$ r.v. r.v.

$g(X) \to$ r.v.

$$\mathbb{E}[g(X)] = \int g(x) f_X(x)\, dx$$

$x = c$ w.p. 1 $\quad p_X(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{o.w.} \end{cases}$

$X \equiv c \to$ constant. $\quad \mathbb{E}[c] = \sum_x x\, p_X(x) = c$

$\to \mathbb{E}[aX + bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]$

~~linearity~~ of $\int$ or $\sum$.

$Y = g(X) \sim p_Y(y)$

$x \to g(x) = y_1 \quad \sum_{x:g(x)=y_1}$

$x \to g(x) = y_2 \quad \sum_{g(x)=y_2}$

$$\mathbb{E}[Y] = \sum_y y\, \boxed{p_Y(y)} = \sum_y y \left[\sum_{x:\, g(x)=y} p_X(x)\right]$$

$$= \boxed{\sum_y \sum_{x:\, g(x)=y}} g(x) p_X(x) = \boxed{\sum_x} g(x) p_X(x)$$

$$= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$\sum_y \sum_{x:\, g(x)=y} \equiv \sum_x$

$y = 1, 2, 3 \quad \sum_{x:g(x)=1} + \sum_{x:g(x)=2} + \sum_{x:g(x)=3}$

$1, 3, 5 \qquad 2, 4, 10 \qquad 6, 7, 8, 9$

$x \in \{1, \dots, 10\}$

$g(1) = g(3) = g(5) = 1$

$g(2) = g(4) = g(10) = 2$

X

$X_1, X_2, \dots, X_n$

$$\frac{1}{n}\sum_{i=1}^n X_i \underset{n\to\infty}{\longrightarrow} \simeq \mathbb{E}[X] \quad \text{SLLN}$$

$$\mathbb{E}X^2 = \int x^2 f(x)\, dx$$

$$0 \leq \operatorname{Var} X = \mathbb{E}[\overset{\geq 0}{(X - \mathbb{E}X)^2}] = \mathbb{E}[X^2 - 2X\,\mathbb{E}[X] + (\mathbb{E}[X])^2]$$

Y - r.v.

$$= \mathbb{E}[X^2] - 2\,\mathbb{E}[X]\cdot\mathbb{E}[X] + (\mathbb{E}[X])^2$$

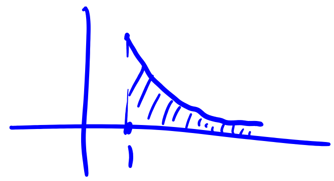
1 - r.v. $\quad = \mathbb{E}[X^2] - 2\mathbb{E}[X]\cdot\mathbb{E}[X] + (\mathbb{E}[X])^2$

$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$$

$$\text{s.d.} = \sqrt{\operatorname{var}(X)}$$

$$f_X(x) = \begin{cases} \frac{1}{x^2} & 1 \leq x < \infty \\ 0 & \text{o.w.} \end{cases}$$



$$\mathbb{E}X = \underbrace{\int_{-\infty}^{\infty} x f_X(x)\,dx}_{\parallel} = \int_1^\infty \frac{x}{x^2}\,dx = \int_1^\infty \frac{1}{x}\,dx$$

$$= \log x \Big|_1^\infty$$

$$= \infty$$

$$= \int_{-\infty}^{0} x f_X(x)\,dx + \int_0^\infty x f_X(x)\,dx$$

$$= -\underbrace{\int_{-\infty}^{0} |x| f_X(x)\,dx}_{\substack{\geq 0 \\ b}} + \underbrace{\int_0^\infty |x| f_X(x)\,dx}_{\substack{\geq 0 \\ a}} = \overset{+\infty}{a} - \overset{+\infty}{b}$$

$$f(x) = \frac{1}{\pi(1+x^2)} \qquad \mathbb{E}X = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d(1+x^2)}{1+x^2}$$

$$-\int_{-\infty}^{0} = \frac{1}{2\pi}\log(1+x^2)\Big|_{-\infty}^{0} = \infty$$

$$\int_0^\infty = \frac{1}{2\pi}\log(1+x^2)\Big|_0^\infty = \infty$$

$\mathbb{E}[|X|] < \infty$ "integrable" = expectation $\mathbb{E}[X]$ exists.

$X \qquad g(X)$

$$\boxed{\mathbb{1}_{\{X\in A\}}(\omega)} = \begin{cases} 1 & \text{if } X(\omega) \in A \\ 0 & \text{o.w.} \end{cases}$$

$\mathbb{1}_{\{x \in A\}}$ $\qquad$ $0 \quad$ o.w.

indicator function

r.v. $\mathbb{1}_{\{X \geq a\}} = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{if } X < a \end{cases}$

$$\mathbb{E}[\mathbb{1}_{\{X \geq a\}}] = \int_{-\infty}^{\infty} \mathbb{1}_{\{x \geq a\}} f_X(x)\, dx$$

$$= \int_a^{\infty} f_X(x)\, dx$$

$$= \mathbb{P}(X \geq a)$$

$$\mathbb{E}[\mathbb{1}_{\{X \in A\}}] = \mathbb{P}(X \in A)$$

$X \geq 0$, $a > 0$

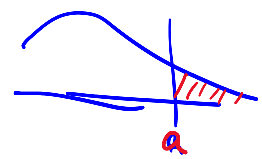
$$a \mathbb{1}_{\{X \geq a\}} \leq \mathbb{1}_{\{X \geq a\}} X \leq X$$

$X \leq Y$

$\mathbb{E}X \leq \mathbb{E}Y$

$$\mathbb{E}[a \mathbb{1}_{\{X \geq a\}}] \leq \mathbb{E}[X]$$

$$a \mathbb{P}(X \geq a) \leq \mathbb{E}[X]$$



$X \geq 0$ $\qquad \mathbb{P}(X \geq a) \leq \frac{1}{a} \mathbb{E}[X]$

Markov Inequality

$X \geq 0$, $\varepsilon \geq 0$

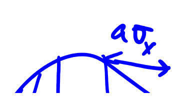
Let $a = \varepsilon \mathbb{E}[X] > 0$ $\qquad$ "="

$$\mathbb{P}(X \geq \varepsilon \mathbb{E}[X]) \boxed{\leq} \frac{1}{\varepsilon \mathbb{E}[X]} \mathbb{E}[X] = \frac{1}{\varepsilon}$$

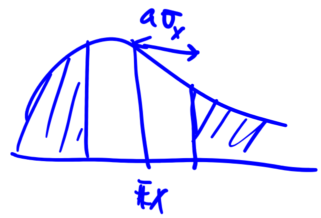
$$X = \begin{cases} \varepsilon \mathbb{E}X & \text{w.p. } \frac{1}{\varepsilon} \\ 0 & \text{w.p. } 1 - \frac{1}{\varepsilon} \end{cases}$$

$$\mathbb{P}(X \geq \varepsilon \mathbb{E}X) = \frac{1}{\varepsilon}$$

Chebyshev Inequality $\qquad \sigma_X^2 = \operatorname{var} X$



Chebyshev Inequality $\quad \sigma_X^2 = \operatorname{var} X$



$$\mathbb{P}(|X - \mathbb{E}X| \geq a\sigma_X) \leq \frac{1}{a^2}$$

$$Y = |X - \mathbb{E}X|^2 \geq 0 \qquad \mathbb{E}Y = \operatorname{var} X = \sigma_X^2$$

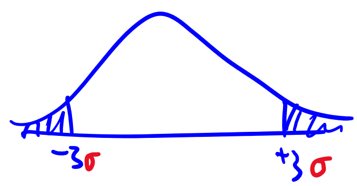
$$\mathbb{P}(Y \geq a^2\sigma_X^2) \leq \frac{1}{a^2\sigma_X^2}\mathbb{E}[Y] = \frac{1}{a^2}$$

$$\| $$

$$\mathbb{P}(|X - \mathbb{E}X|^2 \geq a^2\sigma_X^2)$$

$$\Updownarrow$$

$$|X - \mathbb{E}X| \geq a\sigma_X$$



$$\mathbb{E}[g(X,Y)] = \iint g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

Correlation $\quad \mathbb{E}[XY]$

covariance $\quad \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$

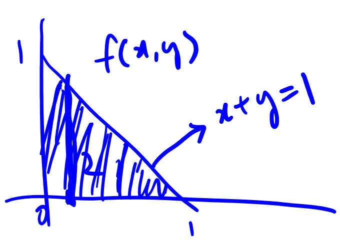
Cov(X,Y)

$$= \mathbb{E}[XY - X\mathbb{E}Y - Y\mathbb{E}X + \mathbb{E}X \cdot \mathbb{E}Y]$$

$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y]$$

$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\operatorname{cov}(X,X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \operatorname{var}(X)$$



$$\mathbb{E}[X] = \int x\, f_X(x)\, dx$$

$$= \int x \int f(x,y)\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} x \cdot 2\, dy\, dx$$

Cov(X,Y) $\quad 1 \cdot \frac{1}{2} = \frac{1}{a}$

$$\int_0 \int_0 \ x \ 2 \ dy \ dx$$

$$= \frac{1}{3}$$

$\text{Cov}(X,Y)$ $\quad \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$$= \underbrace{\mathbb{E}[XY]}_{= \iint xy f_{X,Y}(x,y)\,dy\,dx} - \mathbb{E}X\,\mathbb{E}Y$$

$$\iint xy f_{X,Y}(x,y)\,dy\,dx = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx = \frac{1}{12}$$

$\text{Cov}(X,Y) = 0$ $\quad X, Y$ are uncorrelated

$\boxed{X \perp\!\!\!\perp Y \Rightarrow \text{Cov}(X,Y) = 0}$ $\quad$ Prove: $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

$$\mathbb{E}[XY] = \iint xy \underbrace{f_{X,Y}(x,y)}_{= f_X(x) f_Y(y)}\,dx\,dy$$

$$= \iint xy f_X(x) f_Y(y)\,dx\,dy$$

$$= \int x f_X(x)\,dx \cdot \int y f_Y(y)\,dy$$

$$= \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

$\text{Cov}(X,Y) = 0 \Rightarrow X \perp\!\!\!\perp Y$? FALSE.

$$p(x,y) = \begin{cases} \frac{2}{5} & (x,y) = (+1,+1), (-1,-1) \\ \frac{1}{10} & (x,y) = (+2,-2), (-2,+2) \\ 0 & \text{o.w.} \end{cases}$$

$|X| = |Y|$

$p(x,y) \neq p(x)p(y)$

$$\mathbb{E}[X] = \sum_x x\,p(x) = \sum_x x \sum_y p(x,y)$$

$$= 1 \cdot \frac{2}{5} - 1 \cdot \frac{2}{5} + 2 \cdot \frac{1}{10} - 2\frac{1}{10}$$

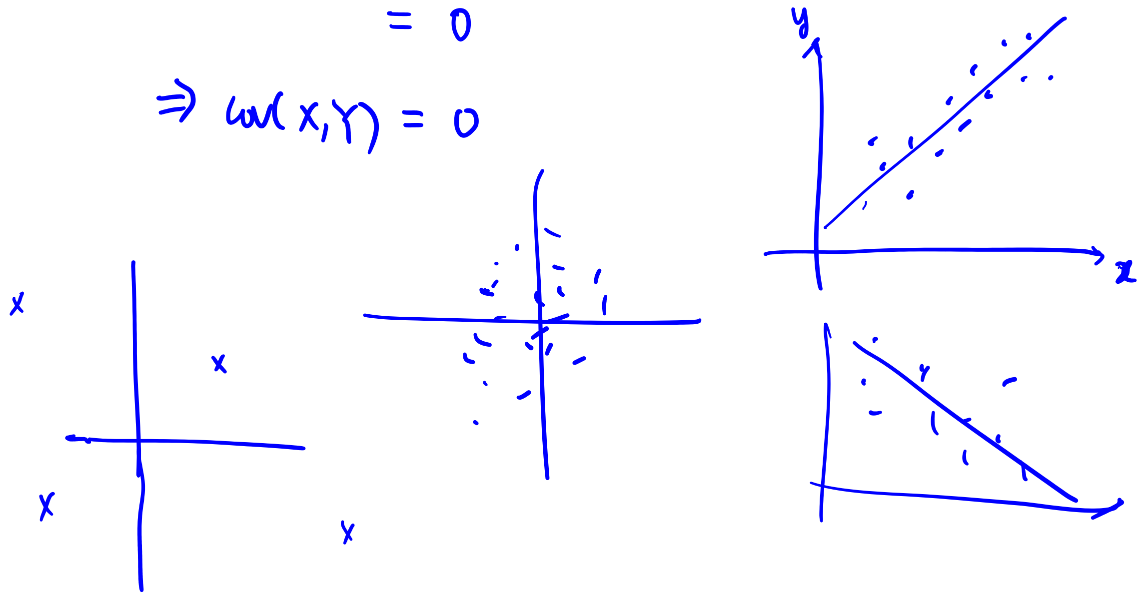
$$= 0$$

$\mathbb{E}[X]\mathbb{E}[Y] = 0$

$$\underline{\mathbb{E}[XY]} = (+1)(+1)\frac{2}{5} + (-1)(-1)\frac{2}{5} + (+2)(-2)\frac{1}{10} + (-2)(+2)\frac{1}{10}$$

$$= 0$$

$$\Rightarrow \operatorname{cov}(X,Y) = 0$$



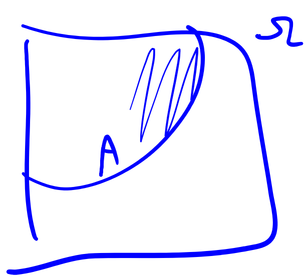
$$\rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}} \qquad -1 \leq \rho_{X,Y} \leq +1$$

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \checkmark$$

$$\{X=x\} \cap \{X \in A\}$$

discrete

$$p_{X|A}(x) = \underline{\mathbb{P}(X=x \mid X \in A)} = \frac{\mathbb{P}(\{X=x\}, \{X \in A\})}{\mathbb{P}(X \in A)}$$



$$= \begin{cases} \dfrac{\mathbb{P}(X=x)}{\mathbb{P}(X \in A)} & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

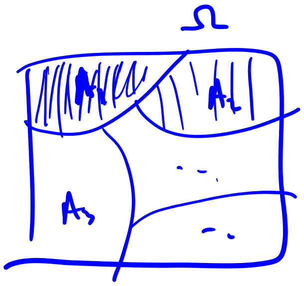
$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(X \in A)} & \underline{\text{if } x \in A} \\ 0 & \text{o.w.} \end{cases}$$

$$A = \{X > a\} \qquad \mathbb{P}(X \in A) = \mathbb{P}(X > a) = e^{-\lambda a}$$

$$f_{X|A}(x) = \begin{cases} \lambda e^{-\lambda(x-a)}, & x > a \end{cases} \qquad f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 \end{cases}$$

$$f_{X|A}(x) = \begin{cases} \lambda e^{-\lambda(x-a)}, & x > a \\ 0 & \text{o.w.} \end{cases} \qquad f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

$$\mathbb{E}[g(X) \mid A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\, dx$$



$$\Omega = \bigcup_{i=1}^{n} A_i \qquad A_i \cap A_j = \phi, \ i \neq j$$

$$\mathbb{E}[g(X)] = \boxed{\sum_{i=1}^{n} \mathbb{P}(X \in A_i)\, \mathbb{E}[g(X) \mid A_i]}$$

$$\mathbb{E}[g(X) \mid A_i] = \int g(x) f_{X|A_i}(x)\, dx$$

$$\sum_{i=1}^{n} \mathbb{P}(X \in A_i) \int g(x) f_{X|A_i}(x)\, dx$$

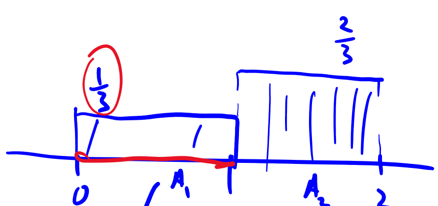
$$= \int g(x) \sum_{i=1}^{n} \mathbb{P}(X \in A_i) f_{X|A_i}(x)\, dx$$

$$f_{X|A_i}(x) = \frac{f_X(x)}{\mathbb{P}(X \in A_i)} \mathbb{1}_{\{x \in A_i\}}$$

$$= \int g(x) \sum_{i=1}^{n} f_X(x) \mathbb{1}_{\{x \in A_i\}}\, dx$$

$$\sum_{i=1}^{n} \mathbb{1}_{\{x \in A_i\}} \equiv 1$$

$$= \int g(x) f_X(x)\, dx = \mathbb{E}[g(X)]$$



$$\mathbb{E}[X \mid A_1] = \frac{1}{2} \qquad \mathbb{E}[X \mid A_2] = \frac{1+2}{2} = \frac{3}{2}$$

$$\mathbb{E}[X^2 \mid A_1] = \qquad \mathbb{E}[X^2 \mid A_2]$$

$$\mathbb{E}[X^2 \mid A_1] \qquad \mathbb{E}[X^2 \mid A_2] = \frac{7}{3} \ (\text{Ex})$$

$$= \int x^2 f_{X|A_1}(x)\, dx$$

$$= \int_0^1 x^2 \boxed{\frac{\frac{1}{3}}{\frac{1}{3}}}\, dx = \frac{1}{3}$$

$$f_{X|A_1}(x) = \frac{f_X(x)}{\mathbb{P}(X \in A_1)} \quad x \in A_1$$

$$\mathbb{E}[X] = \mathbb{P}(X \in A_1)\mathbb{E}[X \mid A_1] + \mathbb{P}(X \in A_2)\mathbb{E}[X \mid A_2]$$

$$= \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{3}{2} = \frac{7}{6}$$

$$\mathbb{E}[X^2] = \mathbb{P}(X \in A_1)\mathbb{E}[X^2 \mid A_1] + \mathbb{P}(X \in A_2)\mathbb{E}[X^2 \mid A_2]$$

$$\mathbb{E}[X^2] = \mathbb{P}(X \in A_1)\,\mathbb{E}[X^2 \mid A_1] + \mathbb{P}(X \in A_2)\,\mathbb{E}[X^2 \mid A_2]$$
$$= \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{7}{3} = \frac{15}{9}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$\mathbb{E}[g(X) \mid Y = \overset{\text{const}}{y}] = \int g(x)\, f_{X|Y}(x|\overset{\text{const}}{y})\, dx$$

$$\mathbb{E}[g(X,Y) \mid Y = y] = \int g(x,y)\, f_{X|Y}(x|y)\, dx$$

$$\mathbb{E}[XY \mid Y = y] = \int x y\, f_{X|Y}(x|y)\, dx$$

nm

$$= y \int x\, f_{X|Y}(x|y)\, dx$$
$$= y\, \mathbb{E}[X \mid Y = y]$$

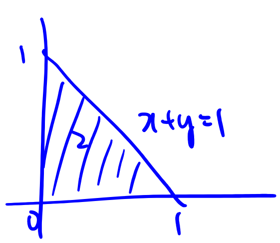
$$\mathbb{E}[X \mid Y = y] = h(y) \qquad Y \text{ r.v.}$$

$$\mathbb{P}(Y = y) \rightarrow \mathbb{E}[X \mid Y = y]$$

$$h(Y) = \mathbb{E}[X \mid Y] \rightarrow \text{r.v.}$$

$$\underset{\text{r.v.}}{X} \rightarrow \underset{\text{r.v.}}{\mathbb{E}[X \mid Y]}$$

expectation operator



$$f_{X|Y}(x|y)$$

$$\mathbb{E}[X \mid Y = y] \qquad \mathbb{E}[X \mid Y]$$

$$= \int x\, f_{X|Y}(x|y)\, dx$$

$$= \int_0^{1-y} \frac{x}{1-y}\, dx$$

$$\frac{1}{\ } \int^{1-y} \ $$

$$\int_0 \frac{\;}{1-y}\,dx$$

$$= \frac{1}{1-y}\int_0^{1-y} x\,dx$$

$$= \frac{1}{1-y}\,\frac{(1-y)^2}{2}$$

$$= \frac{1-y}{2} \qquad 0 \le y < 1$$

$$\mathbb{E}[XY \mid Y=y] = y\,\mathbb{E}[X \mid Y=y] = \frac{y(1-y)}{2}$$

$$\mathbb{E}[X \mid Y] = \boxed{\frac{1-Y}{2}} = Z$$

r.v.

pdf ?

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \le y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$Z = \frac{1-Y}{2} = g(Y)$$

$$z = \frac{1-y}{2}$$

$$y = 1-2z$$

$$g'(y) = -\frac{1}{2}$$

$$f_Z(z) = \frac{f_Y(y)}{|g'(y)|}$$

$$= \frac{f_Y(1-2z)}{\frac{1}{2}} \qquad 0 < z \le \frac{1}{2}$$

$$= 2 \cdot 2(1-1+2z)$$

$$= 8z$$

$$\mathbb{E}[Z] = \mathbb{E}[\overset{\text{r.v.}}{\mathbb{E}[X \mid Y]}] = \mathbb{E}[X]$$

$$\mathbb{E}\big[\mathbb{E}[g(X,Y) \mid Y]\big] = \mathbb{E}[g(X,Y)]$$

$$= \int \boxed{\mathbb{E}[g(X,Y) \mid Y=y]}\, f_Y(y)\,dy$$

$$= \int \boxed{\int g(x,y)\, f_{X|Y}(x|y)\,dx}\; f_Y(y)\,dy$$

$$= \iint g(x,y)\, f_{X|Y}(x|y)\, f_Y(y)\,dx\,dy$$

$$= \iint g(x,y)\, f_{X,Y}(x,y)\,dx\,dy = \mathbb{E}[g(X,Y)]$$

$P \sim f_P(p) = 2(1-p)$, $0 \le p \le 1$

N = # of heads $\qquad X_i = \begin{cases} 1 & \text{if } i\text{th toss is head} \\ 0 & \text{o.w.} \end{cases}$

$X_1, X_2, \ldots, X_n \qquad N = \sum_{i=1}^{n} X_i$

$\mathbb{P}(X_i = 1 \mid P) = P \qquad \mathbb{E}[X_i \mid P] = P$

$$\mathbb{E}[N \mid P] = \mathbb{E}\left[\sum_{i=1}^{n} X_i \mid P\right]$$
$$= \sum_{i=1}^{n} \mathbb{E}[X_i \mid P] = nP$$

$$\mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N \mid P]] = \mathbb{E}[nP]$$
$$= n\,\mathbb{E}[P]$$
$$= n \int_0^1 p\, 2(1-p)\, dp = \frac{n}{3}$$

$\mathbb{E}[X \mid Y] = Y^2 \qquad Y \sim \text{Unif}[0,1]$

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$$
$$= \mathbb{E}[Y^2] = \int_0^1 y^2 \cdot 1\, dy = \frac{1}{3}$$

r.v.

$$\mathrm{var}(X \mid Y = y) = \mathbb{E}[(X - \mathbb{E}[X \mid Y = y])^2 \mid Y = y]$$
$$= \mathbb{E}[X^2 \mid Y = y] - (\mathbb{E}[X \mid Y = y])^2$$

$f_{X|Y}(x \mid y)$

$$\to \mathbb{E}[\mathrm{var}(X \mid Y)] = \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X \mid Y])^2]$$

$$\to \mathrm{var}(\mathbb{E}[X \mid Y]) = \mathbb{E}[(\mathbb{E}[X \mid Y] - \mathbb{E}[X])^2]$$

r.v.

$$= \mathbb{E}[\mathbb{E}[X \mid Y]^2 - 2\,\mathbb{E}[X]\,\mathbb{E}[X \mid Y] + \mathbb{E}[X]^2]$$
$$= \mathbb{E}[\mathbb{E}[X \mid Y]^2] - \mathbb{E}[X]^2$$

$$= \mathbb{E}[\mathbb{E}[X|Y]^2 - 2\mathbb{E}[X]\mathbb{E}[X|Y] + \mathbb{E}[X]^2]$$

$$= \mathbb{E}[\mathbb{E}[X|Y]^2] - \mathbb{E}[X]^2$$

"bias" "variance"

$$\mathbb{E}[\mathrm{var}(X|Y)] + \mathrm{var}(\mathbb{E}[X|Y]) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathrm{var}(X)$$

$X \longrightarrow Y$ observed.

$X|Y$?

$$\hat{X} = \mathbb{E}[X|Y]$$