NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2019-2020

EE7401 – PROBABILITY AND RANDOM PROCESSES

November / December 2019

Time Allowed: 3 hours

INSTRUCTIONS

1. This paper contains 5 questions and comprises 5 pages.
2. Answer all 5 questions.
3. All questions carry equal marks.
4. This is a closed book examination.
5. Unless specifically stated, all symbols have their usual meanings.

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1. Two random variables X and Y are specified by their joint probability density function (pdf)

$$f(x,y)=\begin{cases} cxy & x\geq 0, y\geq 0, x+y\leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the marginal pdf $f_Y(y)$.

(5 Marks)

(b) Find the conditional pdf $f_{X|Y}(x\mid y)$.

(5 Marks)

(c) Estimate the value of X based on the observed value of $Y=y$ so that the mean square error (MSE) is minimized.

(5 Marks)

(d) Compute the MSE for a given observed value $Y=y$.

(5 Marks)

2. Table 1 completely specifies two discrete random variables X and Y.

Table 1

y \ x	4	5	6
1	0	0.15	0.15
3	0.08	0.08	0.24
7	0.09	0.18	0.03

(a) Does Table 1 specify the probability mass function (PMF) $p_{X,Y}(x, y)$, $p_{Y|X}(y \mid x)$, or $p_{X|Y}(x \mid y)$? Give the reasons for your answer.

(6 Marks)

(b) What PMF ($p_{??}(??)$) is specified by the 3 numbers (0.08, 0.08, 0.24) drawn from the second row of Table 1?

(2 Marks)

(c) If we normalize the above 3 numbers by the sum of them, i.e., (0.08, 0.08, 0.24) / (0.08+0.08+0.24), what PMF ($p_{??}(??)$) is specified by the results ? Give the reasons for your answer.

(6 Marks)

(d) If we divide the 3 numbers (0.08, 0.08, 0.24), respectively by the sum of the numbers in the corresponding column of Table 1, what PMF ($p_{??}(??)$) is specified by the results? Give the reasons for your answer.

(6 Marks)

3. (a) A 2-digit integer x, $10 \le x \le 99$, is randomly chosen. All possible choices are equally likely.

(i) What is the probability of event A that the ones-digit of x has the maximum value and what is the probability of event B that the tens-digit of x has the maximum value?

(ii) What is the probability of event C that at least one digit of x has the maximum value?

(10 Marks)

Note: Question No. 3 continues on page 3.

(b) $\mathbf{x}[n]$ is a real wide-sense stationary (WSS) discrete-time random process with autocorrelation $R_{xx}[m]=2e^{-0.5|m|}$. Let $\mathbf{y}[n]=\mathbf{x}[n]-\mathbf{x}[n-3]$.

(i) Find $\eta_y[n]=E\{\mathbf{y}[n]\}$, the mean of $\mathbf{y}[n]$.

(ii) Find $E\{\mathbf{y}^2[n]\}$, the average power of $\mathbf{y}[n]$.

(iii) Find $R_{yy}[m]$, the autocorrelation of $\mathbf{y}[n]$. Hence, show that for $|m|\geq 3$, it may be simplified to $R_{yy}[m]=cR_{xx}[m]$, where c is a constant. Find c.

(10 Marks)

4. (a) *Stoopin' Low:* HK Commissioner of Police Stoophen Lo is accused of stoopin' low–using police brutality and violence–in his department's fight against the youth demonstrators. Mr. Lo starts each day with a peaceful mind, but too many demonstrations make him angry. Let $\mathbf{x}(t)$ ={number of demonstrations starting from time 0 to t, where t is measured in hours} be a Poisson process, such that the probability of k demonstrations is given by $P\{\mathbf{x}(t)=k\}=e^{-\lambda t}(\lambda t)^k/k!$. It is known that, on average, there are 6 demonstrations per day, or $E\{\mathbf{x}(24)\}=6$.

(i) Find the variance of $\mathbf{x}(24)$, the number of demonstrations in a day.

(ii) Mr. Lo spends hour 0 to hour 8 each day in his office, and remains peaceful as long as there are 4 or less demonstrations. He gets agitated if there are 5 to 7 demonstrations. Find the probability that Mr. Lo will be agitated in a day, that is, the probability that there are 5 to 7 demonstrations in 0 to 8 hours, or $P\{5\leq\mathbf{x}(8)\leq 7\}$.

(iii) If the number of demonstrations reaches 8 within his 8 hours in office, it makes Stoophen Lo getting angry and stoopin' low. Today, it has been 6 hours in office, and so far, 5 demonstrations have taken place, or $\mathbf{x}(6)=5$. Find the conditional probability that Stoophen Lo will not be stoopin' low today, that is, the conditional probability that there will be less than 8 demonstrations in 8 hours, or $P\{\mathbf{x}(8)<8\,|\,\mathbf{x}(6)=5\}$.

[Hint: Express the conditional probability as the ratio of a joint probability and a marginal probability. Simplify the joint probability using the fact that non-overlapping Poisson intervals are independent.]

(12 Marks)

Note: Question No. 4 continues on page 4.

(b) *Seeing the Doctor:* During an epidemic, Doctor See You Soon from the NTU Health Center classifies each NTU person as healthy, infected (but not sick), or sick.

- The probability that a healthy person becomes infected is p, but a healthy person cannot become sick without getting infected first.

- An infected person may become sick with probability q, or may become healthy with probability p.

- Once a person is sick, he/she sees Doctor See You Soon. After seeing the Doctor, a sick person may become healthy with probability q, while a sick person remains sick and needs to see Doctor See again with a probability $1-q$. A sick person cannot become infected.

(i) Formulate the above problem as a Markoff chain, where the states are healthy, infected, and sick. Draw the Markoff chain and find the transition matrix.

(ii) Find the condition on p and q such that 90% or more of NTU people are healthy on an average, that is, the steady state probability $p_{healthy} \geq 0.9$.

(8 Marks)

5. *Engineering:* To verify the effectiveness of the new air-conditioning system in the School of EEE, Mr. Eng Ine Er placed n temperature sensors randomly in the School. Each sensor output is the difference between the measured temperature and the preset temperature. The ith sensor output is a real zero mean wide-sense stationary (WSS) random process $\mathbf{y}_i(t)$ with autocorrelation $R_{ii}(\tau) = e^{-a_i|\tau|}$, where a_i is a real positive constant. Due to distant placement of sensors, $\mathbf{y}_i(t_1)$ is uncorrelated to $\mathbf{y}_j(t_2)$ for all $i \neq j$ and for all t_1 and t_2. Mr. Eng found the ambient temperature differential of the School by taking the weighted average $\mathbf{z}(t) = \sum_{i=1}^{n} b_i \mathbf{y}_i(t)$, where b_i are real constants.

(a) Mr. Eng experimentally estimated the mean of the ambient temperature differential $\mathbf{z}(t)$ by observing its value for a long time and averaging these values. Unfortunately, Mr. Eng never took a course like EE7401, so he is not aware that such estimation makes sense only if $\mathbf{z}(t)$ is stationary and ergodic.

(i) Find $\eta_z(t)$, the mean of $\mathbf{z}(t)$. Also find $R_{zz}(t_1,t_2)$, the autocorrelation of $\mathbf{z}(t)$. Is $\mathbf{z}(t)$ WSS?

Note: Question No. 5 continues on page 5.

(ii) Find $C_{zz}(\tau)$, the autocovariance of $\mathbf{z}(t)$, and apply Slutsky's theorem. Is $\mathbf{z}(t)$ mean-ergodic?

(iii) For the special case of $a_i = a$ for all i, $\mathbf{z}(t)$ becomes a special process. Which process is it?

[Hint: Check $R_{zz}(\tau)$.]

(13 Marks)

(b) To better engineer the temperature control, Mr. Eng Ine Er wants to predict the future temperature differential $\mathbf{z}(t+\lambda)$ from the present value $\mathbf{z}(t)$, where $\lambda > 0$. You can help Mr. Eng as follows:

(i) Find the optimal mean square predictor that predicts $\mathbf{z}(t+\lambda)$ using $\mathbf{z}(t)$.

(ii) Find the least mean square (LMS) error P for your predictor.

(7 Marks)

END OF PAPER

EE7401 PROBABILITY & RANDOM PROCESSES

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.