EE7401 Homework 3

Instructions:

● Please write your final solutions in the boxes shown below, then scan (if needed) and upload via NTULearn/Assignments/Homework3 by 23:59, 14 November 2023.
● Please do not copy from others, or let others copy your solution.
● Penalties may apply for late submission.
● Homework 3 carries 10% overall marks.

Q1) *When did you start your simulations:* Consider the simulation of a first order auto-regressive, or AR(1), process. A wide-sense stationary real white noise process $\mathbf{x}[n]$ with autocorrelation $R_{xx}[m] = 5\delta[m]$ is passed through the AR(1) filter with $a = 0.5$, such that the output is $\mathbf{y}[n] = \mathbf{x}[n] + 0.5\mathbf{y}[n-1]$.

$$\mathbf{x}[n] \longrightarrow \boxed{H(z) = \frac{1}{1 - 0.5z^{-1}}} \longrightarrow \mathbf{y}[n]$$

1.1) (1.1.a) Assume the above filter operates at all times. Then $\mathbf{y}[n]$ is jointly wide-sense stationary. Find the cross-correlation between $\mathbf{x}[n]$ and $\mathbf{y}[n]$, $R_{xy}[m] = E\{\mathbf{x}[n_1]\mathbf{y}[n_2]\}$, where $m = n_1 - n_2$. [Hint: There are many ways to solve this problem. You may use the frequency domain or the z domain approach, finding $S_{xy}(\omega)$ or $\mathbf{S}_{xy}(z)$ first, and then taking the inverse Fourier or z transform. You may express $\mathbf{y}[n]$ as a convolution of $\mathbf{x}[n]$ with $h[n]$ (impulse response), multiply the equation by $\mathbf{x}[n_1]$, and take the expectation.]

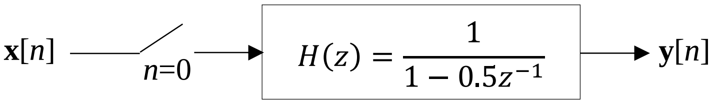
$$R_{xy}[m] = \begin{cases} \boxed{} & \text{for } m > 0 \\ \boxed{} & \text{for } m \le 0 \end{cases}$$

(4 marks)

(1.1.b) Find the autocorrelation of the AR(1) process, $R_{yy}[m]$.

$$R_{yy}[m] = \boxed{}$$

(2 marks)



1.2) Part (1.1) results are true if the simulation is started at $n = -\infty$. However, real-life simulations start at finite times. Consider a real-life simulation of the same AR(1) process as above, where the simulation starts at $n = 0$. $\mathbf{x}[n]$ is still the same wide-sense stationary with the same autocorrelation. However, the AR(1) filter starts at $n = 0$ (meaning, there was no filter before $n = 0$, or the output was zero). Therefore, the output becomes $\mathbf{y}[n] = \begin{cases} 0 & n < 0 \\ \mathbf{x}[n] + 0.5\mathbf{y}[n-1] & n \geq 0 \end{cases}$. Note that $\mathbf{y}[n]$ is no longer stationary. Therefore, the auto/cross-correlations involving $\mathbf{y}[n]$ no longer depend on the time difference m but depend on both times, like $R_{yy}[n_1, n_2]$. As a result, the power spectrums of $\mathbf{y}[n]$ do not exist, and the power spectrum based approaches can no longer be used to find the auto/cross-correlations. The time-domain approach may still be used.

(1.2.a) Express $\mathbf{y}[0]$ using only the input $\mathbf{x}[0]$. Express $\mathbf{y}[1]$ as a sum of only input terms of the form $\mathbf{x}[k]$. There should not be any past output term such as $\mathbf{y}[n-1]$. Continuing as above, express $\mathbf{y}[n_2]$ for any $n_2 \geq 0$ as a sum of only input terms of the form $\mathbf{x}[k]$. There should not be any past output term such as $\mathbf{y}[n_2 - 1]$.

$\mathbf{y}[n_2] =$ $\boxed{}$ for $n_2 \geq 0$ . . . . . eq.(1)

(3 marks)

(1.2.b) Find the cross-correlation between $\mathbf{x}[n]$ and $\mathbf{y}[n]$, $R_{xy}[n_1, n_2] = E\{\mathbf{x}[n_1]\mathbf{y}[n_2]\}$, for all $n_1, n_2 \geq 0$ by multiplying eq.(1) by $\mathbf{x}[n_1]$, and taking the expectation of both sides. [Hint: Find 2 cases: write answers in 2 left boxes, and write the cases in 2 right boxes.]

$$R_{xy}[n_1, n_2] = \begin{cases} \boxed{} & \text{for } n_1 \geq 0, n_2 \geq 0, \boxed{} \\ \boxed{} & \text{for } n_1 \geq 0, n_2 \geq 0, \boxed{} \end{cases}$$

(4 marks)

(1.2.c) Extend your result of (1.2.b) to all possible n_1, n_2 values. [Hint: Find 2 cases: write answers in the 2 left boxes, and write the condition of the primary case in the right box.]

$$R_{xy}[n_1, n_2] = \begin{cases} \boxed{} & \text{for } n_1 \geq 0, n_2 \geq 0, \boxed{} \\ \boxed{} & \text{otherwise} \end{cases}$$

(2 marks)

(1.2.d) Find the autocorrelation of the AR(1) process, $R_{yy}[n_1, n_2]$ for all possible n_1, n_2 values. [Hint: Replace both $\mathbf{y}[n_1]$ and $\mathbf{y}[n_2]$ in $E\{\mathbf{y}[n_1]\mathbf{y}[n_2]\}$ by eq.(1) twice, then take the expectation of this double summation. You should find 3 cases (it is possible to combine 2 primary cases into a single case): write answers in the 3 left boxes, and write the conditions of 2 primary cases in the 2 right boxes.]

$$R_{yy}[n_1, n_2] = \begin{cases} \boxed{} & \text{for } n_1 \geq 0, n_2 \geq 0, \boxed{} \\ \boxed{} & \text{for } n_1 \geq 0, n_2 \geq 0, \boxed{} \\ \boxed{} & \text{otherwise} \end{cases}$$

(6 marks)

(1.2.e) For all non-negative n_1 and n_2, is $R_{xy}[n_1, n_2]$ of (1.2.c) equal to $R_{xy}[m]$ of (1.1.a)? If not, when are they equal?
For all non-negative n_1 and n_2, is $R_{yy}[n_1, n_2]$ of (1.2.d) equal to $R_{yy}[m]$ of (1.1.b)? If not, when are they equal?

(4 marks)

Q2) *Bandlimited process does change with time:* In lecture we have upper bounded the change in value of a bandlimited process over a small time τ. Here we obtain a lower bound (and a new upper bound). First, two intermediate results are obtained as below.

2.1) Assume that the time is bounded by $|\tau| < (\pi/\sigma)$ and that the frequency is bounded by $|\omega| \leq \sigma$. Then, using the fact that if $0 < \varphi < (\pi/2)$, then $(2\varphi/\pi) < \sin\varphi < \varphi$, find a lower bound and an upper bound on $sin^2(\omega\tau/2)$:

$$\boxed{} \leq sin^2(\omega\tau/2) \leq \boxed{}$$

(4 marks)

2.2) Let the power spectrum of $\mathbf{x}(t)$ be $S_{xx}(\omega)$. If $\mathbf{x}(t)$ is passed through a differentiator (frequency response $H(\omega) = j\omega$), then you have already found $S_{x'x'}(\omega)$, the power spectrum of the output $\mathbf{x}'(t)$, in terms of $S_{xx}(\omega)$, in Homework 2 Q(2.b). Copy this $S_{x'x'}(\omega)$ below to obtain the autocorrelation of the output $\mathbf{x}'(t)$ as

$$R_{x'x'}(\tau) = \int_{-\infty}^{\infty} \boxed{} S_{xx}(\omega) e^{j\omega\tau} d\omega/2\pi$$

Now, putting $\tau = 0$, the average power of the output $\mathbf{x}'(t)$ is:

$$E\{|\mathbf{x}'(t)|^2\} = \int_{-\infty}^{\infty} \boxed{} S_{xx}(\omega) d\omega/2\pi$$

(2 marks)

2.3) (2.3.a) Express the expectation of the square of the change in $\mathbf{x}(t)$ over time τ, $E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\}$, using its autocorrelation:

$$E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\} = 2R_{xx}(\boxed{}) - R_{xx}(\boxed{}) - R_{xx}(\boxed{})$$

(5 marks)

(2.3.b) Use (2.3.a), $R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} \frac{d\omega}{2\pi}$, and $1 - cos\theta = 2sin^2(\theta/2)$, to get:

$$E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\} = \int_{-\infty}^{\infty} S_{xx}(\omega) \boxed{} d\omega/2\pi$$

(4 marks)

(2.3.c) Let us say the integral you obtained in part (2.3.b) is $\int_{-\infty}^{\infty} S_{xx}(\omega) g(\omega) d\omega/2\pi$ for some function $g(\omega)$ that you wrote inside the box. Now, since $\mathbf{x}(t)$ is bandlimited, its power spectrum $S_{xx}(\omega) = 0$ for $|\omega| > \sigma$. Therefore, this integral's limits may be changed, $E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\} = \int_{-\infty}^{\infty} S_{xx}(\omega) g(\omega) d\omega/2\pi = \int_{-\sigma}^{\sigma} S_{xx}(\omega) g(\omega) d\omega/2\pi$. Apply the lower and upper bounds on $sin^2(\omega\tau/2)$ from part (2.1) to obtain the lower and upper bounds on the expectation:

$$\int_{-\sigma}^{\sigma} S_{xx}(\omega)\, \omega^2 \boxed{} \frac{d\omega}{2\pi} \leq E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\} \leq \int_{-\sigma}^{\sigma} S_{xx}(\omega)\, \omega^2 \boxed{} \frac{d\omega}{2\pi}$$

(4 marks)

(2.3.d) The difficulty is that, unlike in the lecture, we can't replace ω by σ in the lower bound. Therefore, we need to evaluate $\int_{-\sigma}^{\sigma} S_{xx}(\omega)\omega^2 d\omega/2\pi$. This has already been done in part (2.2) using the differentiated process $\mathbf{x}'(t)$. Use the result of (2.2) on the lower and upper bounds of (2.3.c) to obtain the final result:

$$\boxed{}\, E\{|\mathbf{x}'(t)|^2\} \leq E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\} \leq \boxed{}\, E\{|\mathbf{x}'(t)|^2\}$$

(2 marks)

(2.3.e) Is the upper bound of (2.3.d) smaller, or larger, than the upper bound found in the class?

(4 marks)