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Module 5

MOTION PLANNING & INVERSE KINEMATICS

MOTIVATION & PRELIMINARIES

What is a robot?

- A robot is a machine in a form of mechanically constructed system.
- Programmable by a computer.
- Robots can be articulated, autonomous or semi-autonomous depending on the required application
  - Effecting motion for the robot constitutes an intrinsic element towards autonomy

What is a robot?

- Fundamentals of robotics for effecting motion includes:
  - Kinematics
  - Dynamics
  - Motion planning
  - Computer vision
  - Control

Scope

- An industrial robot manipulator is a feedback controlled, reprogrammable, multipurpose system.
  - It is reprogrammable in three or more degrees of freedom.
  - Robot manipulators are used in processes of industrial automation (ISO 8373 standard)

Scenario

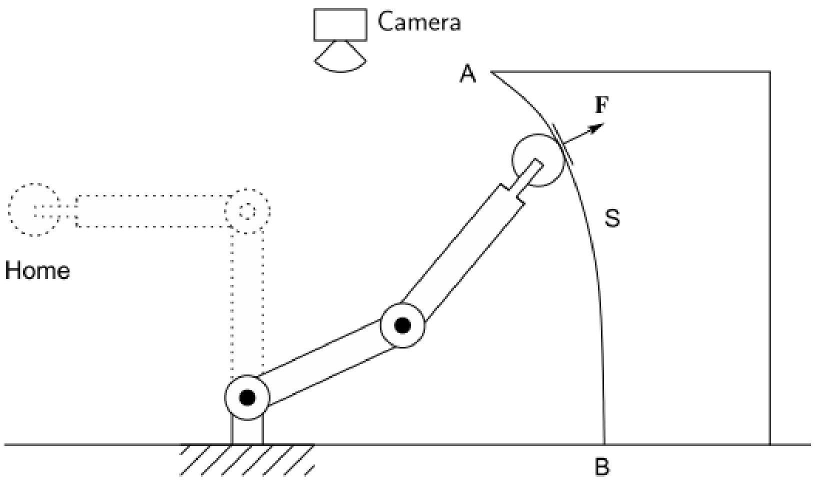
5 typical jobs in manufacturing:

1. Deformation (extrusion, rolling)
2. Material removal (grinding)
3. Solidification (moulding)
4. Assembly
5. Material handling

Scenario

- Job: Material removal (Grinding)
  - Task
    - to make use of tool with edge in to remove materials from workpiece to form desired surface profiles
  - Action
    - To attach & rotate tool, to fix workpiece to expose surface, to move and rotate tool until desired surface profile is obtained
  - Motion
    - Sequence of motion descriptions in the form of script languages or library functions
  - Path
    - Set of equations of motion without time constraints
  - Trajectory
    - Set of equations of trajectory with time constraints

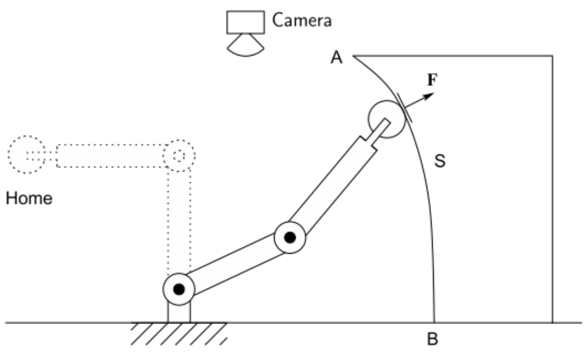
Scenario



- Simple **2-link planar manipulator** with grinding tool to remove metal from a surface.
- *What are the issues to be resolved & what conceptualizations are required to enable programming a robot to perform the tasks?*

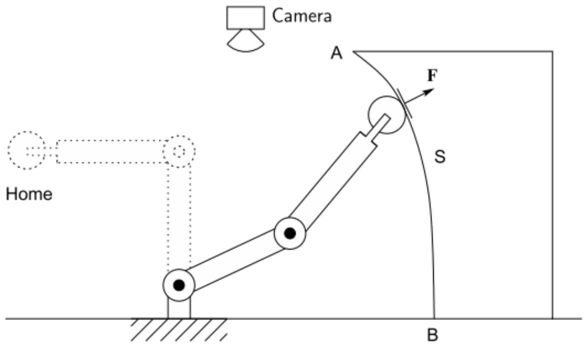
Scenario

- **Forward kinematics**:
  - Describe both position of tool & location *A* & *B* w.r.t. common coordinate system
  - Manipulator can measure joint variables θ_1 & θ_2 with encoders
  - What is needed is to express positions *A* & *B* in terms of these joint angles
  - **Homogeneous coordinates/ transformation DHS**



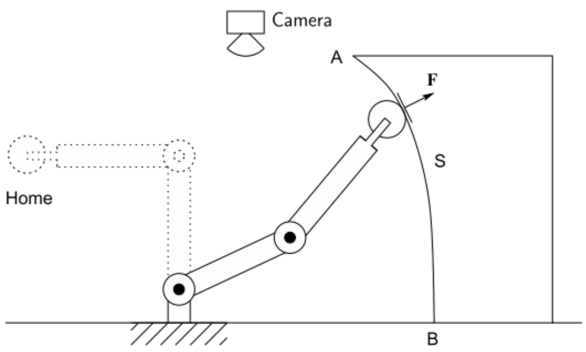
Scenario

- **Inverse kinematics**:
  - To command robot to move to location *A* we need the joint variables θ_1 & θ_2 in terms of the *x* & *y* coordinates.
  - There may be 0 to infinite solutions



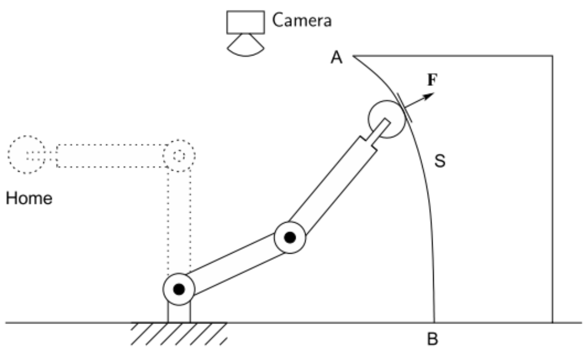
Scenario

- **Velocity kinematics**:
  - To maintain constant chosen velocity, the relationship between the tool & joint velocities must be established
  - To achieve required trajectory.
  - **Jacobian**



Scenario

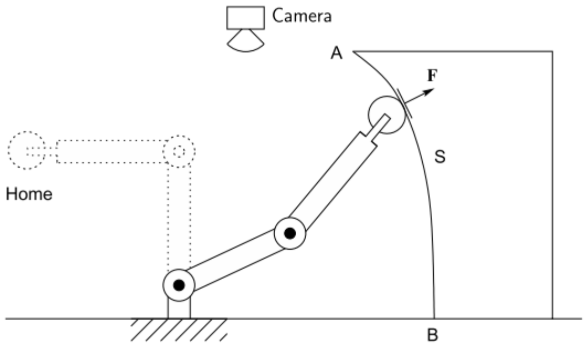
- **Path planning & Trajectory Generation**:
  - Path planning: enable movement of robot to goal position while avoiding collisions with objects in its workspace
    - Path encode position & orientation information without time considerations.
  - Trajectory planning: generate reference trajectories that determine the time history of the manipulator along a given path

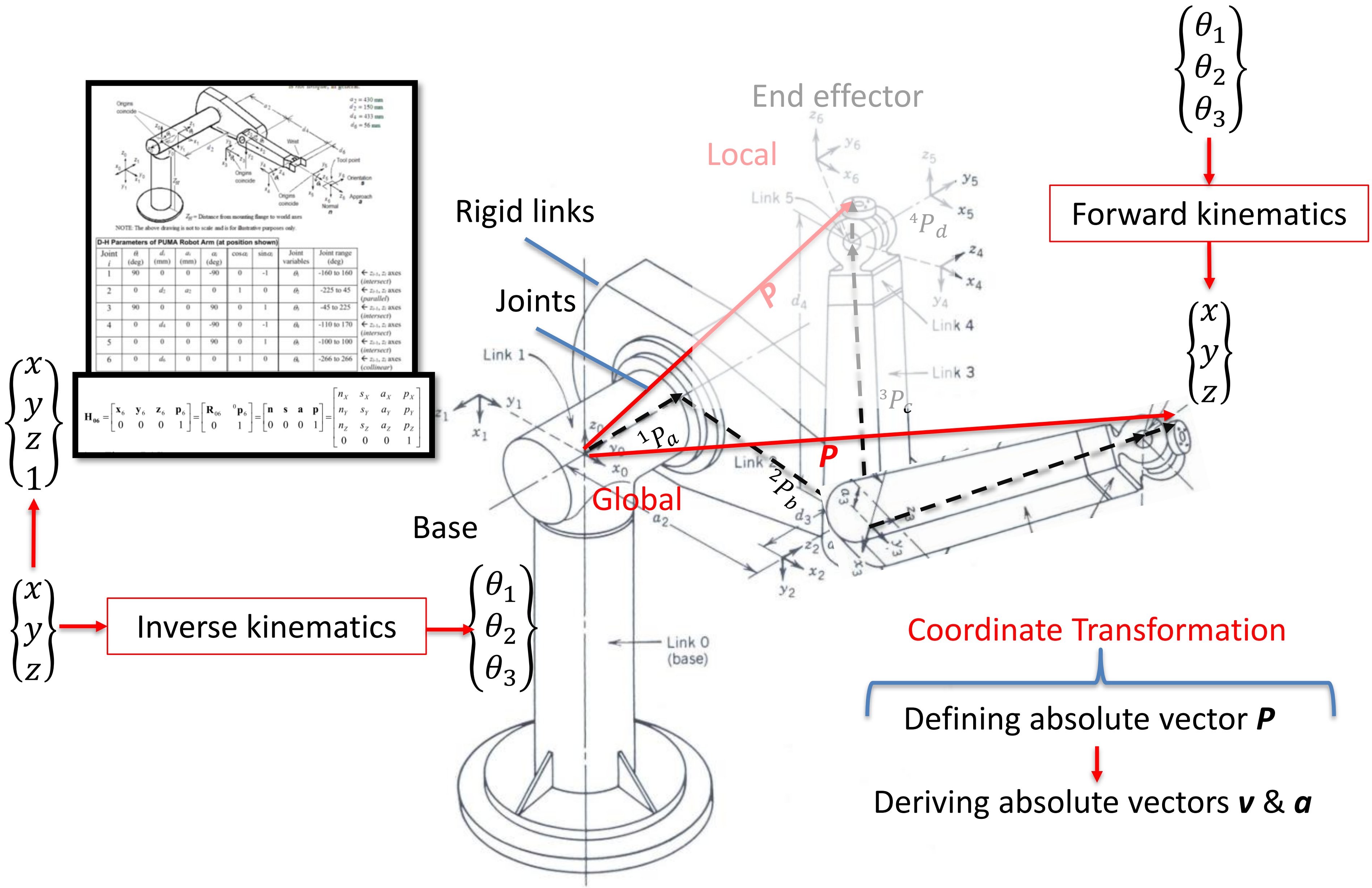


Scenario

...other considerations?

- **Dynamics**
- **Position control**
- **Force control**
- **Vision**
- **Vision-based control**





D-H Parameters of PUMA Robot Arm (at position shown)

Joint i	θ_i (deg)	d_i (mm)	a_i (mm)	α_i (deg)	$\cos\alpha_i$	$\sin\alpha_i$	Joint variables	Joint range (deg)	
1	90	0	0	-90	0	-1	θ_1	-160 to 160	← z_{i-1}, z_i axes (*intersect*)
2	0	d_2	a_2	0	1	0	θ_2	-225 to 45	← z_{i-1}, z_i axes (*parallel*)
3	90	0	0	90	0	1	θ_3	-45 to 225	← z_{i-1}, z_i axes (*intersect*)
4	0	d_4	0	-90	0	-1	θ_4	-110 to 170	← z_{i-1}, z_i axes (*intersect*)
5	0	0	0	90	0	1	θ_5	-100 to 100	← z_{i-1}, z_i axes (*intersect*)
6	0	d_6	0	0	1	0	θ_6	-266 to 266	← z_{i-1}, z_i axes (*collinear*)

$$\mathbf{H}_{06}=\begin{bmatrix}\mathbf{x}_6 & \mathbf{y}_6 & \mathbf{z}_6 & \mathbf{p}_6\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\mathbf{R}_{06} & {}^0\mathbf{p}_6\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{p}\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}n_X & s_X & a_X & p_X\\ n_Y & s_Y & a_Y & p_Y\\ n_Z & s_Z & a_Z & p_Z\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Course Outline

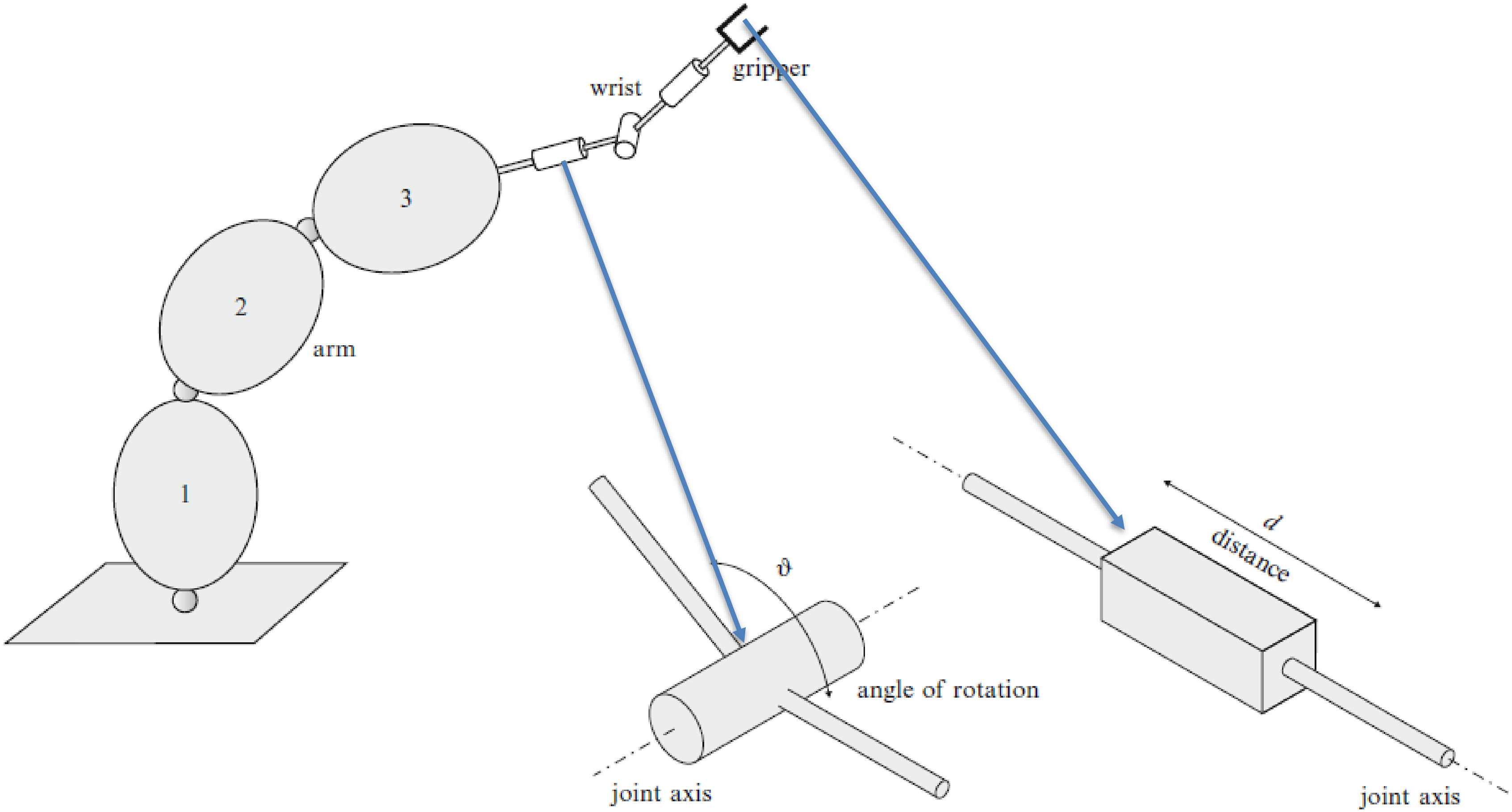
- Rigid body motion
- Coordinate transformation for rotation
- General motion: Forward & Inverse Kinematics
- Homogeneous Transformation & Matrix
- D-H method (parameters & matrix)
- Path planning & Trajectory
- Dynamics

Definitions & Preliminaries

- **Rigid Body**:
  - Solid body with negligible deformation.
  - Distance between 2 points remains constant.
  - Rigid Body Motion: 6 dofs – 3 translations & 3 rotations.
    - 3 translations: Position of the body
    - 3 rotations: Orientation of the body

- **Kinematics**:
  - Describes motion of points, bodies and systems of bodies *without considering the forces* that cause them to move.
    - Trajectories & velocities can be measured by joint sensors described through joint variables
    - Position and orientation of end-effectors is described through external coordinates

- **Objective**:
  - Develop a systematic modelling to describe geometric relationship between rigid bodies & the kinematics of a point of concern in a rigid body using coordinate frames & transformation.
- **Overarching assumptions** (…*rigid bodies*):
  - Industrial robot manipulator modelled as open kinematic chain
  - Consists of rigid links/parts
  - Rotational & translational motion



- **Industrial robot manipulator is modeled as an open kinematic chain**

- **Conceptualization** (…*point of concern*):
  - End gripper/link described thru position vector ***P***
  - Derivation of velocity ***V*** & acceleration ***A***
- **Tools** (…*coordinate frames & transformation*):
  - Defining appropriate global and local coordinate systems/frames
  - Leveraging transformation of coordinate systems to attain full modelling

Mathematical background

Scalars, Vectors and Matrices

A *scalar* quantity is expressible as a single, real number.

A quantity having direction as well as magnitude is called a *vector*. In addition, vectors must have certain transformation properties. For example, vector magnitudes are unchanged after a rotation of axes.

A *matrix* quantity is expressible as a two-dimensional array of numbers.

Scalar	*Vector*	*Matrix*
M---	F----	I------
E-----	M-----	*xxx*------
T----------	V-------	
T---	A-----------	

Scalar (or dot) product

Given $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$, $\boldsymbol{b} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}$

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b} &= a_xb_x + a_yb_y + a_zb_z \\ &\quad + (\ldots.)\,\boldsymbol{i}\cdot\boldsymbol{j} + (\ldots.)\,\boldsymbol{i}\cdot\boldsymbol{k} + (\ldots.)\,\boldsymbol{j}\cdot\boldsymbol{k} \\ &= a_xb_x + a_yb_y + a_zb_z\end{aligned}$$

Vector (or cross) product

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \quad \text{or} \quad \begin{aligned} &(a_yb_z - a_zb_y)\,\boldsymbol{i} \\ &+ (a_zb_x - a_xb_z)\,\boldsymbol{j} \\ &+ (a_xb_y - a_yb_x)\,\boldsymbol{k}\end{aligned}$$

Matrix

A matrix is a set of numbers which are arranged in rows and columns.

A *rectangular* matrix of order $m \times n$ has m rows and n columns and is written in the form

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & - & - & A_{1n} \\ A_{21} & A_{22} & - & - & A_{2n} \\ \vdots & \vdots & - & - & \vdots \\ \vdots & \vdots & - & - & \vdots \\ A_{m1} & A_{m2} & - & - & A_{mn} \end{bmatrix}$$

diagonal matrix:

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix}; \quad \begin{array}{c} D_{ij} = 0, \text{if } \; i \neq j; \\ i, j = 1 \text{ to } 3 \end{array}$$

unit (or identity) matrix:

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

symmetric matrix: a square matrix for which

$$\boldsymbol{A} = \boldsymbol{A}^T \quad (\text{or } A_{ij} = A_{ji})$$

skew-symmetric matrix (anti-symmetric): a square matrix for $A_{ij} = - A_{ji}$. This requires that the elements on the principal diagonal be zero.

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 0 & -A_{21} & -A_{31} \\ A_{21} & 0 & -A_{32} \\ A_{31} & A_{32} & 0 \end{bmatrix}$$

Inverse of matrix

$$\mathbf{A}^{-1} = \frac{\text{adjoint matrix of } \mathbf{A}}{|\mathbf{A}|}$$

Note that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \ \text{(Identity Matrix)}$$

addition: $C_{ij} = A_{ij} + B_{ij}$ (or $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$)

subtraction: $C_{ij} = A_{ij} - B_{ij}$

multiplication: $$\underset{m \times q}{C_{ij}} = \sum_{k=1}^{n} \underset{m \times \boldsymbol{n}}{A_{i\boldsymbol{k}}} \; \underset{\boldsymbol{n} \times q}{B_{\boldsymbol{k}j}}$$

The multiplication is defined only when the two matrices are *conformable*, i.e. the number of columns in the first matrix is equal to the number of rows in the second, i.e. $\boldsymbol{k}=\boldsymbol{k}$.

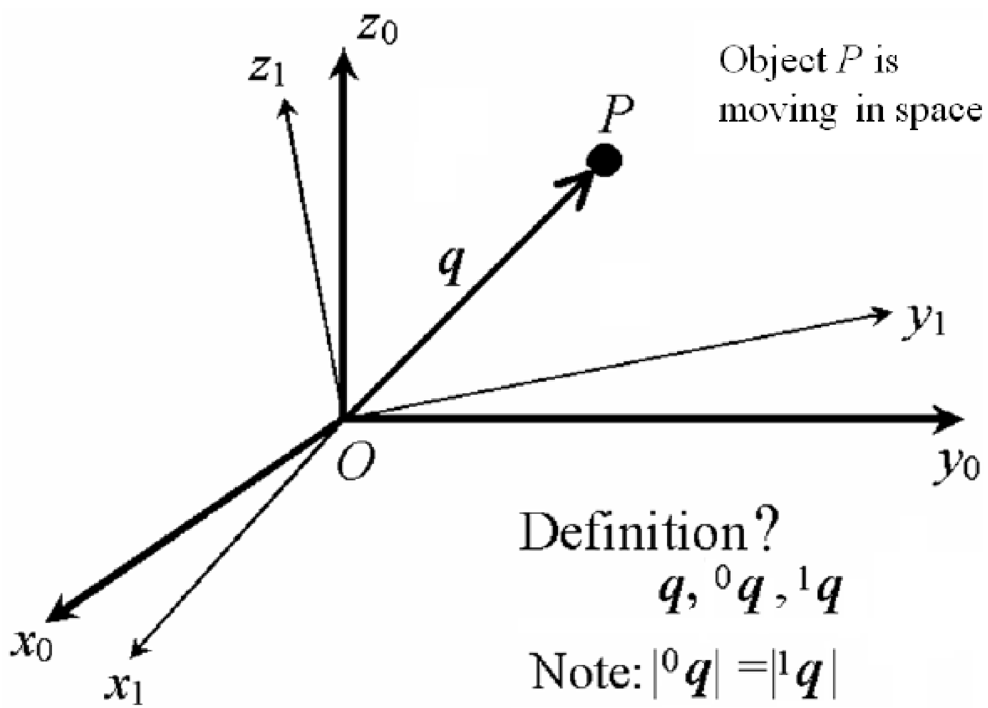
$\boldsymbol{C} = \boldsymbol{AB}$; $\boldsymbol{B}$ is *pre*-multiplied by $\boldsymbol{A}$, or $\boldsymbol{A}$ is *post*-multiplied by $\boldsymbol{B}$.

Note: $\boldsymbol{AB} \neq \boldsymbol{BA}$ [*row*1 x *col*1] ≠ [*row*2 x *col*2]

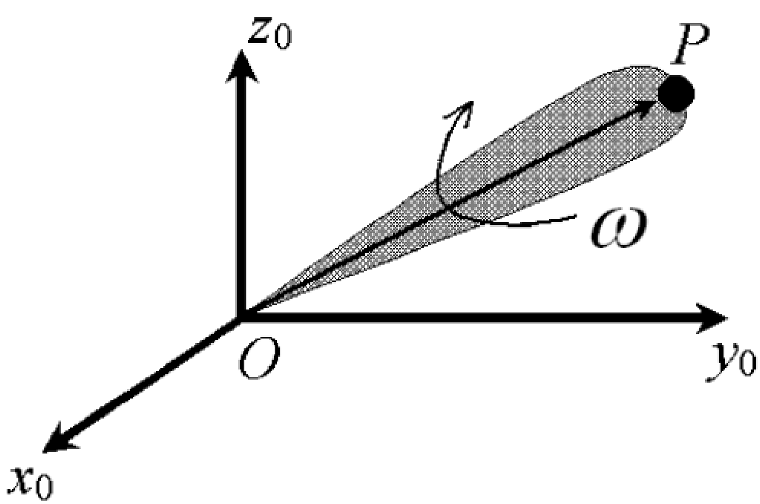
RIGID BODY MOTION

Pure Rotation & Coordinate Systems with Shared Origin

Pure Rotation: an object in two Cartesian coordinate systems *with a common origin*

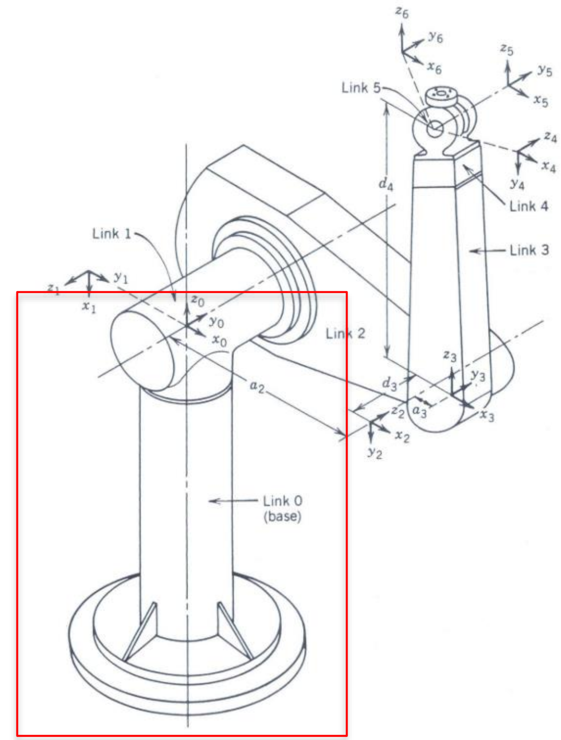


A Single-Link System

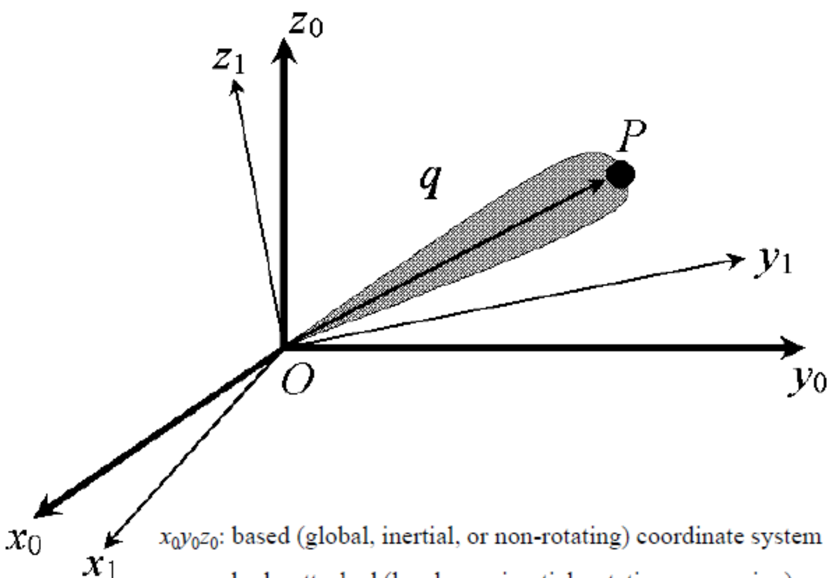


Rigid link OP rotates at ω about point O

$x_0y_0z_0$: Inertial coordinate system

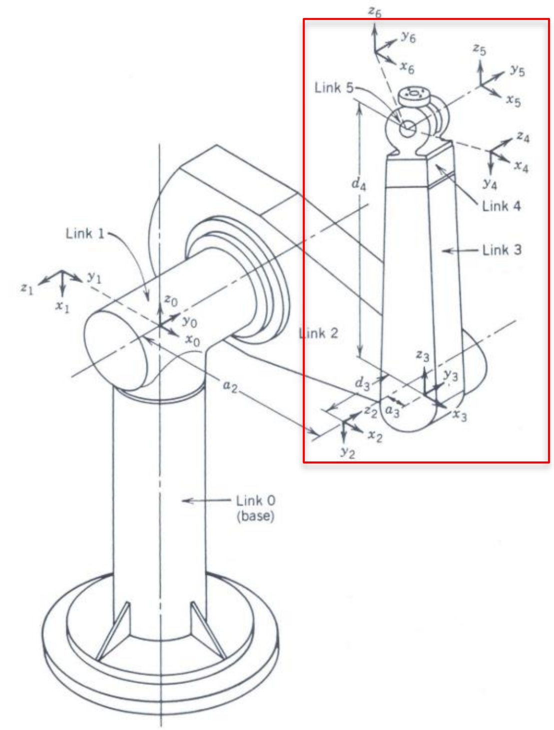


A Single-Link System



$x_0y_0z_0$: based (global, inertial, or non-rotating) coordinate system

$x_1y_1z_1$: body-attached (local, non-inertial, rotating, or moving) coordinate system

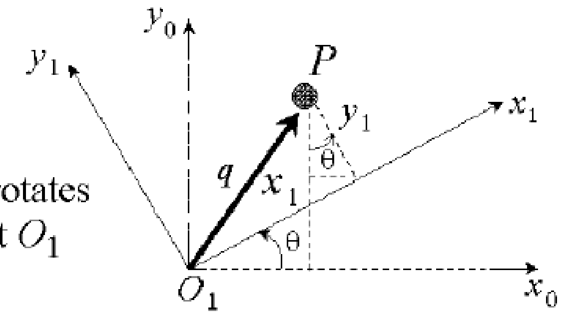


Coordinate Transformation via Rotation Matrix

Pure Rotation

Two-dimensional problem

O_1P rotates about O_1



$$x_0 = x_1 \cos\theta - y_1 \sin\theta \qquad \text{—— (1)}$$
$$y_0 = x_1 \sin\theta + y_1 \cos\theta \qquad \text{—— (2)}$$

or expressed in vector-matrix or matrix form

$$\begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} \quad \begin{matrix} \leftarrow \text{Eq. (1)} \\ \leftarrow \text{Eq. (2)} \end{matrix}$$

where $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{2\text{x}2} \Leftarrow$ rotation matrix, $\mathbf{R}_{01}$

Representation in 3D problem (same origin)

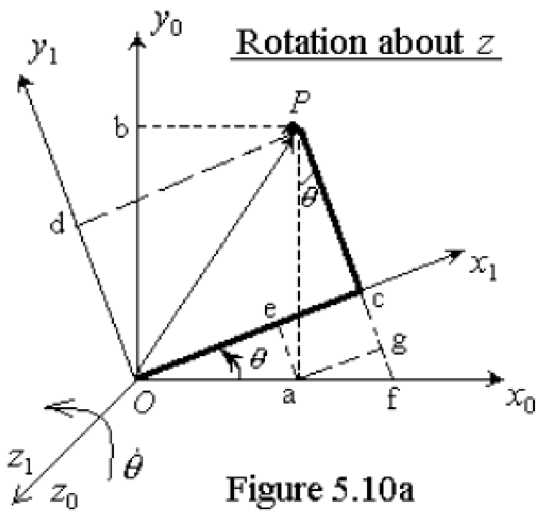


Figure 5.10a

$$x_0 = x_1 \cos\theta - y_1 \sin\theta$$
$$y_0 = x_1 \sin\theta + y_1 \cos\theta$$
$$z_0 = z_1$$

in matrix form

$$\begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}$$

or

$$x_1 = x_0 \cos\theta + y_0 \sin\theta$$
$$y_1 = -x_0 \sin\theta + y_0 \cos\theta$$
$$z_1 = z_0$$

in matrix form

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix}$$

Transformation of Co-ordinates (Chapter 5)

The relations between the components of a vector $\boldsymbol{q}$ (OP in the figure) in various coordinate systems are very useful to many problems of dynamics, especially those in robotics.

Express the vector $\boldsymbol{q}$ in terms of components along two Cartesian sets of axes $\{x_0, y_0, z_0\}$ and $\{x_1, y_1, z_1\}$.

The unit vectors along these axes are denoted by

$\{\boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0\}$ and $\{\boldsymbol{i}_1, \boldsymbol{j}_1, \boldsymbol{k}_1\}$, respectively.

Expression of $\boldsymbol{q}$ in the respective frame

$$\mathbf{q} \Rightarrow \quad {}^{0}\mathbf{q} = x_0\mathbf{i}_0 + y_0\mathbf{j}_0 + z_0\mathbf{k}_0$$

$$\text{or} \quad {}^{1}\mathbf{q} = x_1\mathbf{i}_1 + y_1\mathbf{j}_1 + z_1\mathbf{k}_1$$

Note: $\sqrt{x_0^2 + y_0^2 + z_0^2} = \sqrt{x_1^2 + y_1^2 + z_1^2}$ i.e. constant length OP

Rotation matrix

relation between x_0, x_1; y_0, y_1; and z_0, z_1.

$$\begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}$$

$\boldsymbol{R}_{01}$

or $\;{}^0\mathbf{q} = \mathbf{R}_{01}\,{}^1\mathbf{q}$

$3\times1 \quad 3\times3 \quad 3\times1$

${}^0\mathbf{q} = x_0\mathbf{i}_0 + y_0\mathbf{j}_0 + z_0\mathbf{k}_0$

${}^1\mathbf{q} = x_1\mathbf{i}_1 + y_1\mathbf{j}_1 + z_1\mathbf{k}_1$

$\boldsymbol{R}_{01}$: transformation (or rotation) matrix from

System 1 $\{x_1, y_1, z_1\}$ to System 0 $\{x_0, y_0, z_0\}$

Find ${}^1\boldsymbol{q}$, given ${}^0\boldsymbol{q}$ $\qquad {}^1\mathbf{q} = \mathbf{R}_{10}\,{}^0\mathbf{q} = \mathbf{R}_{01}^{-1}\,{}^0\mathbf{q} = \mathbf{R}_{01}^{T}\,{}^0\mathbf{q}$

Orthogonal property: $\qquad \Rightarrow \mathbf{R}^{-1} = \mathbf{R}^{T}$

Coordinate transformation of *vectors* by making use of $\boldsymbol{R}$

$${}^1\mathbf{q} = \mathbf{R}_{10}\,{}^0\mathbf{q} \quad \text{or} \quad {}^0\mathbf{q} = \mathbf{R}_{10}^{T}\,{}^1\mathbf{q} \quad \text{since} \quad \mathbf{R}_{10}^{-1} = \mathbf{R}_{10}^{T}$$

where

${}^1\mathbf{q}$ $\quad 3\times1$ vector $\mathbf{q}$ expressed in coordinate $\{x_1 y_1 z_1\}$

${}^0\mathbf{q}$ $\quad 3\times1$ vector $\mathbf{q}$ expressed in coordinate $\{x_0 y_0 z_0\}$

Recall ....

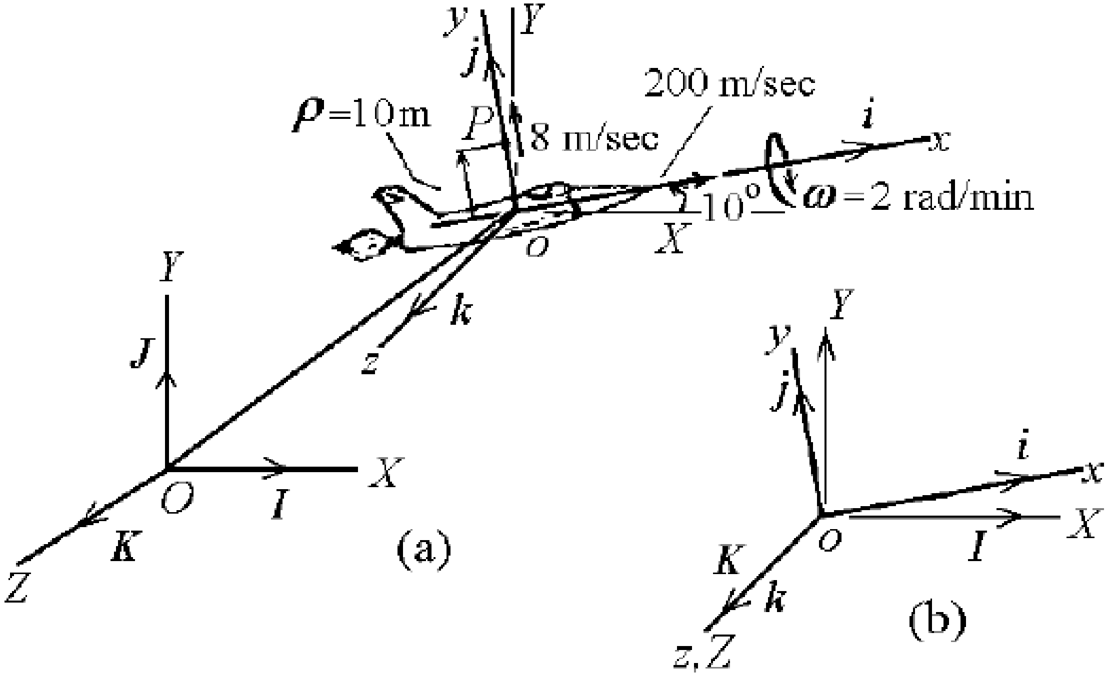
$$\begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}$$

Direction cosine

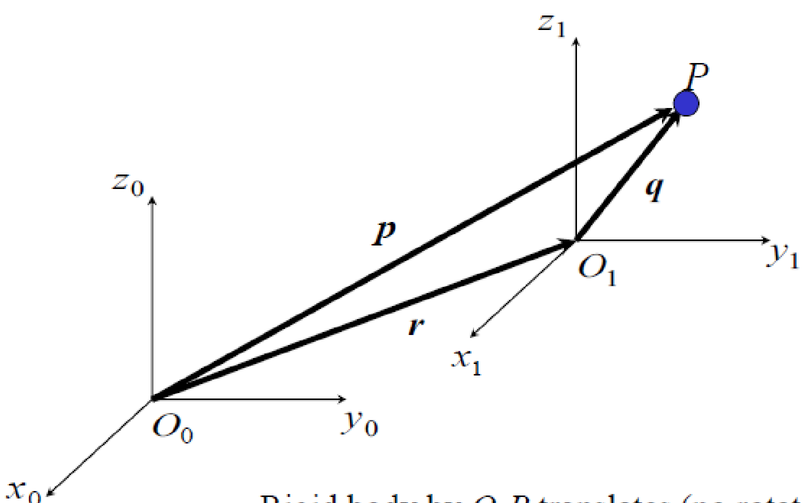
or $\underset{3\times 1}{{}^{0}\mathbf{q}} = \underset{3\times 3}{\mathbf{R}_{01}} \; \underset{3\times 1}{{}^{1}\mathbf{q}}$

Rotation matrix

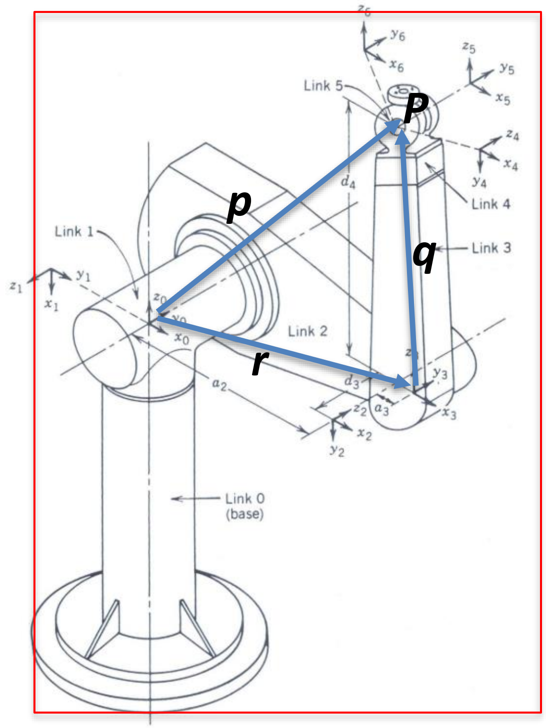
General Motion (Rotation & Translation) & Coordinate Systems with Different Origins



Coordinate transformation:
***pure translation of system* 1 *(w.r.t. system* 0)**



Rigid body by O_1P translates (no rotation)
from O_0 to O_1 along $\boldsymbol{r}$



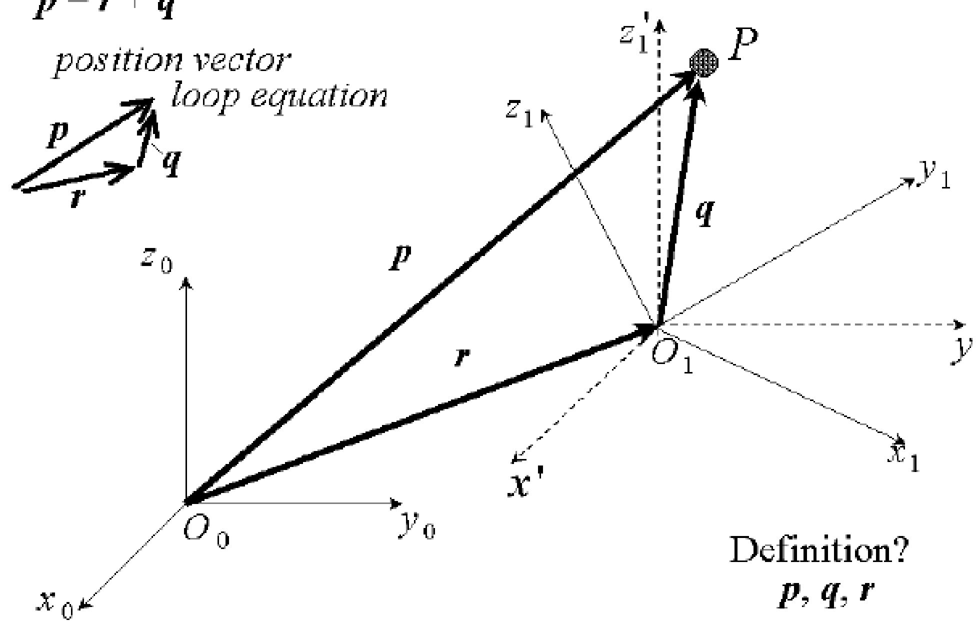
Coordinate transformation: combined translation and rotation

First, rigid body O_1P translates (no rotation) from O_0 to O_1. Next, O_1P rotates about O_1.

Position of point P:

$\boldsymbol{p} = \boldsymbol{r} + \boldsymbol{q}$

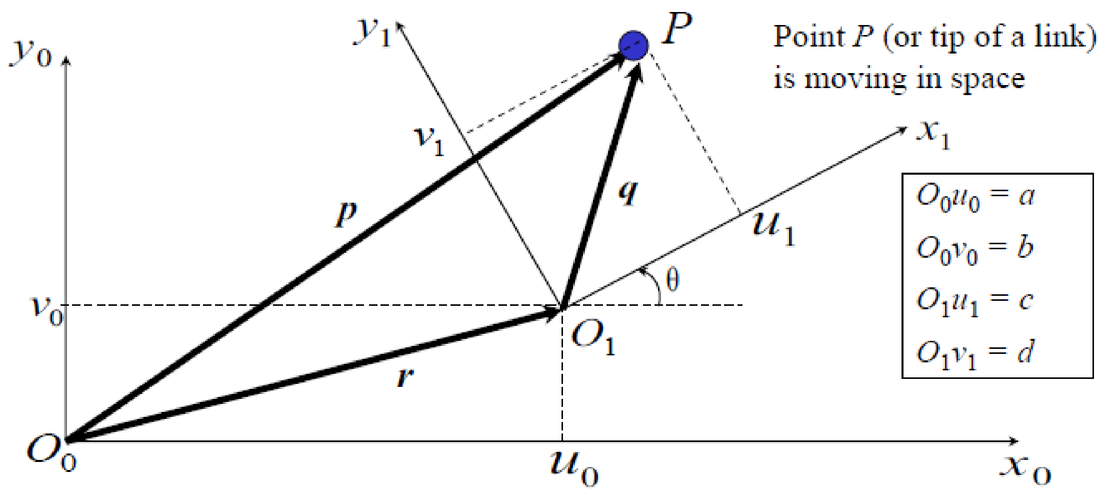
position vector loop equation



Definition?

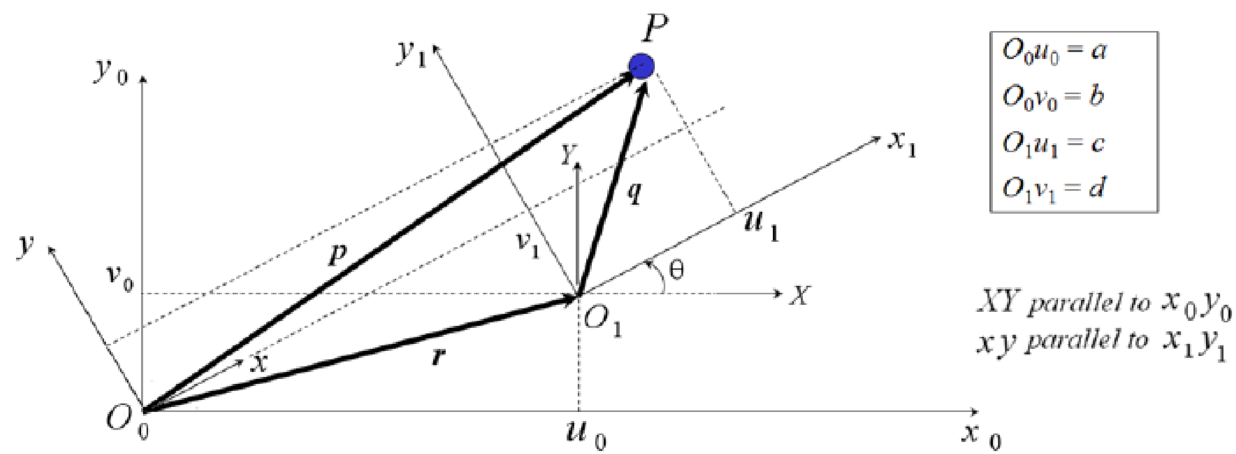
p, q, r

***Explanation*: Position Vector Loop**



Position vector of point P (vector loop equation):

$\boldsymbol{p} = \boldsymbol{r} + \boldsymbol{q}$ (Generic; independent of coordinate systems)



${}^0\boldsymbol{p}$: position vector of Point P expressed in terms of system 0

${}^1\boldsymbol{p}$: position vector of Point P expressed in terms of system 1 (system xy)

Note: ${}^1\boldsymbol{p}$ is the position vector of point P with respect to the fixed point O_0, in which the vector is expressed in terms of coordinate system xy (parallel to x_1y_1 in all times)

Translation and Rotation about axis z

$\boldsymbol{p} = \boldsymbol{r} + \boldsymbol{q}$ (generic)

$${}^0\boldsymbol{p} = {}^0\boldsymbol{r} + {}^0\boldsymbol{q} = {}^0\boldsymbol{r} + \boldsymbol{R}_{01}\,{}^1\boldsymbol{q}$$

Since ${}^0\boldsymbol{p} = p_{x0}\,\hat{\boldsymbol{i}}_0 + p_{y0}\,\hat{\boldsymbol{j}}_0$,

$$p_{x0} = a + c\cos\theta - d\sin\theta$$
$$p_{y_0} = b + c\sin\theta + d\cos\theta$$

or

$$\begin{Bmatrix} p_{x0} \\ p_{y_0} \end{Bmatrix} = \begin{Bmatrix} a \\ b \end{Bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} c \\ d \end{Bmatrix}$$

where

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Leftarrow \text{rotation matrix, } \mathbf{R}_{01}$$

Definition:

$\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, ${}^0\boldsymbol{p}$, ${}^1\boldsymbol{p}$, ${}^0\boldsymbol{q}$, etc.?

How about

${}^1\boldsymbol{p} = {}^1\boldsymbol{r} + {}^1\boldsymbol{q}$?

For example:

${}^0\boldsymbol{p}$: position vector of Point P expressed in terms of system 0

${}^1\boldsymbol{p}$: position vector of Point P expressed in terms of (virtual) system 1

Why expressed in system 1?

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Transport Theorem

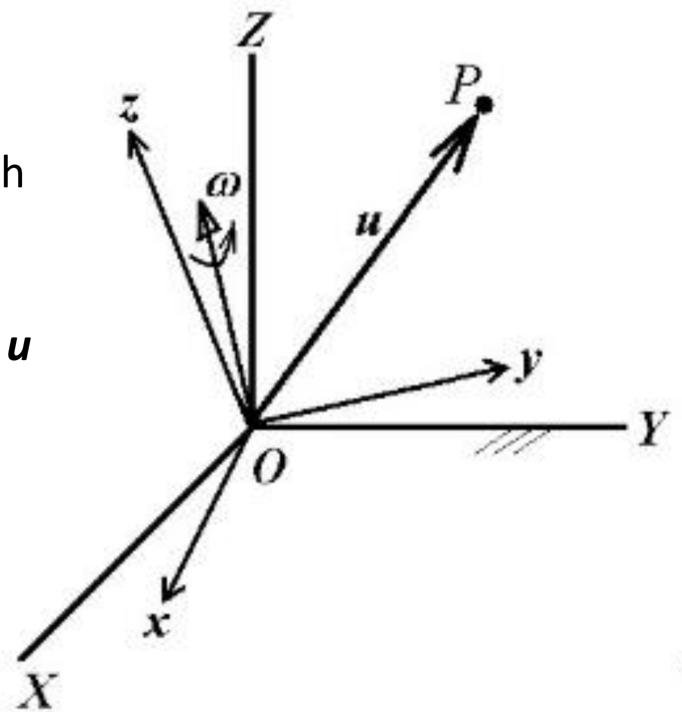
- Formal definition for rate of change of vector expressed in observed reference frame to the desired reference frame.

$$^{A}\left[\frac{d\boldsymbol{u}}{dt}\right] = {}^{B}\left[\frac{d\boldsymbol{u}}{dt}\right] + {}^{A}\boldsymbol{\omega}_{B} \times \boldsymbol{u}$$

- For vector defined in frame B, the rate of change of said vector defined in frame A is equivalent to summation of rate of change of the vector defined in frame B and the vector's angular velocity

• Point in rotating frame

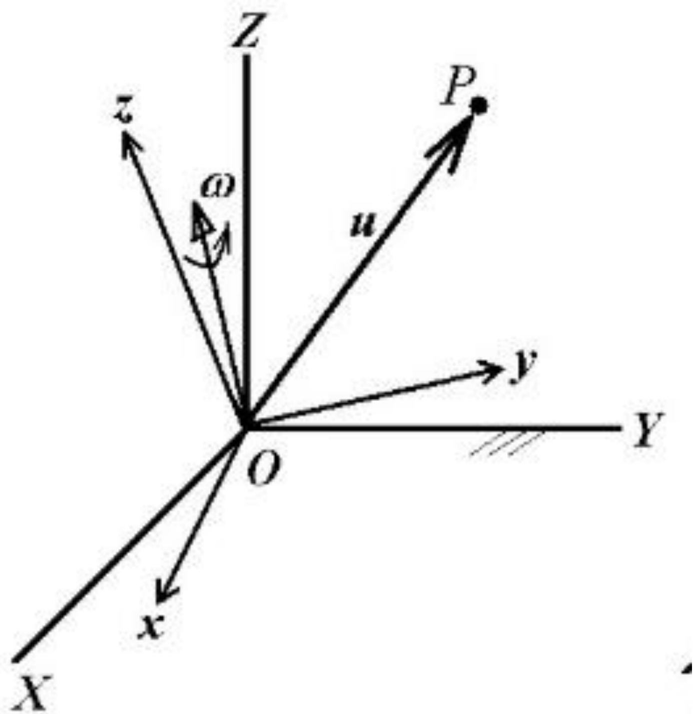
1. Global Frame XYZ
2. Local Frame xyz rotating with absolute angular velocity $\boldsymbol{\omega}$
3. Both frames share origin O
4. Point P with position vector $\boldsymbol{u}$
5. Point P is stationary



• Define absolute position vector $\boldsymbol{u}$

$$\boldsymbol{u} = u_x\,\boldsymbol{i} + u_y\,\boldsymbol{j} + u_z\,\boldsymbol{k}$$

1. Global Frame XYZ
2. Local Frame xyz rotating with $\boldsymbol{\omega}$
3. Both frames share origin *O*
4. Point *P* defined $\boldsymbol{u}$
5. Point *P* is stationary



- Define absolute velocity vector $\dot{\boldsymbol{u}}$

$$\dot{\boldsymbol{u}} = (\dot{u}_x \boldsymbol{i} + \dot{u}_y \boldsymbol{j} + \dot{u}_z \boldsymbol{k}) + (u_x \dot{\boldsymbol{i}} + u_y \dot{\boldsymbol{j}} + u_z \dot{\boldsymbol{k}})$$

$$\dot{\boldsymbol{u}} = (\dot{u}_x \boldsymbol{i} + \dot{u}_y \boldsymbol{j} + \dot{u}_z \boldsymbol{k}) + (u_x \dot{\boldsymbol{i}} + u_y \dot{\boldsymbol{j}} + u_z \dot{\boldsymbol{k}})$$

$$\left(\frac{d\boldsymbol{u}}{dt}\right)_{\text{rel}} = (\dot{u}_x \boldsymbol{i} + \dot{u}_y \boldsymbol{j} + \dot{u}_z \boldsymbol{k}) = (\dot{\boldsymbol{u}})_r$$

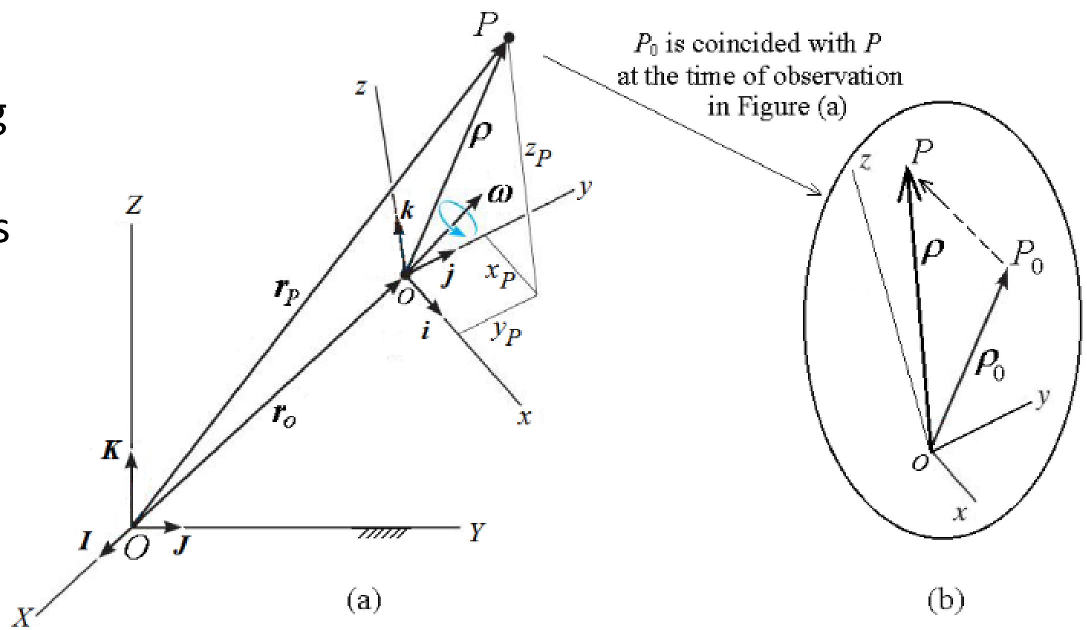
$$\dot{\boldsymbol{i}} = \boldsymbol{\omega} \times \boldsymbol{i}, \quad \dot{\boldsymbol{j}} = \boldsymbol{\omega} \times \boldsymbol{j}, \quad \dot{\boldsymbol{k}} = \boldsymbol{\omega} \times \boldsymbol{k}, \quad \boxed{\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}}$$

$$\begin{aligned}(u_x \dot{\boldsymbol{i}} + u_y \dot{\boldsymbol{j}} + u_z \dot{\boldsymbol{k}}) &= u_x(\boldsymbol{\omega} \times \boldsymbol{i}) + u_y(\boldsymbol{\omega} \times \boldsymbol{j}) + u_z(\boldsymbol{\omega} \times \boldsymbol{k}) \\ &= \boldsymbol{\omega} \times (u_x \boldsymbol{i} + u_y \boldsymbol{j} + u_z \boldsymbol{k}) = \boldsymbol{\omega} \times \boldsymbol{u}\end{aligned}$$

$$\dot{\boldsymbol{u}} = \frac{d\boldsymbol{u}}{dt} = \left(\frac{d\boldsymbol{u}}{dt}\right)_{\text{rel}} + \boldsymbol{\omega} \times \boldsymbol{u} = (\dot{\boldsymbol{u}})_r + \boldsymbol{\omega} \times \boldsymbol{u}$$

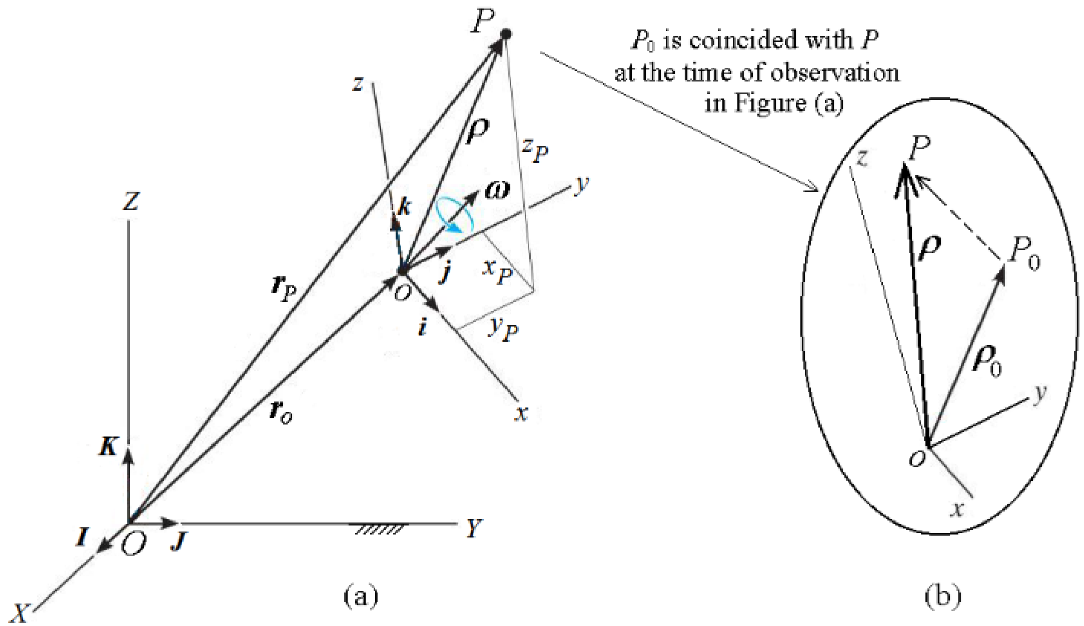
Frame in general motion

1. Global Frame XYZ
2. Local Frame xyz rotating with $\boldsymbol{\omega}$
3. Origin of local frame *o* is offset from *O* by $\boldsymbol{r}_o$
4. Point *P* defined $\boldsymbol{\rho}$



- Define absolute position vector $\boldsymbol{r}_p$

$$\boldsymbol{r}_P = \boldsymbol{r}_o + \boldsymbol{\rho} \qquad \boldsymbol{\rho} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$



(a) (b)

- Define general absolute velocity vector ***v***

$$\boldsymbol{v} = \dot{\boldsymbol{r}}_P = \dot{\boldsymbol{r}}_O + \dot{\boldsymbol{\rho}}$$

$$\boldsymbol{v} = \dot{\boldsymbol{r}}_P = \dot{\boldsymbol{r}}_o + \dot{\boldsymbol{\rho}} \qquad \boldsymbol{\rho} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\dot{\boldsymbol{\rho}} = (\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}) + (x\dot{\boldsymbol{i}} + y\dot{\boldsymbol{j}} + z\dot{\boldsymbol{k}})$$

$$\dot{\boldsymbol{\rho}} = (\dot{\boldsymbol{\rho}})_r + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

$$\underline{\boldsymbol{v} = \dot{\boldsymbol{r}}_P = \dot{\boldsymbol{r}}_o + \boldsymbol{\omega} \times \boldsymbol{\rho} + (\dot{\boldsymbol{\rho}})_r}$$

$\dot{\boldsymbol{r}}_P$ = velocity (absolute) of point P relative <u>to fixed point O</u> (origin of XYZ-system; inertial system)

$\dot{\boldsymbol{r}}_o$ = velocity (absolute) of point o (origin of xyz-system) relative to fixed point O (origin of XYZ-system; inertial system)

$\boldsymbol{\omega}$ = angular velocity (absolute) of xyz-system relative to fixed point O (origin of XYZ-system; inertial system)

$(\dot{\boldsymbol{\rho}})_r$ = velocity of point P relative to point o (origin of xyz-system)

- Define general absolute acceleration vector ***a***

$$\boldsymbol{a} = \frac{d}{dt}(\dot{\boldsymbol{r}}_P) = \frac{d}{dt}[\dot{\boldsymbol{r}}_O + \boldsymbol{\omega} \times \boldsymbol{\rho} + (\dot{\boldsymbol{\rho}})_r]$$

$$\frac{d}{dt}[(\dot{\boldsymbol{\rho}})_r] = (\ddot{\boldsymbol{\rho}})_r + \boldsymbol{\omega} \times (\dot{\boldsymbol{\rho}})_r$$

$$\frac{d}{dt}(\dot{\boldsymbol{r}}_O) = \ddot{\boldsymbol{r}}_O$$

$$\dot{\boldsymbol{\rho}} = (\dot{\boldsymbol{\rho}})_r + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

$$\frac{d}{dt}(\boldsymbol{\omega} \times \boldsymbol{\rho}) = \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}$$

$$\boldsymbol{a} = \ddot{\boldsymbol{r}}_P = \ddot{\boldsymbol{r}}_O + \underbrace{\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}}_{\text{Tangential acceleration}} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})}_{\text{Centripetal acceleration}} + (\ddot{\boldsymbol{\rho}})_r + \underbrace{2\boldsymbol{\omega} \times (\dot{\boldsymbol{\rho}})_r}_{\text{Coriolis}}$$

$$a = \ddot{r}_P = \ddot{r}_o + \underbrace{\dot{\omega} \times \rho}_{\text{Tangential acceleration}} + \underbrace{\omega \times (\omega \times \rho)}_{\text{Centripetal acceleration}} + (\ddot{\rho})_r + \underbrace{2\omega \times (\dot{\rho})_r}_{\text{Coriolis}}$$

Note that

$\ddot{r}_P$ = acceleration (absolute) of point P relative to fixed point O (origin of XYZ-system; inertial system)

in which

$\ddot{r}_o$ = acceleration (absolute) of point o (origin of xyz-system) relative to fixed point O (origin of XYZ-system; inertial system)

$\dot{\omega}$ (or α) = angular acceleration (absolute) of xyz-system relative to fixed point O (origin of XYZ-system; inertial system)

$(\ddot{\rho})_r$ = acceleration of point P relative to point o (origin of xyz-system)

Note that $a = \ddot{r}_o + \dot{\omega} \times \rho + \omega \times (\omega \times \rho)$

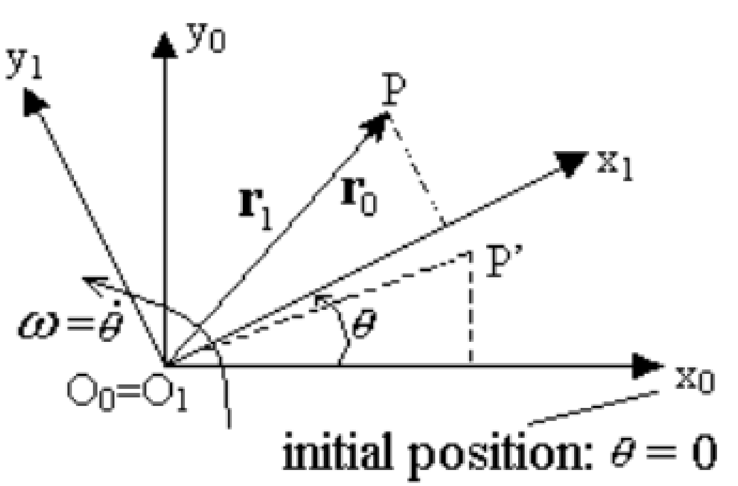
if $(\dot{\rho})_r = (\ddot{\rho})_r = 0$.

COORDINATE TRANSFORMATION FOR ROTATION

Coordinate transform due to rotation

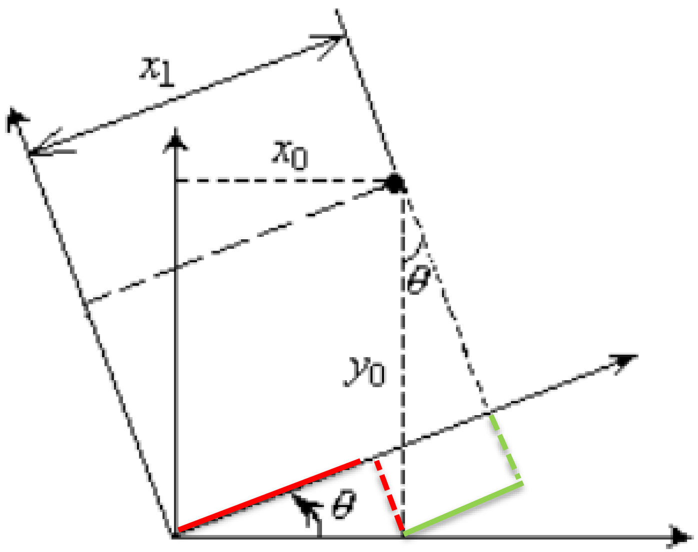
- General motion in rigid body motion consists of translation and rotation.
- Consider and account for rotation.

1. Global Frame $x_o y_o z_o$
2. Local Frame $x_1 y_1 z_1$ rotating with $\boldsymbol{\omega}=\dot{\boldsymbol{\theta}}$ (Right hand rule)
3. Both frames share origin *O*
4. Original point *P'* moves to *P* due to rotation
5. Point *P* defined by $\boldsymbol{r_1}$ and or $\boldsymbol{r_0}$.

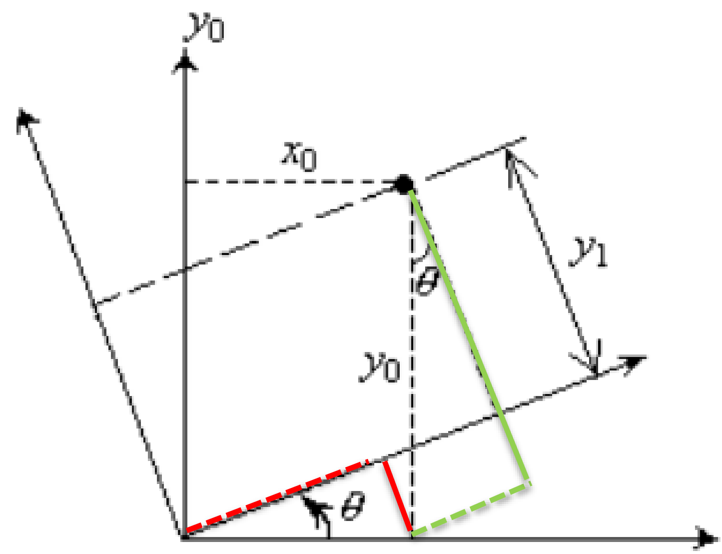


$$r_1 = r_0 \text{ or } x_1\hat{i}_1 + y_1\hat{j}_1 = x_0\hat{i}_0 + y_0\hat{j}_0$$

- **Point in rotating frame**



$$x_1 = x_0 \cos\theta + y_0 \sin\theta$$



$$y_1 = -x_0 \sin\theta + y_0 \cos\theta$$

$$x_1\hat{\boldsymbol{i}}_1 + y_1\hat{\boldsymbol{j}}_1 = (x_0 \cos\theta + y_0 \sin\theta)\hat{\boldsymbol{i}}_1 + (-x_0 \sin\theta + y_0 \cos\theta)\hat{\boldsymbol{j}}_1$$

$$x_1\hat{\boldsymbol{i}}_1 + y_1\hat{\boldsymbol{j}}_1 = (x_0\cos\theta + y_0\sin\theta)\hat{\boldsymbol{i}}_1 + (-x_0\sin\theta + y_0\cos\theta)\hat{\boldsymbol{j}}_1$$

Rotation matrix

$$\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix}$$

$$\boxed{\begin{Bmatrix} \hat{\boldsymbol{i}}_1 \\ \hat{\boldsymbol{j}}_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{i}}_0 \\ \hat{\boldsymbol{j}}_0 \end{Bmatrix}}$$

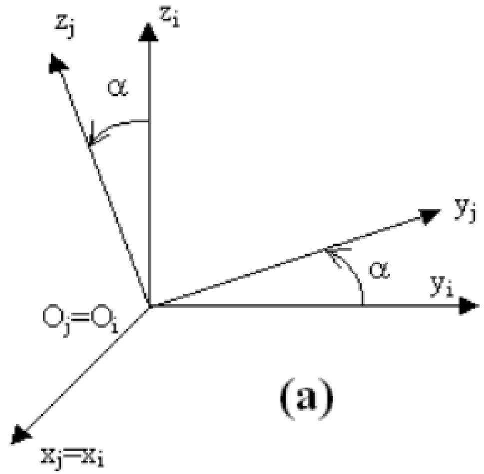
$$\begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}$$

Inverse rotation matrix

$$\boxed{\begin{Bmatrix} \hat{\boldsymbol{i}}_0 \\ \hat{\boldsymbol{j}}_0 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{i}}_1 \\ \hat{\boldsymbol{j}}_1 \end{Bmatrix}}$$

$$\boldsymbol{R}_{ij}(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

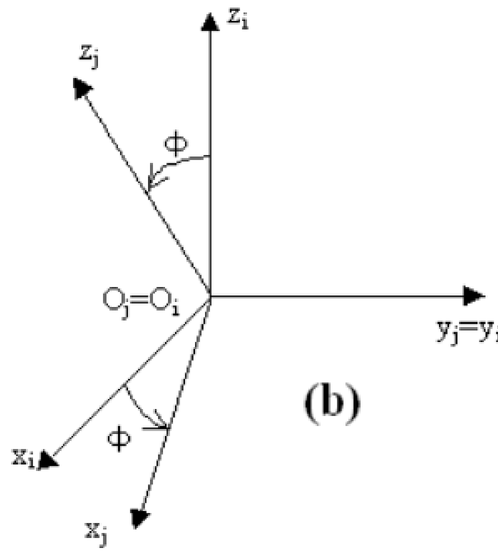
- Basic Rotation Matrix
  - about x-axis



(a)

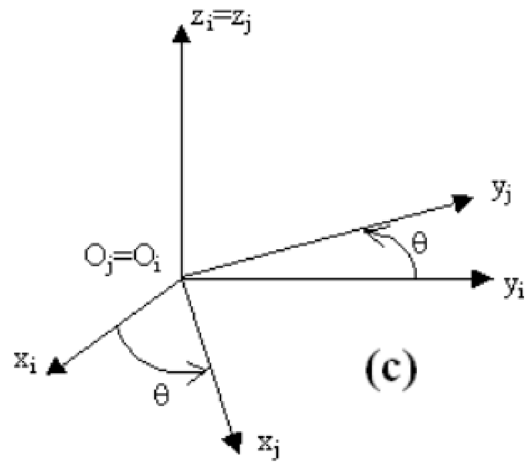
$$\boldsymbol{R}_{ij}(x,\alpha)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix}, \quad i<j$$

- Basic Rotation Matrix
  - about y-axis



$$\boldsymbol{R}_{ij}(y,\phi)=\begin{bmatrix}\cos\phi & 0 & \sin\phi\\ 0 & 1 & 0\\ -\sin\phi & 0 & \cos\phi\end{bmatrix},\quad i<j$$

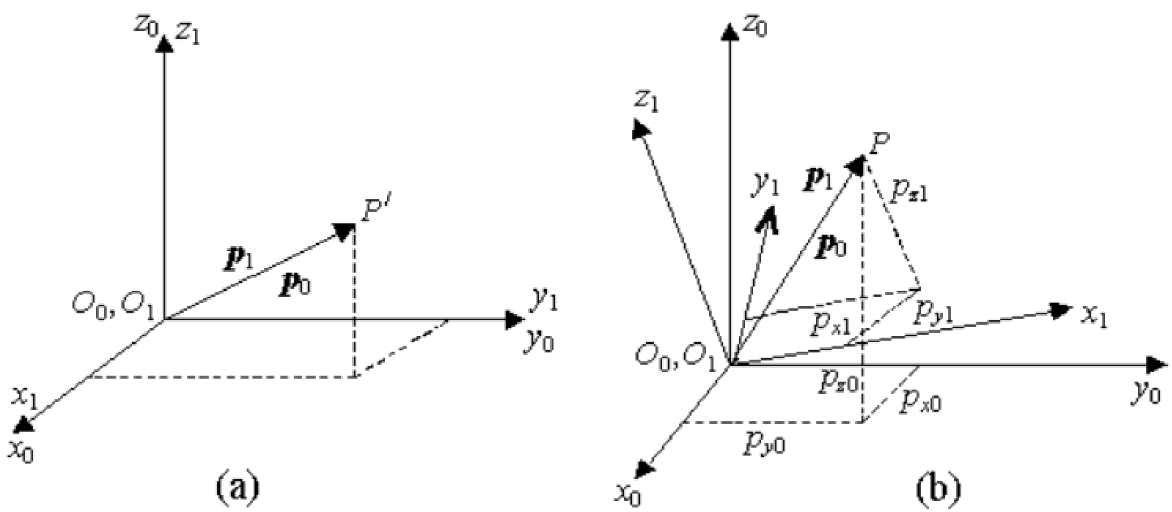
- Basic Rotation Matrix
  - about z-axis



(c)

$$\boldsymbol{R}_{ij}(z,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix},\quad i<j$$

- Generalization of the rotation matrix to 3 dimensions
- Introduction of *direction cosines* (dot product)



$$\boldsymbol{p}_0 = p_{x0}\,\boldsymbol{i}_0 + p_{y0}\,\boldsymbol{j}_0 + p_{z0}\,\boldsymbol{k}_0$$

$$\boldsymbol{p}_1 = p_{x1}\,\boldsymbol{i}_1 + p_{y1}\,\boldsymbol{j}_1 + p_{z1}\,\boldsymbol{k}_1$$

$$\boldsymbol{p}_0 = p_{x0}\,\boldsymbol{i}_0 + p_{y0}\,\boldsymbol{j}_0 + p_{z0}\,\boldsymbol{k}_0$$

$$\boldsymbol{p}_1 = p_{x1}\,\boldsymbol{i}_1 + p_{y1}\,\boldsymbol{j}_1 + p_{z1}\,\boldsymbol{k}_1$$

- What is the difference

$$\begin{aligned} p_{x0} &= \boldsymbol{p}_0 \cdot \boldsymbol{i}_0 = \boldsymbol{p}_1 \cdot \boldsymbol{i}_0 \text{ as } \boldsymbol{p}_0 = \boldsymbol{p}_1 \\ &= \left(p_{x1}\,\boldsymbol{i}_1 + p_{y1}\,\boldsymbol{j}_1 + p_{z1}\,\boldsymbol{k}_1\right)\cdot \boldsymbol{i}_0 \\ &= \left(\boldsymbol{i}_0 \cdot \boldsymbol{i}_1\right) p_{x1} + \left(\boldsymbol{i}_0 \cdot \boldsymbol{j}_1\right) p_{y1} + \left(\boldsymbol{i}_0 \cdot \boldsymbol{k}_1\right) p_{z1} \end{aligned}$$

$$\begin{aligned} p_{y0} &= \boldsymbol{p}_0 \cdot \boldsymbol{j}_0 = \boldsymbol{p}_1 \cdot \boldsymbol{j}_0 \\ &= \left(p_{x1}\,\boldsymbol{i}_1 + p_{y1}\,\boldsymbol{j}_1 + p_{z1}\,\boldsymbol{k}_1\right)\cdot \boldsymbol{j}_0 \\ &= \left(\boldsymbol{j}_0 \cdot \boldsymbol{i}_1\right) p_{x1} + \left(\boldsymbol{j}_0 \cdot \boldsymbol{j}_1\right) p_{y1} + \left(\boldsymbol{j}_0 \cdot \boldsymbol{k}_1\right) p_{z1} \end{aligned}$$

$$\begin{aligned} p_{z0} &= \boldsymbol{p}_0 \cdot \boldsymbol{k}_0 = \boldsymbol{p}_1 \cdot \boldsymbol{k}_0 \\ &= \left(p_{x1}\boldsymbol{i}_1 + p_{y1}\boldsymbol{j}_1 + p_{z1}\boldsymbol{k}_1\right)\cdot \boldsymbol{k}_0 \\ &= \left(\boldsymbol{k}_0 \cdot \boldsymbol{i}_1\right) p_{x1} + \left(\boldsymbol{k}_0 \cdot \boldsymbol{j}_1\right) p_{y1} + \left(\boldsymbol{k}_0 \cdot \boldsymbol{k}_1\right) p_{z1} \end{aligned}$$

$$\begin{Bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix} \begin{Bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{Bmatrix}$$

$$\begin{Bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix} \begin{Bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{Bmatrix}$$

$$\downarrow$$

Define $\boldsymbol{R}_{01} = \begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix}$

$$\boldsymbol{i}_0 \cdot \boldsymbol{i}_1 = \cos\theta_{Xx} \qquad \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 = \cos\theta_{Xy}, \quad \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 = \cos\theta_{Xz}$$

$$\downarrow$$

$$\boldsymbol{R}_{01} = \begin{bmatrix} \cos\theta_{Xx} & \cos\theta_{Xy} & \cos\theta_{Xz} \\ \cos\theta_{Yx} & \cos\theta_{Yy} & \cos\theta_{Yz} \\ \cos\theta_{Zx} & \cos\theta_{Zy} & \cos\theta_{Zz} \end{bmatrix} = \begin{bmatrix} R_{Xx} & R_{Xy} & R_{Xz} \\ R_{Yx} & R_{Yy} & R_{Yz} \\ R_{Zx} & R_{Zy} & R_{Zz} \end{bmatrix}$$

$$\underline{\boldsymbol{p}_0 = \boldsymbol{R}_{01}\boldsymbol{p}_1}$$

- Generalized rotation matrix

$$\boldsymbol{p}_1 = \boldsymbol{R}_{10}\boldsymbol{p}_0 ?$$

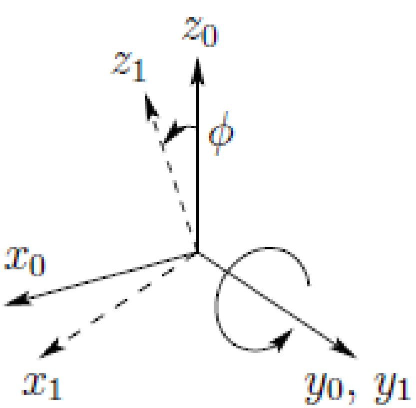
$$\boldsymbol{R}_{10} = \begin{bmatrix} \boldsymbol{i}_1 \cdot \boldsymbol{i}_0 & \boldsymbol{i}_1 \cdot \boldsymbol{j}_0 & \boldsymbol{i}_1 \cdot \boldsymbol{k}_0 \\ \boldsymbol{j}_1 \cdot \boldsymbol{i}_0 & \boldsymbol{j}_1 \cdot \boldsymbol{j}_0 & \boldsymbol{j}_1 \cdot \boldsymbol{k}_0 \\ \boldsymbol{k}_1 \cdot \boldsymbol{i}_0 & \boldsymbol{k}_1 \cdot \boldsymbol{j}_0 & \boldsymbol{k}_1 \cdot \boldsymbol{k}_0 \end{bmatrix} = \begin{bmatrix} R_{xX} & R_{xY} & R_{xZ} \\ R_{yX} & R_{yY} & R_{yZ} \\ R_{zX} & R_{zY} & R_{zZ} \end{bmatrix}$$

$\boldsymbol{R}_{01}$

$$\boxed{\begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix}}$$

- More than 1 rotation
  - Defined about current/localized frame

1. Global frame $x_o y_o z_o$
2. Local frame $x_1 y_1 z_1$ rotating
3. Both frames share origin
4. Rotation from global frame $x_o y_o z_o$ to frame $x_2 y_2 z_2$



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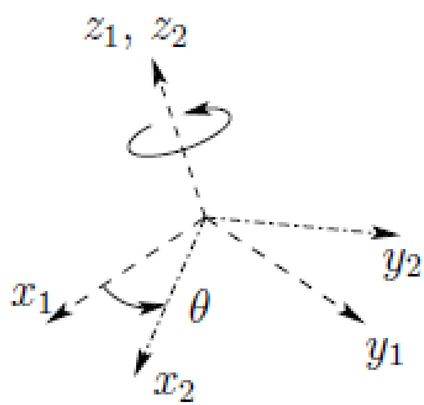

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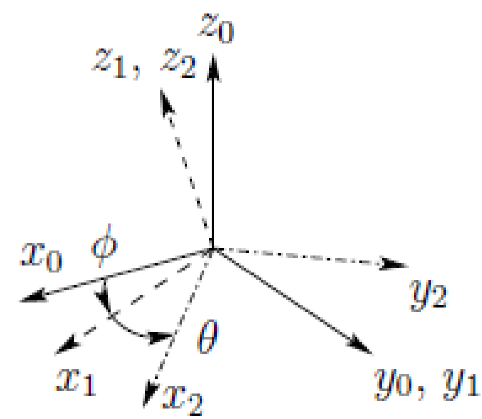


1. Rotation about y_1 axis

$$p_0 = R_{01} p_1 = R_{y_1,\Phi} p_1$$

2. Rotation about z_1 axis

$$p_1 = R_{12} p_2 = R_{z_1,\theta} p_2$$

$$p_0 = R_{01} R_{12} p_2$$

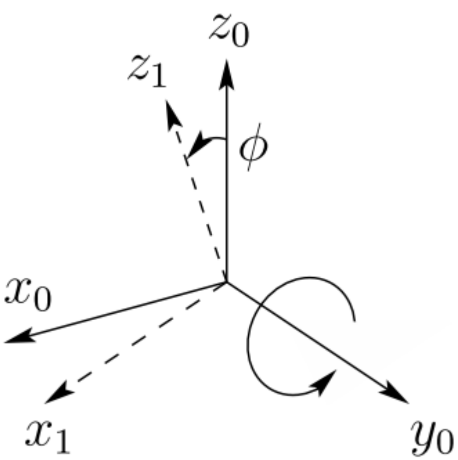
$$R_{02} = R_{01} R_{12} = \boldsymbol{R_{y_1,\Phi}}\ \boldsymbol{R_{z_1,\theta}}$$

- More than 1 rotation
  - Defined about fixed frame

Similarity Transformation

$$B = (R_1^0)^{-1} A R_1^0$$

1. Global frame $x_o y_o z_o$
2. Local frame $x_1 y_1 z_1$ rotating
3. Both frames share origin
4. Rotation from global frame $x_o y_o z_o$ to frame $x_2 y_2 z_2$



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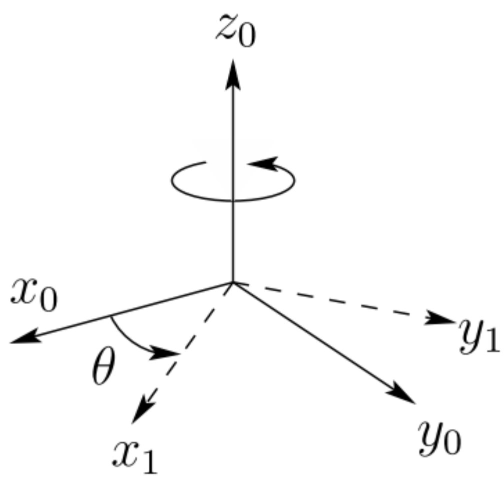

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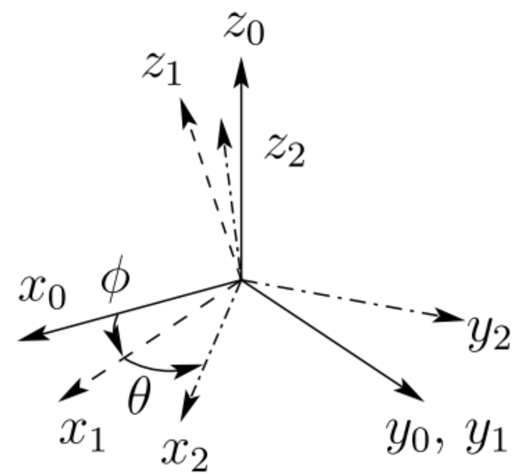


1. Rotation about y_0 axis

$$p_0 = R_{01} p_1 = R_{y_0,\Phi} p_1$$

2. Rotation about z_0 axis

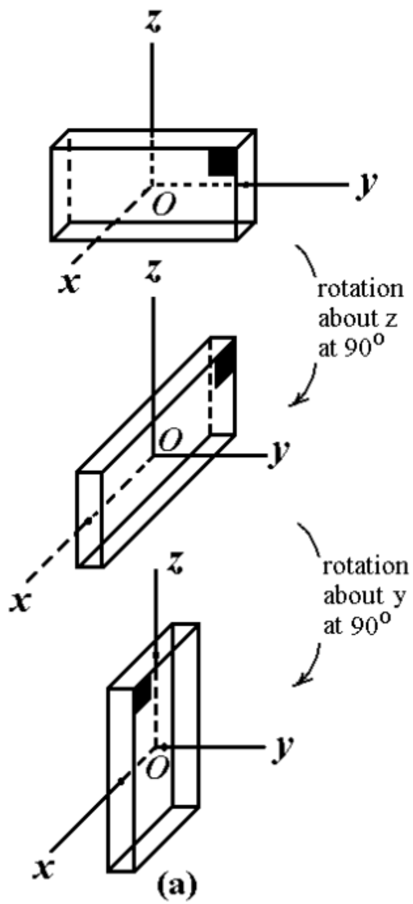
$$p_1 = R_{12} p_2 = R_{y_0,-\Phi} R_{z_0,\theta} R_{y_0,\Phi} p_2$$

$$p_0 = R_{01} R_{12} p_2$$

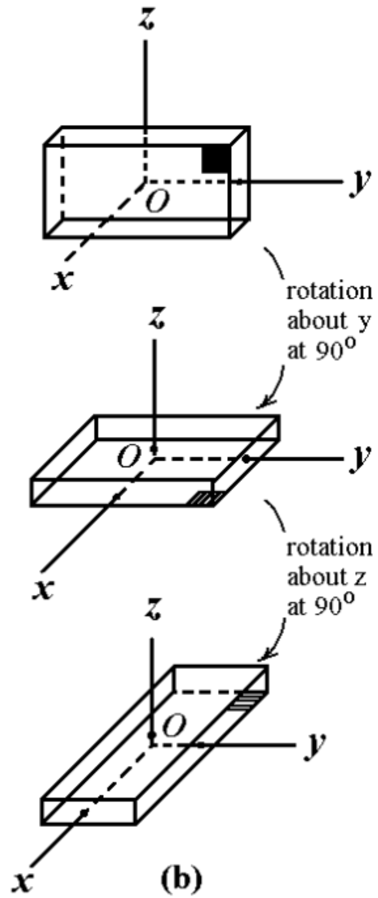
$$R_{02} = R_{\boldsymbol{y_0},\boldsymbol{\Phi}} \left[R_{y_0,-\Phi} R_{z_0,\theta} R_{y_0,\Phi} \right]$$
$$= \boldsymbol{R_{z_0,\theta} R_{y_0,\Phi}}$$

- Sequence of rotations

z - y



y - z



Order of sequence is **non-commutative**

…..sequence of rotation matters!

- Define the sequence of rotation: Cardan Angles & Euler Angles

x-y-x, x-y-z, x-z-x, x-z-y, y-x-y, y-x-z,
y-z-x, y-z-y, z-x-y, z-x-z, z-y-x, z-y-z.

Group 1

xyz , xzy, yxz, yzx, zxy, zyx

xyz

Cardan Angles (roll, pitch, yaw)

Rotation about fixed or current frame

Group 2

xyx , xzx, yxy, yzy, zxz, zyz

zxz

Euler Angles (precision, nutation, spin)

Rotation about current frame

- Deriving the Cardan angles rotational matrix
  - about **fixed frame**

$$
\begin{aligned}
R_{XYZ} &= R_{z,\phi} R_{y,\theta} R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}
\end{aligned}
$$

  - about **current/localized frame**

$$
\begin{aligned}
R_{XYZ} &= R_{x,\Psi} R_{y,\theta} R_{z,\Phi} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\Psi & -s\Psi \\ 0 & s\Psi & c\Psi \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\Phi & -s\Phi & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

- Deriving the Euler angles rotational matrix

$$
\begin{aligned}
R_{ZYZ} &= R_{z,\phi} R_{y,\theta} R_{z,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}
\end{aligned}
$$

$$
\begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} = [R_{01}(\phi)][R_{12}(\theta)][R_{23}(\psi)] \begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = [R_{03}] \begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix}
$$

where

$$
[R_{01}(\phi)] = [R_{10}(\phi)]^T = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

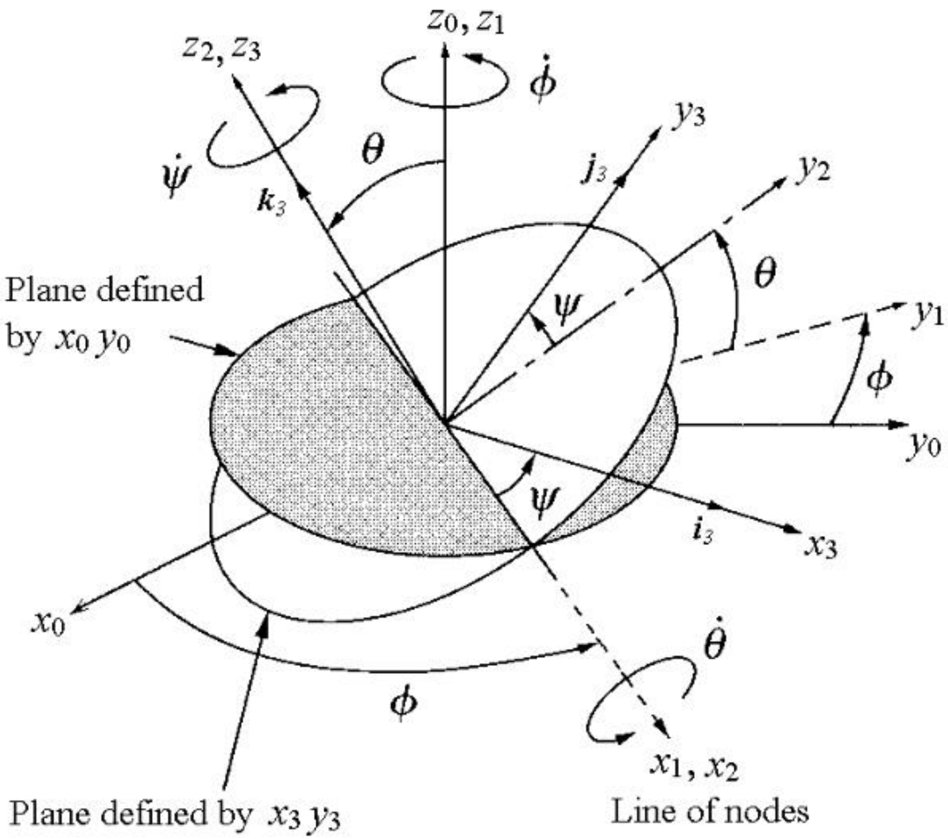
$$
[R_{12}(\theta)] = [R_{21}(\theta)]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}
$$

$$
[R_{23}(\psi)] = [R_{32}(\psi)]^T = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Euler angles rotational matrix are orthogonal

Note that $[R_{03}] = [R_{30}]^T$.

- Euler angle rotation from $O_0x_0y_0z_0$ to $O_3x_3y_3z_3$



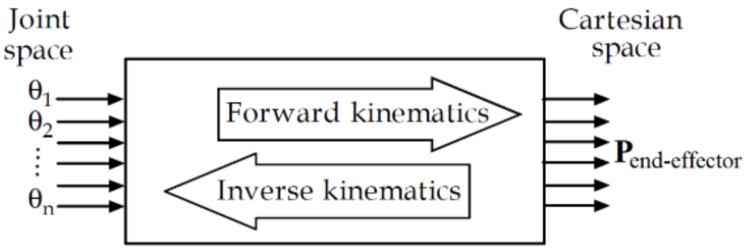
$$
\begin{aligned}
R_{ZYZ} &= R_{z,\phi}R_{y,\theta}R_{z,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}
\end{aligned}
$$

- θ is measured directly between $\boldsymbol{z_0}$ and $\boldsymbol{z_3}$ axes
- x_1 is perpendicular to the plane formed by $\boldsymbol{z_0}$ & $\boldsymbol{z_3}$ axes
- Φ and Ψ are measured from the $\boldsymbol{x_1}$-axis to the $\boldsymbol{x_0}$-axis and to the $\boldsymbol{x_3}$-axis

FORWARD & INVERSE KINEMATICS

Direct Kinematic

- Kinematics & dynamics of a robotic manipulator is fundamental to motion planning & control
- Kinematics relates directly to position, velocity & acceleration relationships among the links of the manipulator…
    - Provides a link between the Cartesian space and the joint space
    - Direct kinematics – locates the position of the end of the manipulator given particular joint angles
    - Inverse kinematics – establishes the required joint angles for a given position of the end of the manipulator.



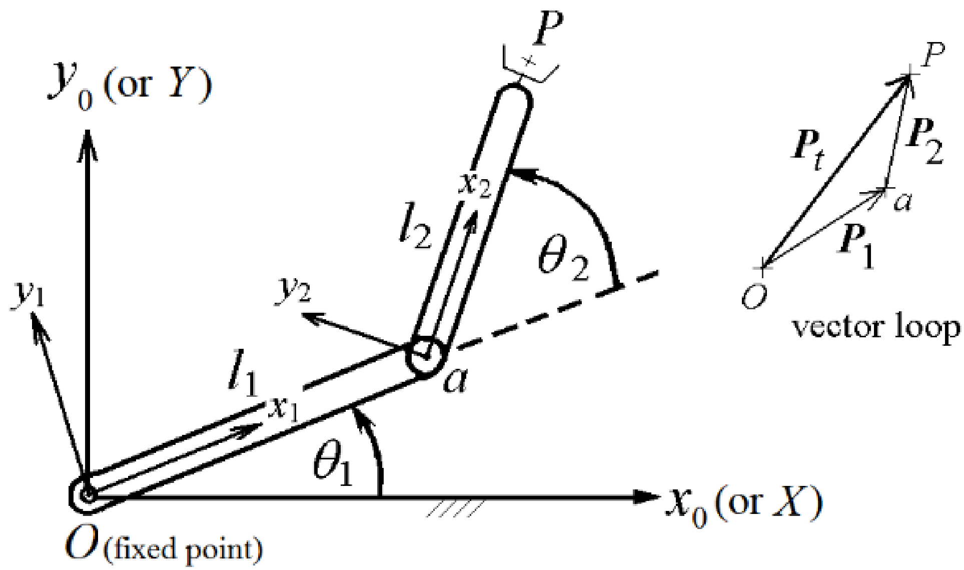
- Problem statement:

.....in order to place the end of the manipulator at a given position in <u>Cartesian space</u>, one must solve for the corresponding joint angles i.e. <u>joint space</u>....

What is the point of direct kinematics?

• Direct kinematics with rotation matrix

1. Global Frame *XYZ*/ $x_0y_0z_0$
2. Consists of link 1 & link 2 rotating Θ_1 ref fixed global frame Θ_2 ref link 1 respectively.
3. Both links rotate about the local z_1 & z_2 axis.
4. Local Frame attached to link 1 & 2 are $x_1y_1z_1$ & $x_2y_2z_2$ respectively
5. Point *P* with position vector ${}^0\boldsymbol{P}_t$



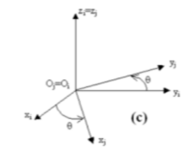
$$ {}^0\mathbf{P}_t = {}^0\mathbf{P}_1 + {}^0\mathbf{P}_2 = \mathbf{R}_{01}\,{}^1\mathbf{P}_1 + \mathbf{R}_{01}\mathbf{R}_{12}\,{}^2\mathbf{P}_2 $$

$$ {}^{0}\mathbf{P}_t = {}^{0}\mathbf{P}_1 + {}^{0}\mathbf{P}_2 = \mathbf{R}_{01}\,{}^{1}\mathbf{P}_1 + \mathbf{R}_{01}\mathbf{R}_{12}\,{}^{2}\mathbf{P}_2 $$

$$ \begin{array}{cccccc} & \mathbf{R}_{01} & {}^{1}\boldsymbol{P}_1 & \mathbf{R}_{01} & \mathbf{R}_{02} & {}^{2}\boldsymbol{P}_2 \\ = & \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{Bmatrix} l_1 \\ 0 \\ 0 \end{Bmatrix} + & \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{Bmatrix} l_2 \\ 0 \\ 0 \end{Bmatrix} \end{array} $$

$$ = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} l_1 \\ 0 \\ 0 \end{Bmatrix} + \underbrace{\begin{bmatrix} c\theta_1 c\theta_2 - s\theta_1 s\theta_2 & -c\theta_1 s\theta_2 - s\theta_1 c\theta_2 & 0 \\ s\theta_1 c\theta_2 + c\theta_1 s\theta_2 & -s\theta_1 s\theta_2 + c\theta_1 c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}_{02}} \begin{Bmatrix} l_2 \\ 0 \\ 0 \end{Bmatrix} $$

- **Basic Rotation Matrix**
  - about z-axis



$$ \boldsymbol{R}_{ij}(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad i<j $$

$$ \mathbf{P}_t = {}^{0}\mathbf{P}_t = \begin{Bmatrix} P_{tX} \\ P_{tY} \\ P_{tZ} \end{Bmatrix} = \overset{{}^{0}\boldsymbol{P}_1}{\begin{Bmatrix} l_1 c\theta_1 \\ l_1 s\theta_1 \\ 0 \end{Bmatrix}} + \overset{{}^{0}\boldsymbol{P}_2}{\begin{Bmatrix} l_2(c\theta_1 c\theta_2 - s\theta_1 s\theta_2) \\ l_2(s\theta_1 c\theta_2 + c\theta_1 s\theta_2) \\ 0 \end{Bmatrix}} = \begin{Bmatrix} l_1 c\theta_1 + l_2 c\theta_{12} \\ l_1 s\theta_1 + l_2 s\theta_{12} \\ 0 \end{Bmatrix} $$

in which $c\theta_1=\cos\theta_1$, $s\theta_1=\sin\theta_1$, $c\theta_{12}=\cos(\theta_1+\theta_2)$, and $s\theta_{12}=\sin(\theta_1+\theta_2)$.

$$ {}^{0}\boldsymbol{P}_t = (l_1 c\theta_1 + l_2 c\theta_{12})\boldsymbol{i}_0 + (l_1 s\theta_1 + l_2 s\theta_{12})\boldsymbol{j}_0 $$

– Deriving velocity:

$$\mathbf{P}_t = \mathbf{R}_{01}\,{}^{1}\mathbf{P}_1 + \mathbf{R}_{01}\mathbf{R}_{12}\,{}^{2}\mathbf{P}_2$$

Product rule

$${}^{0}\dot{\mathbf{P}}_t = \dot{\mathbf{R}}_{01}\,{}^{1}\mathbf{P}_1 + \dot{\mathbf{R}}_{02}\,{}^{2}\mathbf{P}_2 \longrightarrow R_{01}\ {}^{1}\dot{P}_1 \ \&\ R_{02}\ {}^{2}\dot{P}_2?$$

$$= \begin{bmatrix} -\dot{\theta}_1 s\theta_1 & -\dot{\theta}_1 c\theta_1 & 0 \\ \dot{\theta}_1 c\theta_1 & -\dot{\theta}_1 s\theta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} l_1 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} -\dot{\theta}_{12} s\theta_{12} & -\dot{\theta}_{12} c\theta_{12} & 0 \\ \dot{\theta}_{12} c\theta_{12} & -\dot{\theta}_{12} s\theta_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} l_2 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -l_1\dot{\theta}_1 s\theta_1 - l_2\dot{\theta}_{12} s\theta_{12} \\ l_1\dot{\theta}_1 c\theta_1 + l_2\dot{\theta}_{12} c\theta_{12} \\ 0 \end{Bmatrix}$$

Note that $\frac{\mathrm{d}}{\mathrm{d}t}(\sin\theta) = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta) = \dot{\theta}\cos\theta.$

Note also that: ${}^{1}\dot{\mathbf{P}}_t \neq \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{1}\mathbf{P}_t\right).$

where $\dot{\theta}_{12} = \dot{\theta}_1 + \dot{\theta}_2$. Also, ${}^{1}\dot{\mathbf{P}}_t = \mathbf{R}_{10}\,{}^{0}\dot{\mathbf{P}}_t$. Note that the time derivative of ${}^{1}\mathbf{P}_1$ and ${}^{2}\mathbf{P}_2$ are zero as the lengths l_1 and l_2 are constant.

Why is transport theorem not applied here?

– Expression in frame 1:

$$ {}^{1}\mathbf{P}_t = {}^{1}\mathbf{P}_1 + {}^{1}\mathbf{P}_2 = {}^{1}\mathbf{P}_1 + \mathbf{R}_{12}\,{}^{2}\mathbf{P}_2 = \begin{Bmatrix} l_1 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} l_2 \\ 0 \\ 0 \end{Bmatrix} $$

$$ \text{Therefore, } {}^{1}\mathbf{P}_t = \begin{Bmatrix} l_1 + l_2 c\theta_2 \\ l_2 s\theta_2 \\ 0 \end{Bmatrix} \qquad \text{or } {}^{0}\mathbf{P}_t = \mathbf{R}_{01}\,{}^{1}\mathbf{P}_t $$

$$ {}^{1}\boldsymbol{P}_t = (l_1 + l_2 c\theta_2)\boldsymbol{i}_1 + l_2 s\theta_2 \boldsymbol{j}_1 $$

vs

$$ {}^{0}\boldsymbol{P}_t = (l_1 c\theta_1 + l_2 c\theta_{12})\boldsymbol{i}_0 + (l_1 s\theta_1 + l_2 s\theta_{12})\boldsymbol{j}_0 $$

– Deriving acceleration:

$$^0\dot{\mathbf{P}}_t = \dot{\mathbf{R}}_{01}\,{}^1\mathbf{P}_1 + \dot{\mathbf{R}}_{02}\,{}^2\mathbf{P}_2$$

Product rule

$$^0\ddot{\mathbf{P}}_t = \ddot{\mathbf{R}}_{01}\,{}^1\mathbf{P}_1 + \ddot{\mathbf{R}}_{02}\,{}^2\mathbf{P}_2$$

$$\begin{aligned}
^0\ddot{\mathbf{P}}_t &= \ddot{\mathbf{R}}_{01}\,{}^1\mathbf{P}_1 + \ddot{\mathbf{R}}_{02}\,{}^2\mathbf{P}_2 \\
&= \begin{bmatrix} -\ddot{\theta}_1 s\theta_1 - \dot{\theta}_1^2 c\theta_1 & -\ddot{\theta}_1 c\theta_1 + \dot{\theta}_1^2 s\theta_1 & 0 \\ \ddot{\theta}_1 c\theta_1 - \dot{\theta}_1^2 s\theta_1 & -\ddot{\theta}_1 s\theta_1 - \dot{\theta}_1^2 c\theta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} l_1 \\ 0 \\ 0 \end{Bmatrix} \\
&\quad + \begin{bmatrix} -\ddot{\theta}_{12} s\theta_{12} - \dot{\theta}_{12}^2 c\theta_{12} & -\ddot{\theta}_{12} c\theta_{12} + \dot{\theta}_{12}^2 s\theta_{12} & 0 \\ \ddot{\theta}_{12} c\theta_{12} - \dot{\theta}_{12}^2 s\theta_{12} & -\ddot{\theta}_{12} s\theta_{12} - \dot{\theta}_{12}^2 c\theta_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} l_2 \\ 0 \\ 0 \end{Bmatrix}
\end{aligned}$$

Therefore, $\ddot{\mathbf{P}}_t = {}^0\ddot{\mathbf{P}}_t = \begin{Bmatrix} -l_1\ddot{\theta}_1 s\theta_1 - l_1\dot{\theta}_1^2 c\theta_1 - l_2\ddot{\theta}_{12} s\theta_{12} - l_2\dot{\theta}_{12}^2 c\theta_{12} \\ l_1\ddot{\theta}_1 c\theta_1 - l_1\dot{\theta}_1^2 s\theta_1 + l_2\ddot{\theta}_{12} c\theta_{12} - l_2\dot{\theta}_{12}^2 s\theta_{12} \\ 0 \end{Bmatrix}$

in which $\ddot{\theta}_{12} = \ddot{\theta}_1 + \ddot{\theta}_2$ and $\dot{\theta}_{12}^2 = (\dot{\theta}_1 + \dot{\theta}_2)^2$. Note that the second time derivative of $^1\mathbf{P}_1$ and $^2\mathbf{P}_2$ are also zero as the lengths l_1 and l_2 are constant.

- **Direct kinematics with Jacobian matrix**

Velocity - $\dot{\boldsymbol{P}}_t = \boldsymbol{J}\dot{\boldsymbol{\theta}}$

$$\overset{\dot{\boldsymbol{P}}_t}{\begin{Bmatrix} \dot{P}_{tX} \\ \dot{P}_{tY} \end{Bmatrix}} = \overbrace{\begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix}}^{\text{Jacobian matrix } \boldsymbol{J}} \overset{\dot{\boldsymbol{\theta}}_t}{\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix}}$$

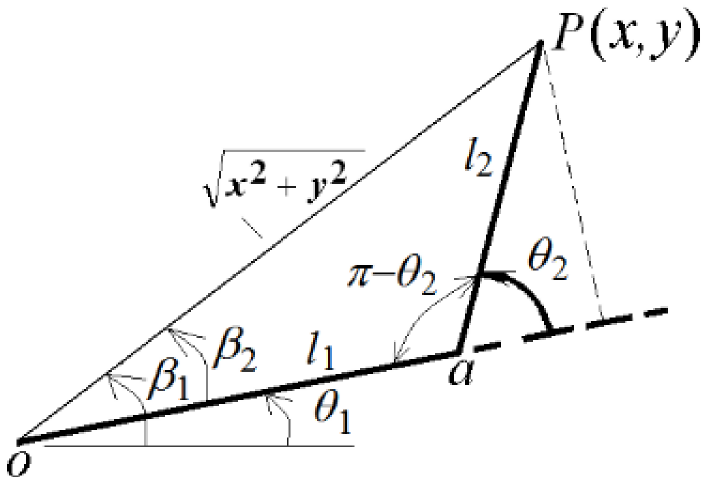
Acceleration - $\ddot{\boldsymbol{P}}_t = \dot{\boldsymbol{J}}\dot{\boldsymbol{\theta}} + \boldsymbol{J}\ddot{\boldsymbol{\theta}}$

$$\ddot{\boldsymbol{P}}_t = \begin{bmatrix} -l_1\dot{\theta}_1\cos\theta_1 - l_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2) & -l_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2) \\ -l_1\dot{\theta}_1\sin\theta_1 - l_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2) & -l_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2) \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix}$$

Inverse Kinematic

- Inverse kinematics – establishes the required joint angles for a given position of the end of the manipulator.
- Position $\{\Theta_1, \Theta_2\}$

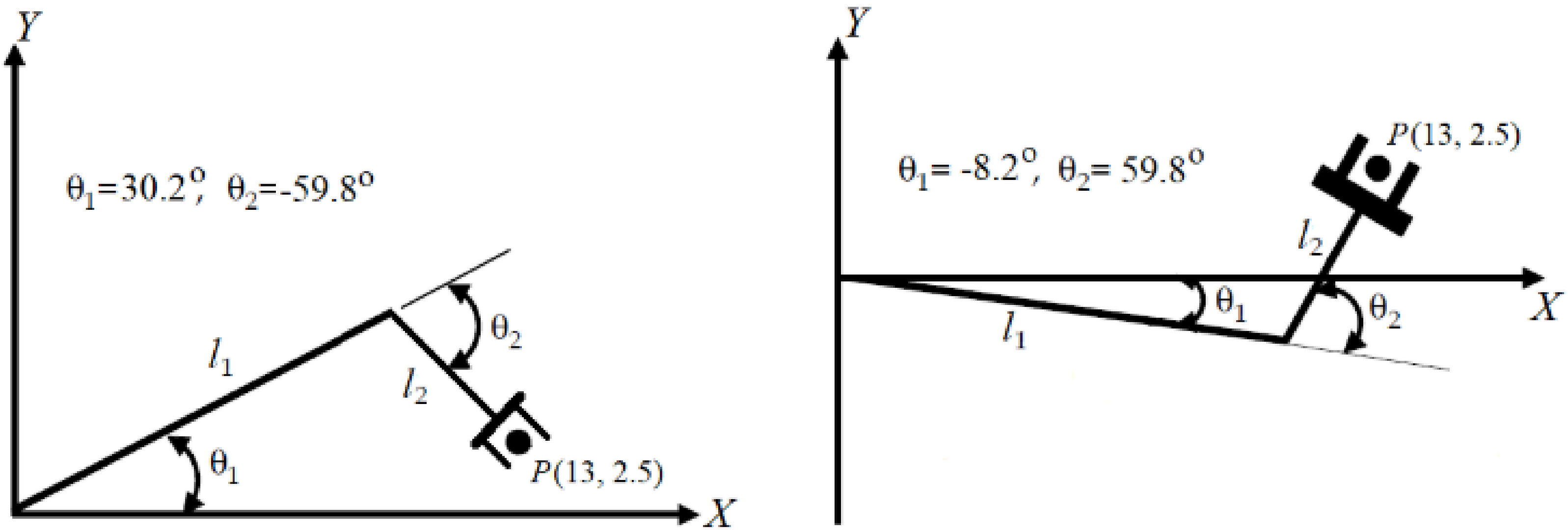


$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta_2}{l_1 + l_2\cos\theta_2}\right) \quad \text{or} \quad \theta_1 = \beta_1 - \beta_2$$

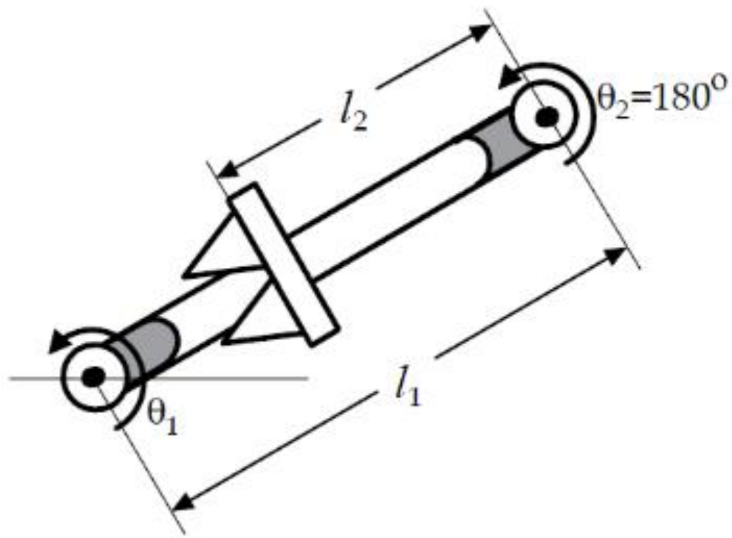
$$-\cos(\pi - \theta_2) = \frac{\left(x^2 + y^2\right) - l_1^2 - l_2^2}{2l_1 l_2}$$

$$\Rightarrow \theta_2 = \cos^{-1}\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}\right)$$

- Position $\{\Theta_1, \Theta_2\}$ – configuration is not unique



- Position $\{\Theta_1, \Theta_2\}$ – clashing



- Velocity $\{\dot{\theta}_1, \dot{\theta}_2\}$ – leveraging inverse Jacobian $\boldsymbol{J}^{-1}$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1} \dot{\boldsymbol{P}}_t$$

$$\boldsymbol{J} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$\textbf{Adj =} \begin{bmatrix} l_2 \cos(\theta_1 + \theta_2) & l_2 \sin(\theta_1 + \theta_2) \\ -l_1 \cos\theta_1 - l_2 \cos(\theta_1 + \theta_2) & -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

$$\frac{1}{determinant} = \frac{1}{\big(l_2 \cos(\theta_1 + \theta_2) \cdot (-l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2))\big) - \big(-l_2 \sin(\theta_1 + \theta_2) \cdot (l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2))\big)}$$

$$= \frac{1}{l_1 l_2 \sin\theta_2}$$

$$\boldsymbol{J}^{-1} = \frac{1}{l_1 l_2 \sin\theta_2} \begin{bmatrix} l_2 \cos(\theta_1 + \theta_2) & l_2 \sin(\theta_1 + \theta_2) \\ -l_1 \cos\theta_1 - l_2 \cos(\theta_1 + \theta_2) & -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \frac{1}{l_1 l_2 \sin\theta_2} \begin{bmatrix} l_2 \cos(\theta_1 + \theta_2) & l_2 \sin(\theta_1 + \theta_2) \\ -l_1 \cos\theta_1 - l_2 \cos(\theta_1 + \theta_2) & -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix}$$

- Acceleration $\{\ddot{\theta}_1, \ddot{\theta}_2\}$ – leveraging inverse Jacobian $\boldsymbol{J}^{-1}$

$$\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} = [B] \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + [\dot{B}] \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} \tag{6.15}$$

where

$$[B] = [J]^{-1} = \frac{1}{l_1 l_2 \sin\theta_2} \begin{bmatrix} l_2 \cos(\theta_1 + \theta_2) & l_2 \sin(\theta_1 + \theta_2) \\ -l_1 \cos\theta_1 - l_2 \cos(\theta_1 + \theta_2) & -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

Homogeneous Coordinates

- **Background:**
  - Purpose: relating 2 coordinate systems with <u>different origins.</u>
  - Homogenous matrix able to define <u>translation & orientation</u>.
  - 3D rigid body motion is represented in $\mathbb{R}^4$

$$\underset{\text{physical}\,\mathbf{R}^3}{\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}} \Leftrightarrow \underset{\text{homogeneous}\,\mathbf{R}^4}{\begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}}$$

Scaling factor

Homogeneous Coordinates

- **Definition:**

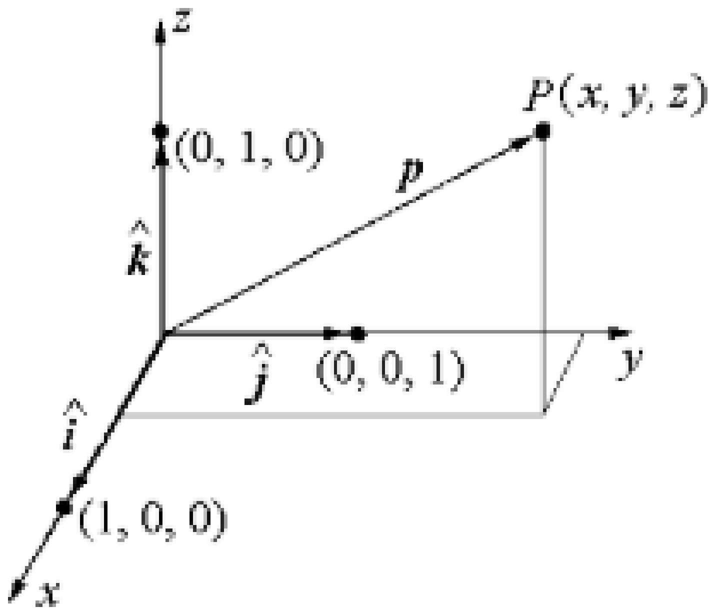
Let $\boldsymbol{x} = \{x_1, \dots x_n\}$ be the Cartesian coordinates of a point in $\boldsymbol{R}^n$. Then any set of (n+1) numbers (y_1, y_2,…, y_n, y_{n+1}) where $y_{n+1} \neq 0$ for which

$$\frac{y_1}{y_{n+1}} = x_1, \frac{y_2}{y_{n+1}} = x_2, \dots \frac{y_n}{y_{n+1}} = x_n$$

Scaling factor

is called a set of homogeneous coordinates for the point x.

- **Vector representation**



$$\boldsymbol{p} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$$ Cartesian coordinates

$$\hat{\boldsymbol{p}} = \{x, y, z, 1\}^T = \{wx, wy, wz, w\}^T = \{\hat{x}, \hat{y}, \hat{z}, w\}^T$$

Homogeneous coordinates

$$\hat{\boldsymbol{p}} = \{x, y, z, 1\}^T = \{wx, wy, wz, w\}^T = \{\hat{x}, \hat{y}, \hat{z}, w\}^T$$

Scaling factor

- **Scaling factor** – a factor that scales the size of the vector which may take any number.
  - *w*<1: coordinates scales up, *w*>1: coordinates scales down
  - w=1: coordinate remains, default for application in robotics
- **Direction** – when *w* is set to 0 then the vector in homogeneous coordinates represents a direction.

$$\hat{\boldsymbol{p}} = \begin{Bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ w \end{Bmatrix} \Rightarrow \quad \hat{\boldsymbol{i}} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \hat{\boldsymbol{j}} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad \hat{\boldsymbol{k}} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

- **Origin of coordinate frame –** the null vector representing a point at the origin of a reference frame is defined as $\{0\ 0\ 0\ w\}^T$ where the scaling factor *w* can be any number other than zero.

For the general homogeneous coordinate representation, there are a number of unintended/unrequired consequences:

- Non-unique representation of vector:

$$\boldsymbol{p} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = 3\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$$

Homogeneous representations:

$$\hat{\boldsymbol{p}} = \begin{Bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ w \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4 \\ 5 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 8 \\ 10 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 15 \\ 20 \\ 25 \\ 5 \end{Bmatrix} = \begin{Bmatrix} -3 \\ -4 \\ -5 \\ -1 \end{Bmatrix} \Leftarrow \begin{Bmatrix} wx \\ wy \\ wz \\ w \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

- Representation of direction with *w*=0

$$\hat{\boldsymbol{p}} = \begin{Bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ w \end{Bmatrix} \Rightarrow \quad \hat{\boldsymbol{i}} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \hat{\boldsymbol{j}} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad \hat{\boldsymbol{k}} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

Consequently, for application in DH representation, we let *w*=1 permanently

HOMOGENEOUS TRANSFORMATION & MATRIX

Homogeneous Transformation

- **Homogeneous transformation matrix *H***
  - 4x4 matrix
  - 4 sub-matrices

$$\boldsymbol{H} = \left[\begin{array}{c|c} \boldsymbol{R}_{3\times3} & \boldsymbol{p}_{3\times1} \\ \hline \boldsymbol{\beta}_{1\times3} & s_{1\times1} \end{array}\right] = \left[\begin{array}{c|c} \text{rotation} & \text{position} \\ \text{matrix} & \text{vector} \\ \hline \text{perspective} & \text{scaling} \\ \text{transformation} & \text{factor} \end{array}\right]$$

- **Principle diagonal**
  - 1st 3 elements *a b c :* Local scaling

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix} = \begin{Bmatrix} ax \\ by \\ cz \\ 1 \end{Bmatrix}$$

  - 4th element *w=s* : Global scaling

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \\ s \end{Bmatrix} \quad \Leftarrow \begin{Bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ w \end{Bmatrix}$$

$$\hat{x} = \frac{x}{s}, \quad \hat{y} = \frac{y}{s}, \quad \hat{z} = \frac{z}{s}, \quad w = \frac{s}{s} = 1$$

$1<s$ – global reduction
$0<s<1$ – global enlargement

- **Basic Homogeneous Transformation**

$$\boldsymbol{H}_{\mathrm{rot}}(x,\alpha) = \boldsymbol{H}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{H}_{\mathrm{rot}}(y,\phi) = \boldsymbol{H}_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{H}_{\mathrm{rot}}(z,\theta) = \boldsymbol{H}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Upper left 3x3 submatrix rotates while the upper right 3x1 submatrix translates

$$\boldsymbol{H}_{\text{trans}} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Represents a translated coordinate system *oxyz* with axes parallel to the reference coordinate system *OXYZ* but origin at d_x d_y d_z

- Geometric Interpretation of Homogeneous Transformation Matrices

$$\boldsymbol{H}_{12} = \begin{bmatrix} n_X & s_X & a_X & p_X \\ n_Y & s_Y & a_Y & p_Y \\ n_Z & s_Z & a_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{a} & \boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or

$$\boldsymbol{H}_{12} = \begin{bmatrix} & \boldsymbol{R}_{12} & & {}^{1}\boldsymbol{p}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{H}_{12} = \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{Bmatrix} = \begin{array}{c} \begin{array}{cccc} x_2 & y_2 & z_2 & \\ \downarrow & \downarrow & \downarrow & \end{array} \\ \begin{bmatrix} n_X & s_X & a_X & p_X \\ n_Y & s_Y & a_Y & p_Y \\ n_Z & s_Z & a_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{Bmatrix} \begin{array}{l} \longleftarrow x_1 \\ \longleftarrow y_1 \\ \longleftarrow z_1 \\ \end{array}$$

- Each of the 3 column vectors ***n***, ***s*** & ***a*** represent the direction vector of coordinate system 2
- The 4th column vector represents the location of coordinate system 2 origin

- Inverse of the homogeneous rotation matrix

$$\boldsymbol{H}^{-1}=\begin{bmatrix} n_X & n_Y & n_Z & -\boldsymbol{n}^T\boldsymbol{p} \\ s_X & s_Y & s_Z & -\boldsymbol{s}^T\boldsymbol{p} \\ a_X & a_Y & a_Z & -\boldsymbol{a}^T\boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} & \boldsymbol{n}^T & & -\boldsymbol{n}^T\boldsymbol{p} \\ & \boldsymbol{s}^T & & -\boldsymbol{s}^T\boldsymbol{p} \\ & \boldsymbol{a}^T & & -\boldsymbol{a}^T\boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} & & & -\boldsymbol{n}^T\boldsymbol{p} \\ & \boldsymbol{R}_{3\times 3}^T & & -\boldsymbol{s}^T\boldsymbol{p} \\ & & & -\boldsymbol{a}^T\boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Rigid Body Motion

- Pure Translation

$$\hat{\mathbf{a}}_{XYZ} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 4(1)+1(5) \\ 3(1) \\ 2(1)+1(-3) \\ 1(1) \end{Bmatrix} = \begin{Bmatrix} 9 \\ 3 \\ -1 \\ 1 \end{Bmatrix}$$

$$\hat{\mathbf{b}}_{XYZ} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 5 \\ 2 \\ 4 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 2 \\ 1 \\ 1 \end{Bmatrix}$$

- Pure rotation

$$\mathbf{H}_{01} = \begin{bmatrix} a_{x1} & b_{x1} & c_{x1} & 0 \\ a_{y1} & b_{y1} & c_{y1} & 0 \\ a_{z1} & b_{z1} & c_{z1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with the result $\boldsymbol{r}_0 = \boldsymbol{H}_{01}\boldsymbol{r}_1$ or in matrix form

$$\begin{Bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{Bmatrix} = \begin{bmatrix} a_{x1} & b_{x1} & c_{x1} & 0 \\ a_{y1} & b_{y1} & c_{y1} & 0 \\ a_{z1} & b_{z1} & c_{z1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} x_1 a_{x1} + y_1 b_{x1} + z_1 c_{x1} \\ x_1 a_{y1} + y_1 b_{y1} + z_1 c_{y1} \\ x_1 a_{z1} + y_1 b_{z1} + z_1 c_{z1} \\ 1 \end{Bmatrix}$$

Homogeneous Transformation Matrix

$$\boldsymbol{H} = \left[\begin{array}{c|c} \boldsymbol{R}_{3\times3} & \boldsymbol{p}_{3\times1} \\ \hline \boldsymbol{\beta}_{1\times3} & s_{1\times1} \end{array}\right] = \left[\begin{array}{c|c} \text{rotation matrix} & \text{position vector} \\ \hline \text{perspective transformation} & \text{scaling factor} \end{array}\right]$$

<u>Rigid body motion</u>

Rotation

$$\boldsymbol{H}_{\mathrm{rot}}(z,\theta) = \boldsymbol{H}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation

$$\boldsymbol{H}_{\mathrm{trans}} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>Geometric interpretation</u>

$$\boldsymbol{H}_{12} = \begin{bmatrix} n_X & s_X & a_X & p_X \\ n_Y & s_Y & a_Y & p_Y \\ n_Z & s_Z & a_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{a} & \boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or

$$\boldsymbol{H}_{12} = \begin{bmatrix} \boldsymbol{R}_{12} & {}^{1}\boldsymbol{p}_2 \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} n_X \\ n_Y \\ n_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad y_1 = x_2 \qquad \begin{Bmatrix} s_X \\ s_Y \\ s_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad x_1 = -y_2$$

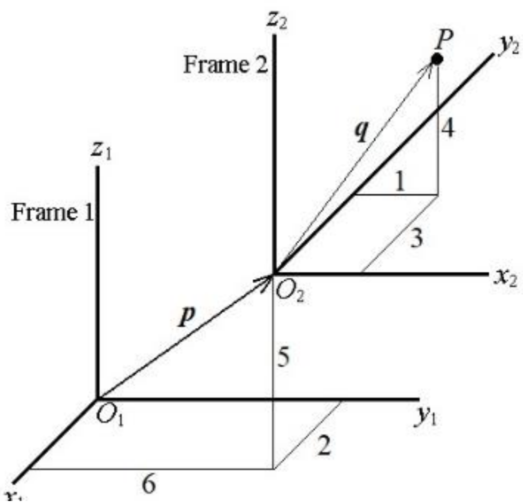
$$\begin{Bmatrix} a_X \\ a_Y \\ a_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \quad \text{or} \quad z_1 = z_2 \qquad \begin{Bmatrix} p_X \\ p_Y \\ p_Z \\ 1 \end{Bmatrix} = \begin{Bmatrix} a \\ b \\ c \\ 1 \end{Bmatrix} \quad \text{or} \quad \begin{cases} p_x = a \\ p_y = b \\ p_z = c \end{cases}$$

O_2

- Rigid body motion

$$\boldsymbol{H}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{H}_{\text{trans}} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$\boldsymbol{R}_{3x3}$:rotation

$\boldsymbol{P}_{3x1}$: translation

$$^1\boldsymbol{q} = \boldsymbol{H}_{12}\, {}^2\boldsymbol{q}$$

$$\begin{Bmatrix} q_{x1} \\ q_{y1} \\ q_{z1} \\ 1 \end{Bmatrix} = [H_{12}] \begin{Bmatrix} q_{x2} \\ q_{y2} \\ q_{z2} \\ 1 \end{Bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 3 \\ 4 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 7 \\ 9 \\ 1 \end{Bmatrix} \Rightarrow \mathbf{H}_{0n} = \prod_{i=1}^{n} \mathbf{H}_{(i-1),i}$$

which implies: $^1\boldsymbol{q} = -1\, \boldsymbol{i}_1 + 7\, \boldsymbol{j}_1 + 9\, \boldsymbol{k}_1$.

Using rotational matrix and physical coordinates:

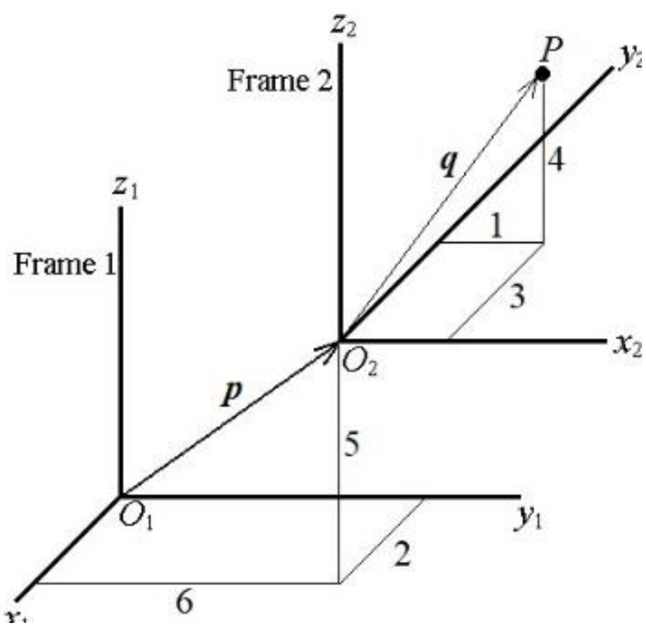
$$^1\overrightarrow{O_1P} = \boldsymbol{p} + \boldsymbol{q} = \boldsymbol{R}_{12}\ {}^2\boldsymbol{p} + \boldsymbol{R}_{12}\ {}^2\boldsymbol{q} \Rightarrow {}^i\boldsymbol{p}_j = \sum_{k=i}^{j} \boldsymbol{R}_{ij}\ {}^j\boldsymbol{p}_j$$

Vector loop: translation

Rotation matrix: rotation

- Geometrical Interpretation

$$\boldsymbol{H}_{12} = \begin{bmatrix} n_X & s_X & a_X & p_X \\ n_Y & s_Y & a_Y & p_Y \\ n_Z & s_Z & a_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{Bmatrix} n_X \\ n_Y \\ n_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad y_1 = x_2$$

$$\begin{Bmatrix} s_X \\ s_Y \\ s_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad x_1 = -y_2$$

$$\begin{Bmatrix} a_X \\ a_Y \\ a_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \quad \text{or} \quad z_1 = z_2$$

$$\begin{Bmatrix} p_X \\ p_Y \\ p_Z \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 6 \\ 5 \\ 1 \end{Bmatrix} O_2$$

$$^1\boldsymbol{p} = \begin{Bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \\ 1 \end{Bmatrix} = [H_{12}] \begin{Bmatrix} O_{2x2} \\ O_{2y2} \\ O_{2y2} \\ 1 \end{Bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 6 \\ 5 \\ 1 \end{Bmatrix}$$

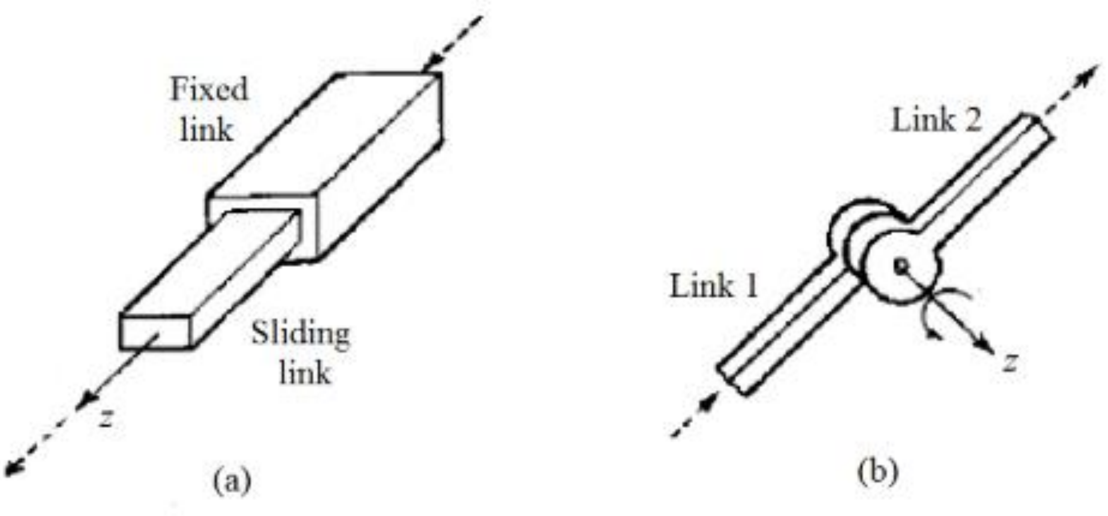
D-H METHOD

Denavit-Hartenberg (D-H) Representation

- Developed in 1955 as method for **kinematic modelling** of *lower pair mechanisms* based on matrices.
- Technique applied in 1981 as a standard way of **representing robots** & **modelling their motion**.
  - Use of Joint/link classification

Adoption of this convention provide a systematic procedure which standardize and simplify

- <u>Defining Links, Joints and their Parameters</u>
  - Links (0 to *n*+1) – series of rigid bodies connected by joints
  - Joints (1 to *n*) – either revolute or prismatic (associated with an actuator)



Joints (a) Prismatic joint, and (b) revolute joint

- Robot with *n* joints will have *n*+1 links
  - Joint *i* connects link *i*-1 to link *i*
  - Actuation of joint *i* moves link *i*
  - Link 0 is the base/fixed and does not move

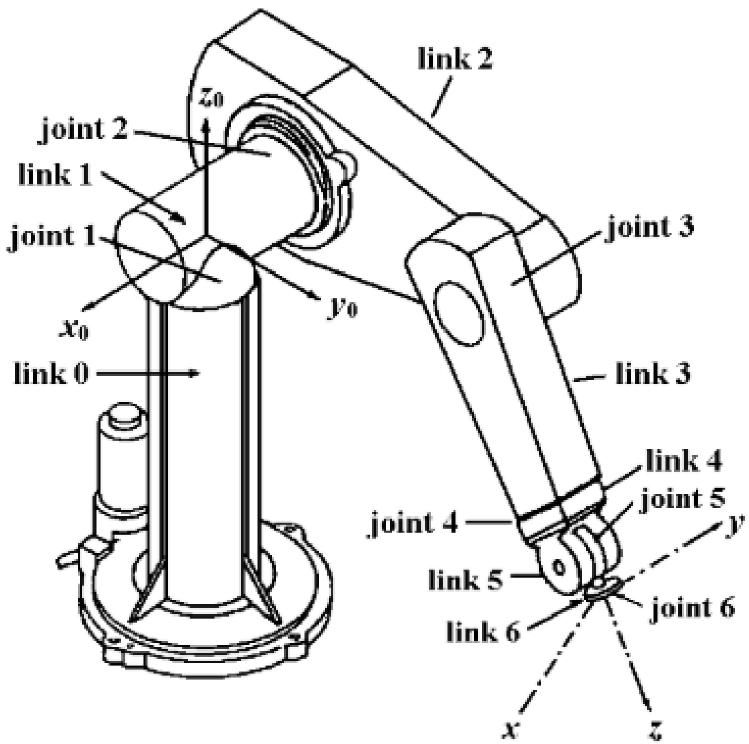


Figure 7.6 PUMA robot arm with joints (1 to 6) and links (0, 1 to 6; 0 is the base).

- Robot arms are modelled as open kinematic chain
  - A single DOF of the robot is associated/effected by a single joint/actuation

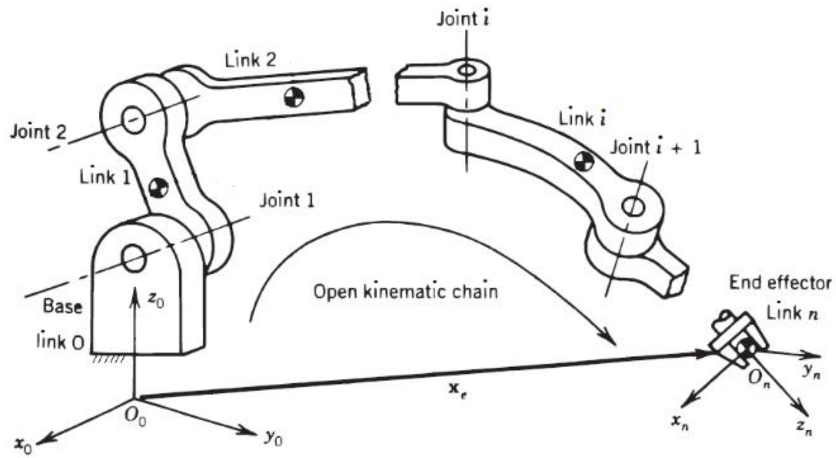


Figure 7.7b Robotic Arm formed by open kinematic chain.

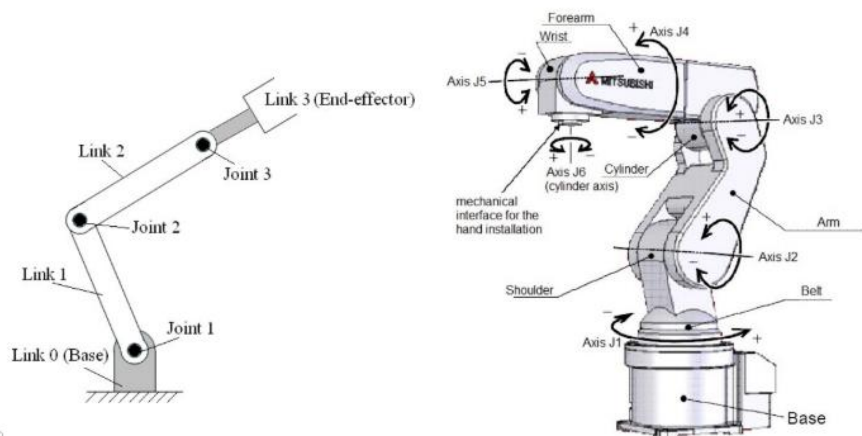
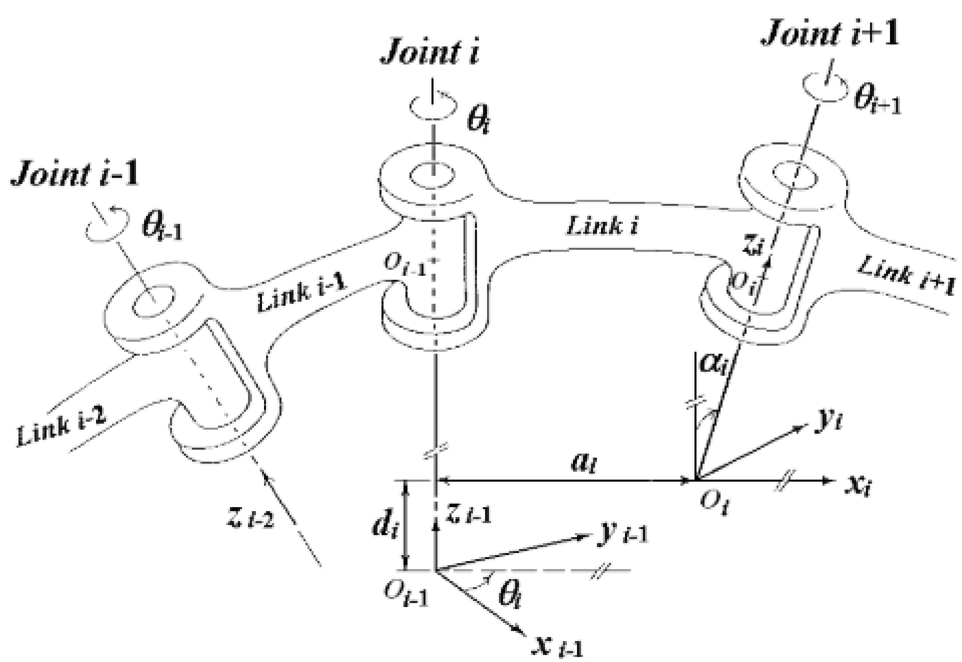
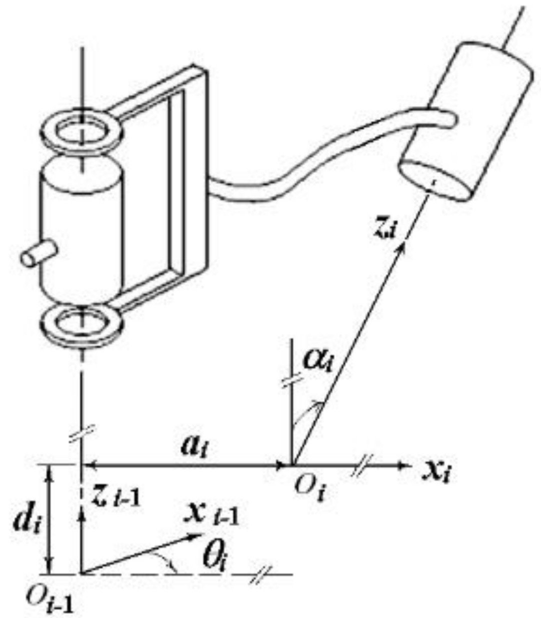


Figure 7.7c Robots with joints and links.

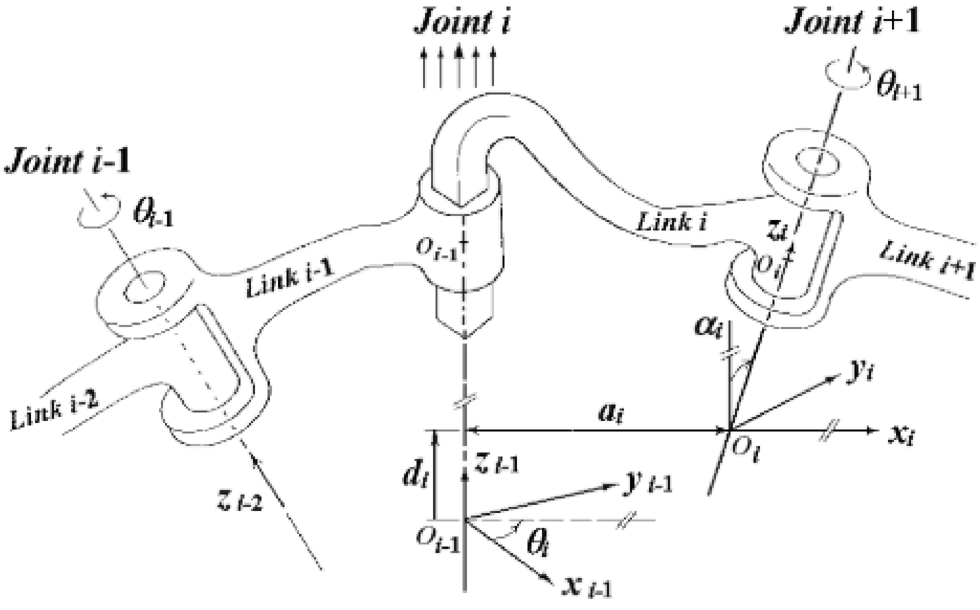
- DH Parameters for joints and links
  - Each joint has a joint axis z_i and 2 normal x_{i-1} and x_i with respect to each link connected
  - d_i and θ_i associated with distance and angle between adjacent links (i.e. relative position)
    - d_i : distance measured along the joint axis between normals x_{i-1} and x_i
    - θ_i : joint angle measured between the normal x_{i-1} and x_i in a plane normal to the joint axis



- DH Parameters for joints and links
  - Each link maintains a fixed configuration between their joints based of the *length* and *twist angle* of link *i*
  - a_i and α_i associated with distance and angle between adjacent links
    - a_i : shortest distance measured along the common normal between joint axes z_{i-1} and z_i
    - α_i : angle measured between the joint axes z_{i-1} and z_i perpendicular to a_i



- DH Parameters for joints and links
  - Link parameters a_i and α_i determine the structure of the link
  - Joint parameters d_i and θ_i determine the relative position of neighbouring links

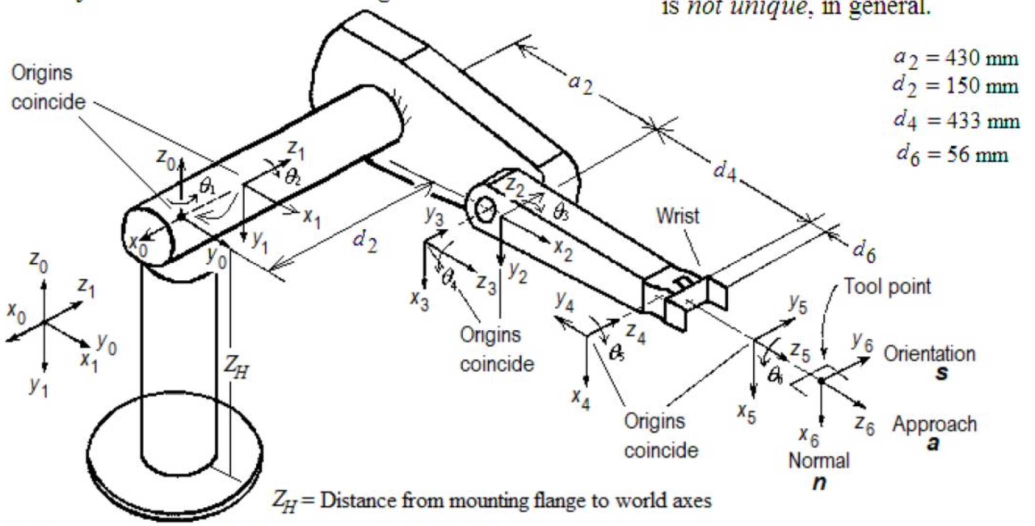


(b)

• Assigning D-H Coordinate Systems

The robot is placed in a configuration with joitn variables of 0 or 90 degree.

* assignment of coordinate systems is *not unique*, in general.



NOTE: The above drawing is not to scale and is for illustrative purposes only.

1. Establishing Link Coordinate Systems

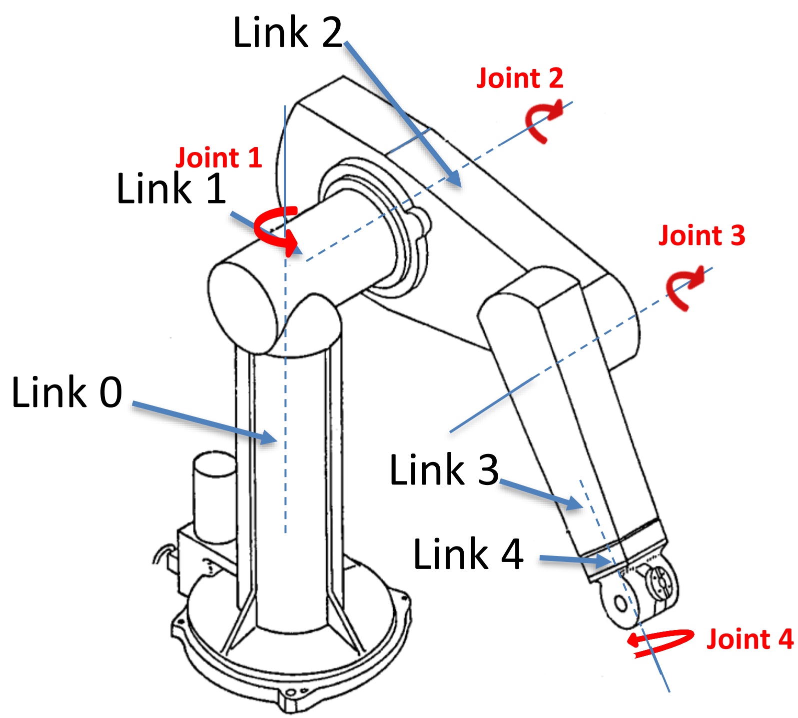
2. Define the D-H Parameters

D-H Parameters of PUMA Robot Arm (at position shown)

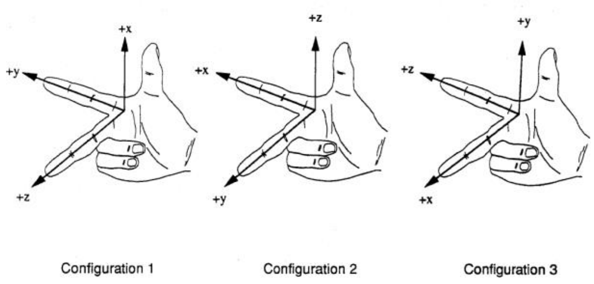
Joint i	θ_i (deg)	d_i (mm)	a_i (mm)	α_i (deg)	$\cos\alpha_i$	$\sin\alpha_i$	Joint variables	Joint range (deg)	
1	90	0	0	-90	0	-1	θ_1	-160 to 160	← z_{i-1}, z_i axes (*intersect*)
2	0	d_2	a_2	0	1	0	θ_2	-225 to 45	← z_{i-1}, z_i axes (*parallel*)
3	90	0	0	90	0	1	θ_3	-45 to 225	← z_{i-1}, z_i axes (*intersect*)
4	0	d_4	0	-90	0	-1	θ_4	-110 to 170	← z_{i-1}, z_i axes (*intersect*)
5	0	0	0	90	0	1	θ_5	-100 to 100	← z_{i-1}, z_i axes (*intersect*)
6	0	d_6	0	0	1	0	θ_6	-266 to 266	← z_{i-1}, z_i axes (*collinear*)

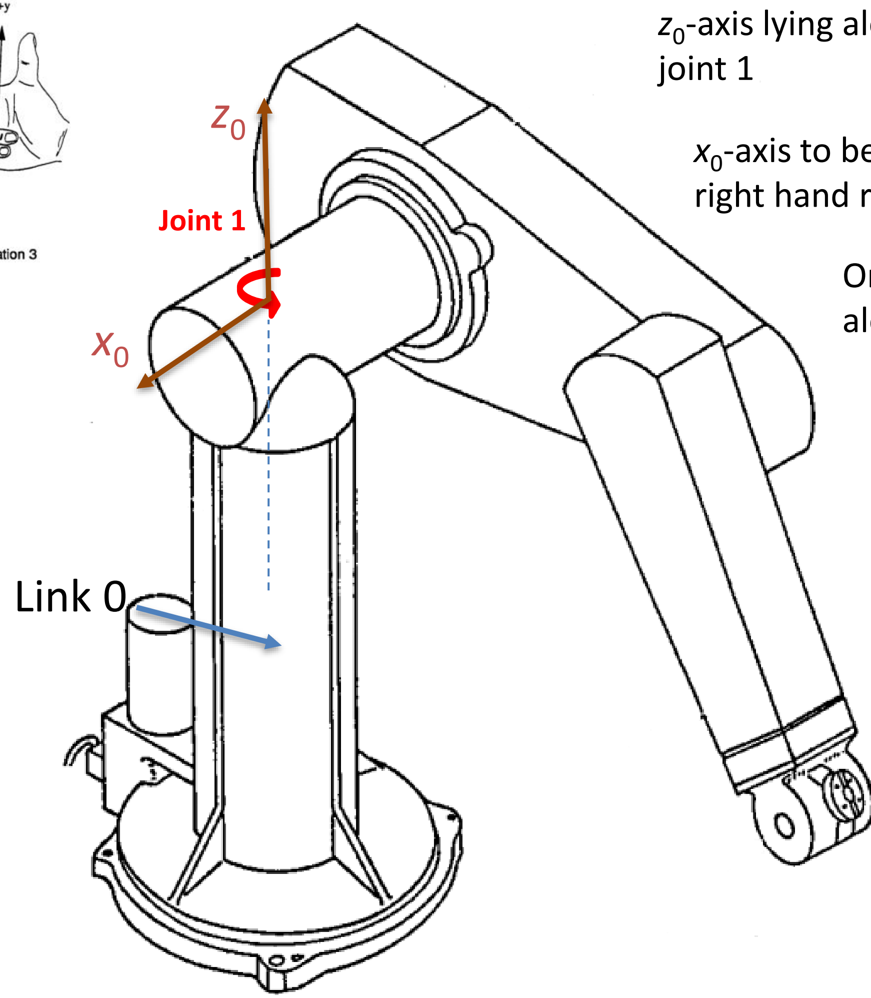
Note: plus-minus of the D-H parameters depending on the rotation-direction/distance from axis x_{i-1} to axis x_i (based on the right-hand coordinate system)

- Establishing Link Coordinate Systems
  - To assign Cartesian coordinate frames for the links in a serial link *n-joints* manipulator.
    - **Each joint represents 1 DOF**
    - **Each joint is either revolute or prismatic.**
  - Manipulator will have *n*+1 links.



- Establishing Link Coordinate Systems
  - Establish base coordinate system *XYZ* to the base of the manipulator for 1st link



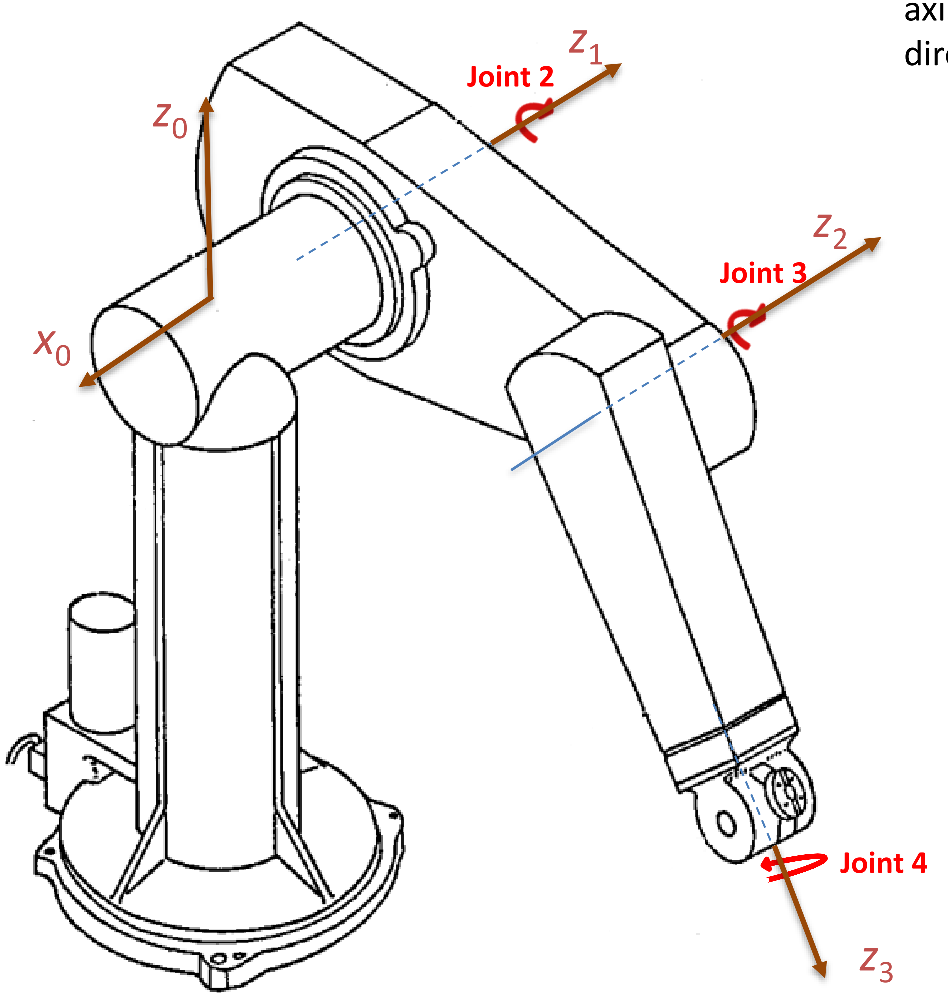


z_0-axis lying along axis of the motion of joint 1

x_0-axis to be conveniently established by right hand rule

Origin placed anywhere along z_0-axis

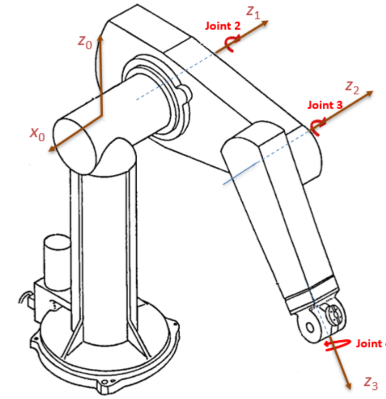
- Establishing Link Coordinate Systems
  - Establish link coordinates 1 to 4



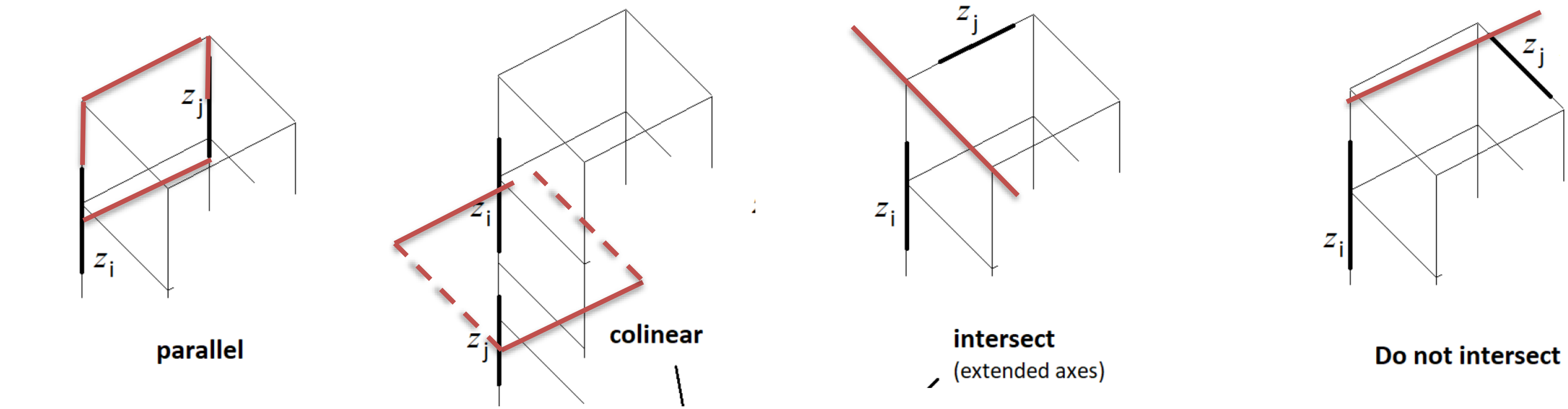
Align z_{i-1}-axis about the joint *i* axis i.e. axis of revolution for revolute and direction of linear motion for prismatic

Establish the **origin** of the *i*th coordinate: a) @ intersection of the z_{i-1}-axis and the common normal or b) @intersection of z_{i-1} & z_i axes

- Establishing Link Coordinate Systems
  - Establish link coordinates 1 to 4
    - Establish x_i-axis is normal to the z_{i-1} axis
    - The x_i-axis is normal to both z_i & z_{i-1} axes



Relationship between two z axes



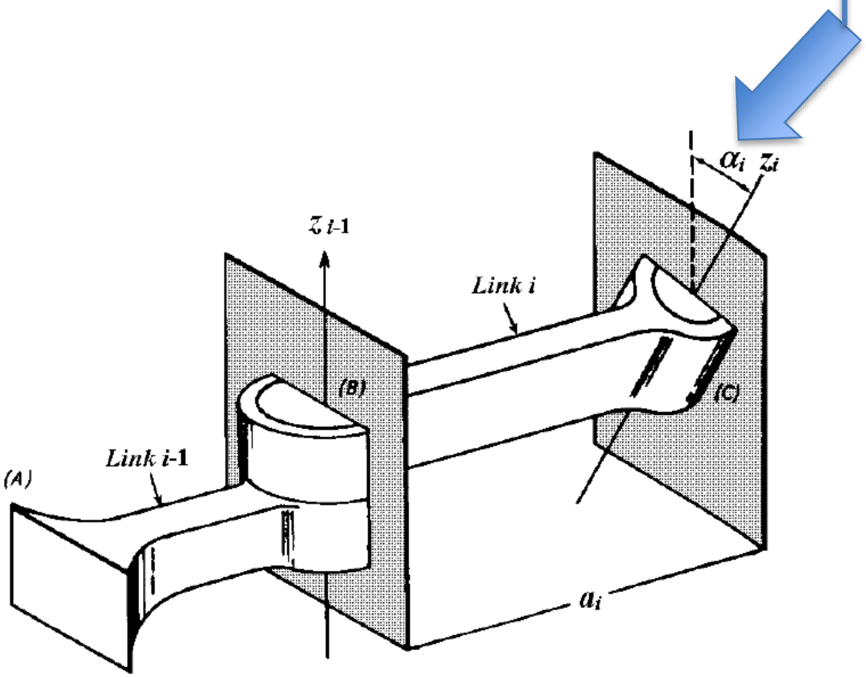
- Infinite no. of common normal
- Direction is defined from z_{i-1} towards z_i
- x_i will be the common normal passing thru O_i

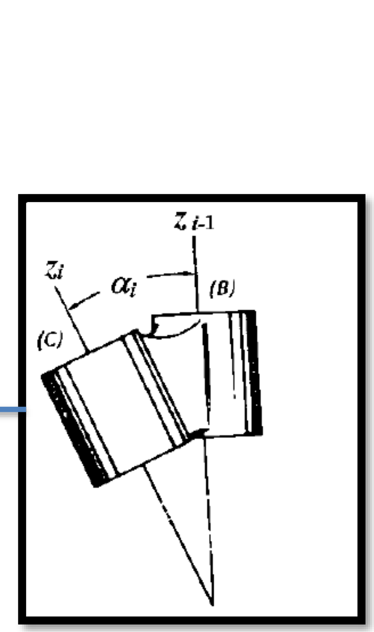
- x_i can be chosen anywhere in the plane perpendicular to z_i and z_{i-1}

- No common normal connecting z_i and z_{i-1}
- x_i can be chosen along either of 2 opposite directions perpendicular to z_i and z_{i-1}

- x_i is along the common normal to z_i and z_{i-1}s
- Direction is defined from z_{i-1} towards z_i

- Establishing Link Coordinate Systems
  - Establish link coordinates 1 to 4
    - Establish x_i-axis



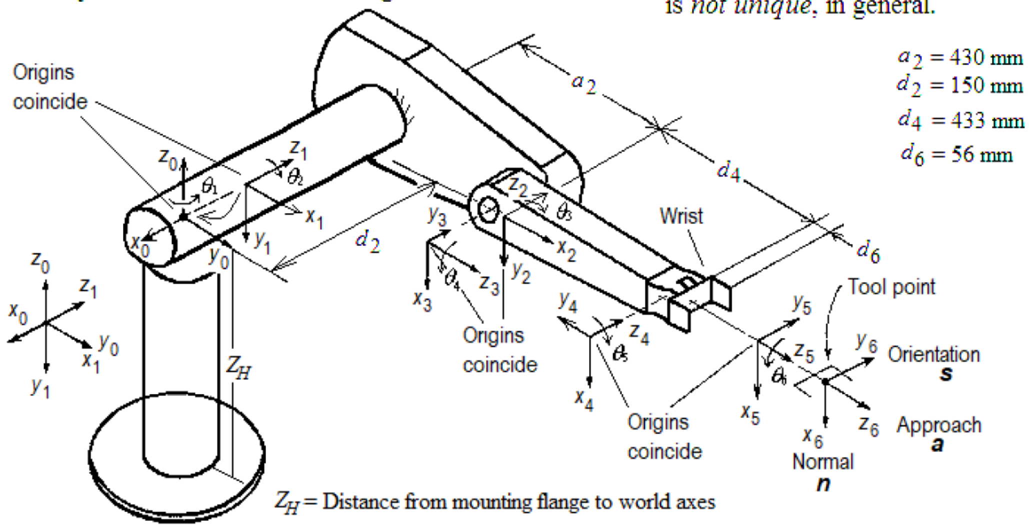


The robot is placed in a configuration with joitn variables of 0 or 90 degree.

* assignment of coordinate systems is *not unique*, in general.

z-axes (extended, if required)

skewlines: $\alpha_i \neq 0$ and $a_i \neq 0$
parallel: $\alpha_i = 0$ and $a_i \neq 0$
collinear: $\alpha_i = 0$ and $a_i = 0$
intersecting: $\alpha_i \neq 0$ and $a_i = 0$



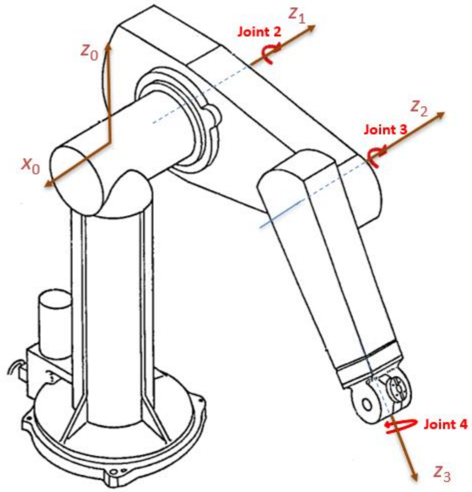
Z_H = Distance from mounting flange to world axes

NOTE: The above drawing is not to scale and is for illustrative purposes only.

D-H Parameters of PUMA Robot Arm (at position shown)

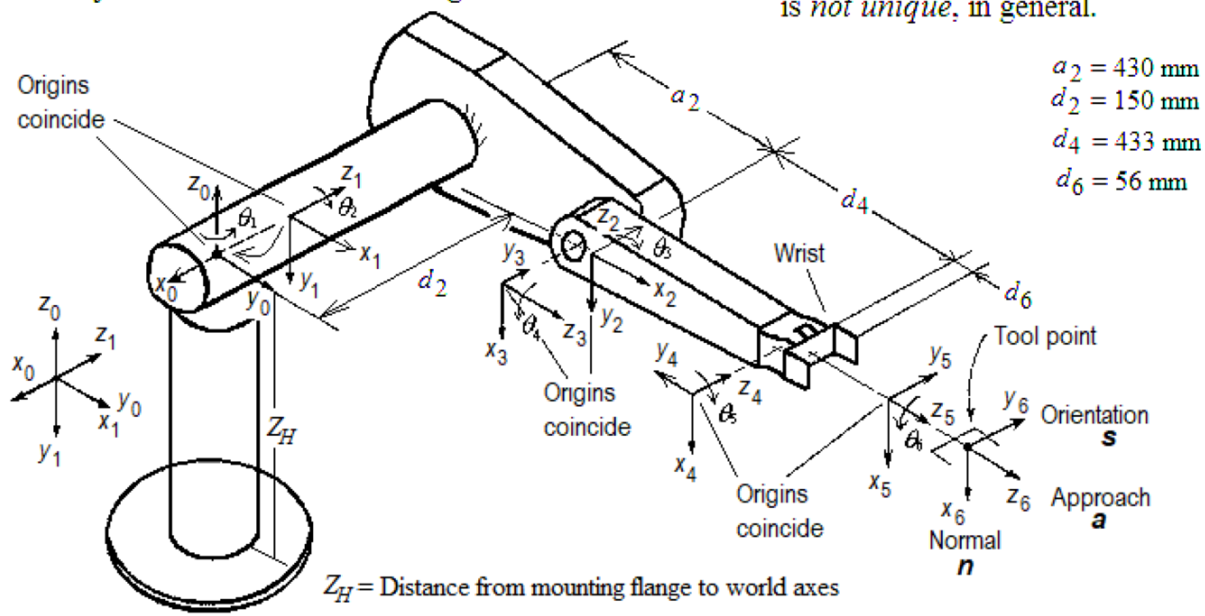
Joint i	θ_i (deg)	d_i (mm)	a_i (mm)	α_i (deg)	$\cos\alpha_i$	$\sin\alpha_i$	Joint variables	Joint range (deg)	
1	90	0	0	-90	0	-1	θ_1	-160 to 160	← z_{i-1}, z_i axes (*intersect*)
2	0	d_2	a_2	0	1	0	θ_2	-225 to 45	← z_{i-1}, z_i axes (*parallel*)
3	90	0	0	90	0	1	θ_3	-45 to 225	← z_{i-1}, z_i axes (*intersect*)
4	0	d_4	0	-90	0	-1	θ_4	-110 to 170	← z_{i-1}, z_i axes (*intersect*)
5	0	0	0	90	0	1	θ_5	-100 to 100	← z_{i-1}, z_i axes (*intersect*)
6	0	d_6	0	0	1	0	θ_6	-266 to 266	← z_{i-1}, z_i axes (*collinear*)

Note: plus-minus of the D-H parameters depending on the rotation-direction/distance from axis x_{i-1} to axis x_i (based on the right-hand coordinate system)



The robot is placed in a configuration with joitn variables of 0 or 90 degree.

* assignment of coordinate systems is *not unique*, in general.

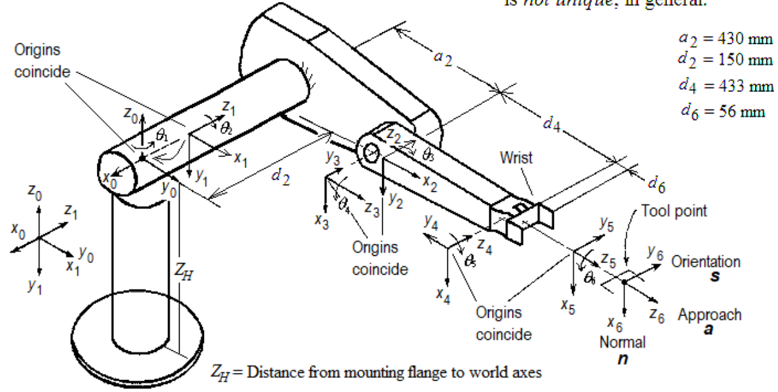


Z_H = Distance from mounting flange to world axes

NOTE: The above drawing is not to scale and is for illustrative purposes only.

The robot is placed in a configuration with joitn variables of 0 or 90 degree.

* assignment of coordinate systems is *not unique*, in general.



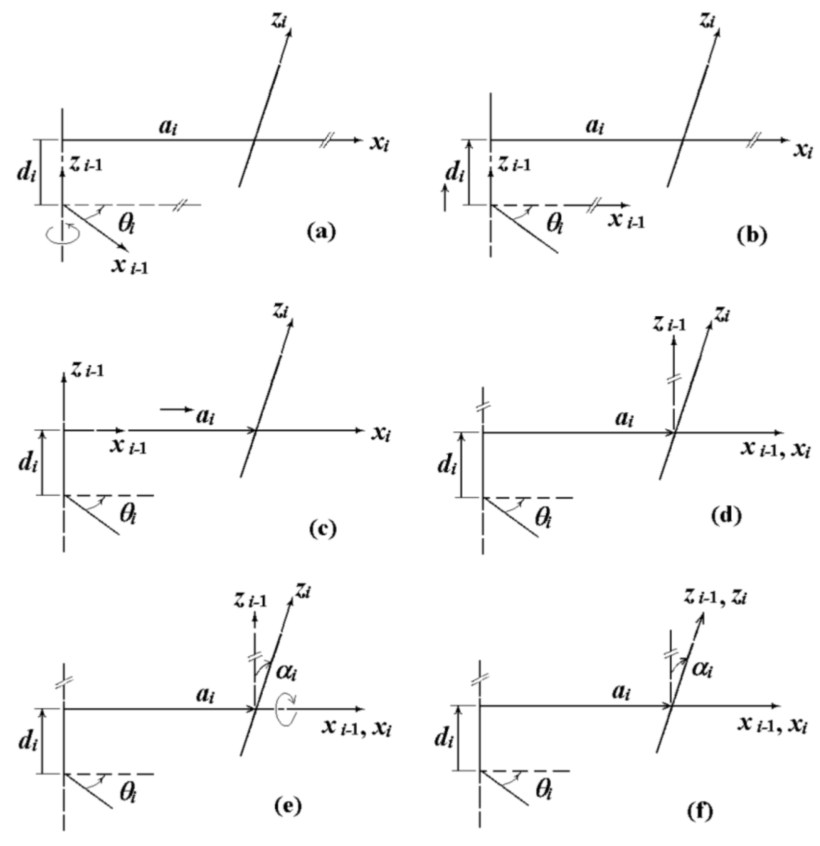
NOTE: The above drawing is not to scale and is for illustrative purposes only.

D-H Parameters of PUMA Robot Arm (at position shown)

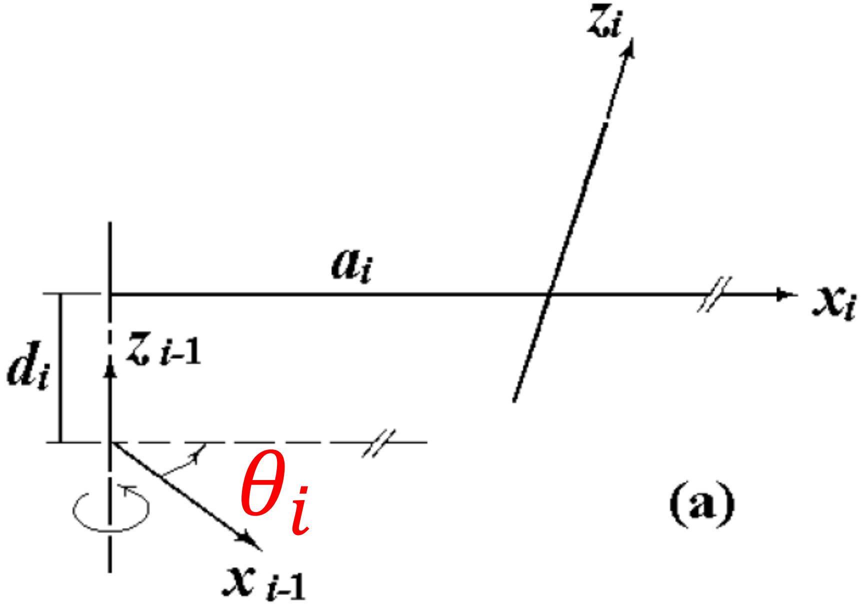
Joint i	θ_i (deg)	d_i (mm)	a_i (mm)	α_i (deg)	$\cos\alpha_i$	$\sin\alpha_i$	Joint variables	Joint range (deg)	
1	90	0	0	-90	0	-1	θ_1	-160 to 160	← z_{i-1}, z_i axes (*intersect*)
2	0	d_2	a_2	0	1	0	θ_2	-225 to 45	← z_{i-1}, z_i axes (*parallel*)
3	90	0	0	90	0	1	θ_3	-45 to 225	← z_{i-1}, z_i axes (*intersect*)
4	0	d_4	0	-90	0	-1	θ_4	-110 to 170	← z_{i-1}, z_i axes (*intersect*)
5	0	0	0	90	0	1	θ_5	-100 to 100	← z_{i-1}, z_i axes (*intersect*)
6	0	d_6	0	0	1	0	θ_6	-266 to 266	← z_{i-1}, z_i axes (*collinear*)

Note: plus-minus of the D-H parameters depending on the rotation-direction/distance from axis x_{i-1} to axis x_i (based on the right-hand coordinate system)

- D-H Homogeneous Matrix
  - Generation of $\left[H_{(i-1),i}\right]$ where $\{P\}_{i-1} = \left[H_{(i-1),i}\right]\{P\}_i$
    - Exercise in 'physically shifting' Frame *i*-1 into Frame *i*

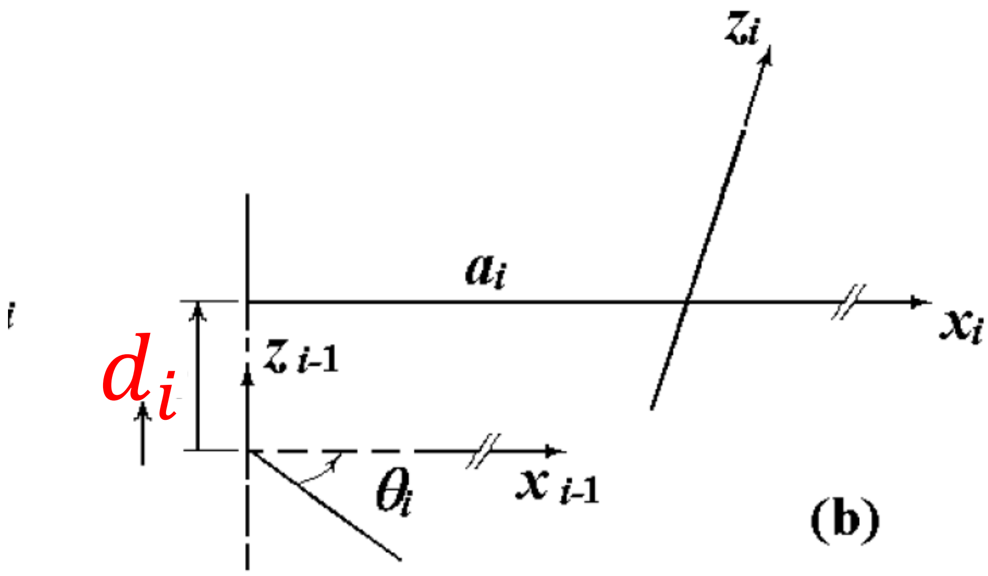


- <u>Step 1</u>
  - Rotate about $z_{i\text{-}1}$ axis θ_i



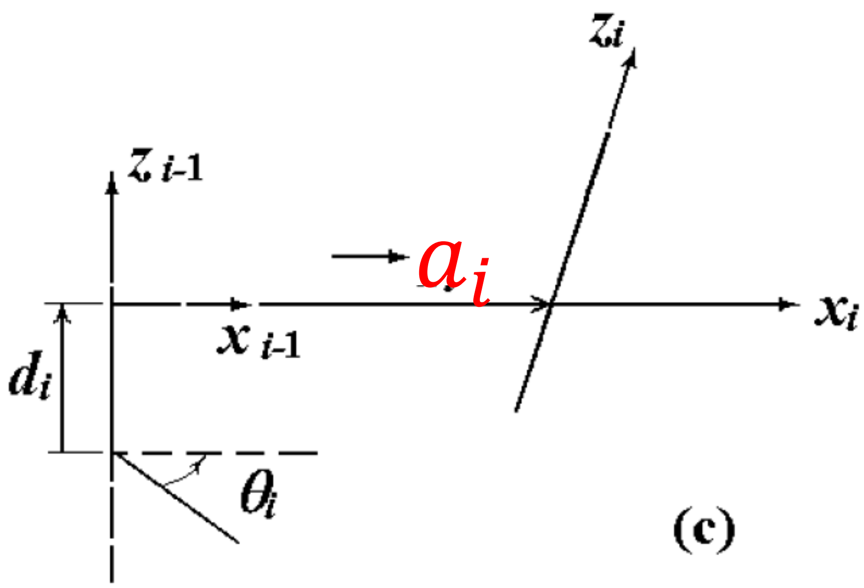
$x_{i\text{-}1}$ axis and x_i axis will become parallel

- <u>Step 2</u>
  - Translate along the $z_{i\text{-}1}$ axis d_i



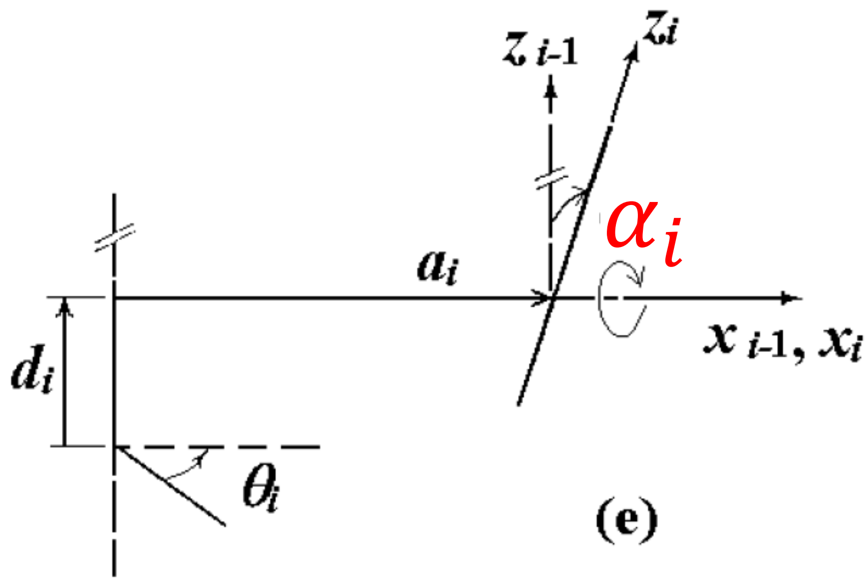
$x_{i\text{-}1}$ axis and x_i axis will become collinear

- Step 3
  - Translate along x_{i-1} axis a_i



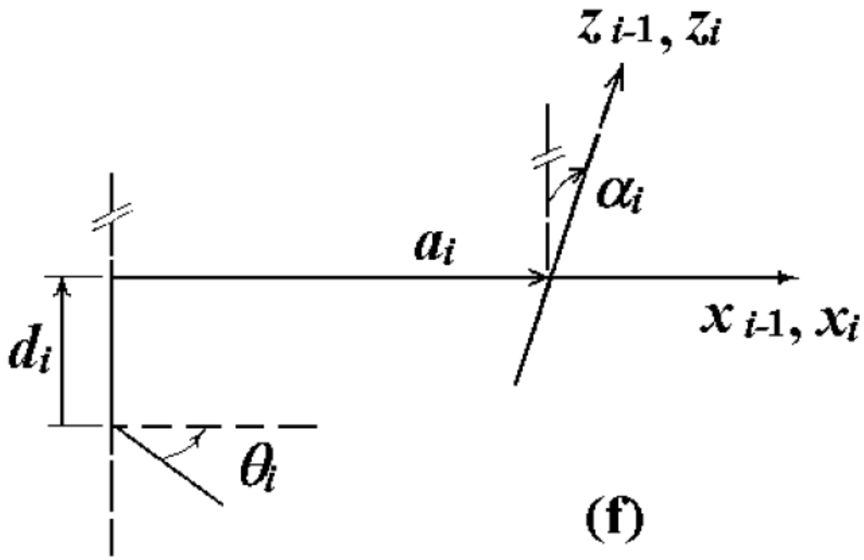
Origins of frame *i*-1 and *i* will collocated

- Step 4
  - Rotate $z_{i\text{-}1}$ axis α_i about x_i axis



Frame i-1 and Frame i will be exactly the same

- <u>End</u>
  - Transformation of Frame *i*-1 to Frame *i* completed



(f)

- In effect the 4 steps result in the matrix ***T***

$$\boldsymbol{T}_{(i-1),i} = \left[H_{(i-1),i}\right] = [H]_{z_{i-1},\theta_i}[H]_{z_{i-1},d_i}[H]_{x_i,a_i}[H]_{x_i,\alpha_i}$$

$$= \underbrace{\begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Rotation } \theta_i \text{ about } z_{i-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Translation } d_i \text{ along } z_{i-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Translation } a_i \text{ along } x_{i-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Rotation } \alpha_i \text{ about } x_i} \quad (7.23)$$

$$\left[H_{(i-1),i}\right] = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{(7.24)}$$

- For example, the transformation between joints 2 & 3 would be:

$$\boldsymbol{T}_{23} = [H_{23}] = \begin{bmatrix} \cos\theta_3 & -\cos\alpha_3 \sin\theta_3 & \sin\alpha_3 \sin\theta_3 & a_3 \cos\theta_3 \\ \sin\theta_3 & \cos\alpha_3 \cos\theta_3 & -\sin\alpha_3 \cos\theta_3 & a_3 \sin\theta_3 \\ 0 & \sin\alpha_3 & \cos\alpha_3 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (7.25)$$

- The total transformation between base of the robot and the hand:

$$\boldsymbol{T}_{RH} = \boldsymbol{T}_{01}\boldsymbol{T}_{12}\boldsymbol{T}_{23} \ \ldots \boldsymbol{T}_{(n-1),n} \qquad (7.26)$$

- The inverse of $\left[H_{(i-1),i}\right]$ is

$$\left[H_{i,(i-1)}\right] = \left[H_{(i-1),i}\right]^{-1} = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & -a_i \\ -\cos\alpha_i \sin\theta_i & \cos\alpha_i \cos\theta_i & \sin\alpha_i & -d_i \sin\alpha_i \\ \sin\alpha_i \sin\theta_i & -\sin\alpha_i \cos\theta_i & \cos\alpha_i & -d_i \cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.27)$$

$$\boldsymbol{H}^{-1} = \begin{bmatrix} n_X & n_Y & n_Z & -\boldsymbol{n}^T\boldsymbol{p} \\ s_X & s_Y & s_Z & -\boldsymbol{s}^T\boldsymbol{p} \\ a_X & a_Y & a_Z & -\boldsymbol{a}^T\boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \boldsymbol{n}^T & & -\boldsymbol{n}^T\boldsymbol{p} \\ & \boldsymbol{s}^T & & -\boldsymbol{s}^T\boldsymbol{p} \\ & \boldsymbol{a}^T & & -\boldsymbol{a}^T\boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & -\boldsymbol{n}^T\boldsymbol{p} \\ & \boldsymbol{R}_{3\times 3}^T & & -\boldsymbol{s}^T\boldsymbol{p} \\ & & & -\boldsymbol{a}^T\boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- <u>D-H Forward Kinematics</u>
  - The homogeneous matrix $H_{0,i}$
    - The chain product of successive coordinate transformation matrices of $\left[H_{(i-1),i}\right]$

$$\mathbf{H}_{0i} = \mathbf{H}_{01}\mathbf{H}_{12}\dots\mathbf{H}_{i-1,i} = \prod_{j=1}^{i}\mathbf{H}_{j-1,j} \quad \text{for} \quad i = 1,2,\dots,n$$
$$= \begin{bmatrix} \mathbf{x}_i & \mathbf{y}_i & \mathbf{z}_i & \mathbf{p}_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{0i} & {}^{\mathbf{0}}\mathbf{p}_i \\ 0 & 1 \end{bmatrix} \tag{7.28}$$

where

$[\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i]$ = rotation matrix of the ith coordinate system established at link i with respect to the base coordinate system. It is the upper left 3×3 partitioned matrix of $\mathbf{H}_{0i}$.

$\{\mathbf{p}_i\}$ = position vector that points from the origin of the base coordinate system to the origin of the ith coordinate system. It is the upper right 3×1 partitioned matrix of $\mathbf{H}_{0i}$.

- The forward matrix $H_{0,n}$
  - A manipulator with *n* links and joints
    - The base reference frame will be $O_0x_0y_0z_0$ and the last reference frame at the hand will be $O_nx_ny_nz_n$

$$\mathbf{H}_{0n} = \prod_{i=1}^{n} \mathbf{H}_{(i-1),i}$$

Is there a simpler way?

  - Recall the duality of the homogeneous transformation matrix:
    - A class of matrix operators that can perform simultaneous vector operations resulting in translation &/or rotation
    - Description of the geometric relationship between a body-attached frame *oxyz* and another reference coordinate frame *OXYZ*

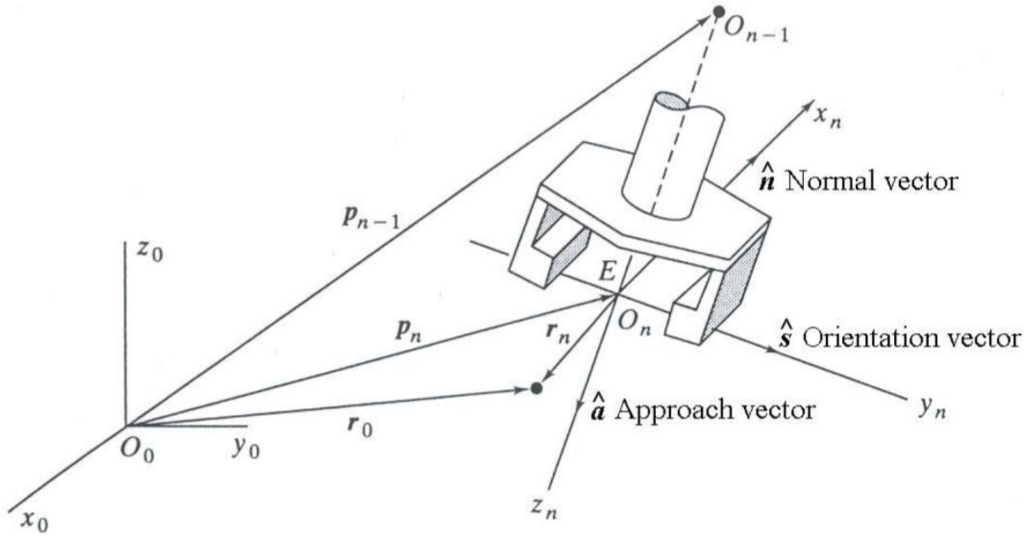


Figure 7A Hand frame of a manipulator with respect to the fixed system 0.

$$\text{Again,}\quad \boldsymbol{H} = \begin{bmatrix} n_X & s_X & a_X & p_X \\ n_Y & s_Y & a_Y & p_Y \\ n_Z & s_Z & a_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{a} & \boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \boldsymbol{R} & & \boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.12}$$

where $\boldsymbol{n}$ = *normal* vector; $\boldsymbol{s}$ = *sliding* (or *orientation*) vector, $\boldsymbol{a}$ = *approach* vector, and $\boldsymbol{p}$ = *position* vector.

- <u>Summary</u>

Step 1: Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

Step 2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-hand frame.

For $i = 1, \ldots, n-1$, perform Steps 3 to 5.

Step 3: Locate the origin O_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate O_i at this interse and z_{i-1} are parallel, locate O_i in any convenient position along z_i.

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This of the tool frame expressed in

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through O_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 5: Establish y_i to complete a right-hand frame.

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n-th joint is revolute, set $z_n = \boldsymbol{a}$ along the direction z_{n-1}. Establish the origin O_n conveniently along z_n, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = \boldsymbol{s}$ in the direction of the gripper closure and set $x_n = \boldsymbol{n}$ as $\boldsymbol{s} \times \boldsymbol{a}$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-hand frame.

Step 7: Create a table of link parameters a_i, d_i, α_i, θ_i.

$a_i =$ distance along x_i from O_i to the intersection of the x_i and z_{i-1} axes.

$d_i =$ distance along z_{i-1} from O_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

$\alpha_i =$ the angle between z_{i-1} and z_i measured about x_i (see Figure 3.3).

$\theta_i =$ the angle between x_{i-1} and x_i measured about z_{i-1} (see Figure 3.3). θ_i is variable if joint i is revolute.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into (3.10).

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

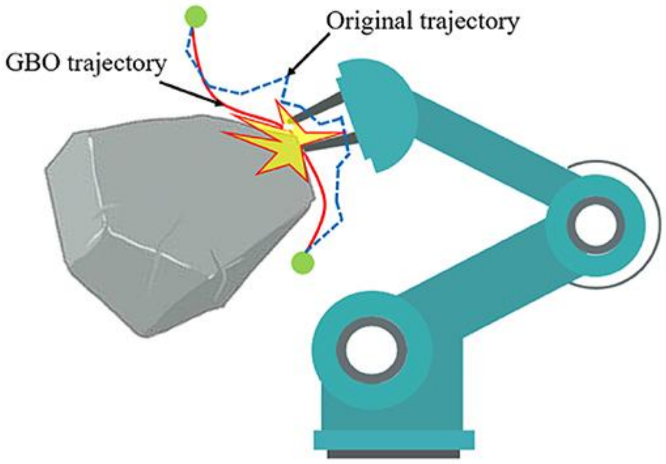
PATH PLANNING & TRAJECTORY

Motivation & Background

- Path planning: provides a geometric description of robot motion
  - Grows in computational complexity with no. of dofs.
  - does
- Path planning is treated as a search problem.

Motivation & Background

- Forward & inverse kinematics is concerned with the intrinsic geometry of robots.
- Motion planning deals with the extrinsic problem of collision within the workspace.
  - consists of Path & Trajectory Planning



1. Assume that the **initial** & **final configurations** are specified.
2. Find a **collision free path** for the robot that connects them.

Motivation & Background

- Outline:
  - Introduction to **configuration space**
  - Introduction to path planning methods
  - Introduction to trajectory planning

Configuration space

- A complete specification of the location of every point on the robot is referred to as a configuration
  - The set of all possible configurations is referred to as the configuration space
  - Does not specify dynamic aspects of motion (i.e. velocity, accelerations)
  - is treated as a search problem.
- A configuration could refer to the vector of joint variables q.
  - E.g. for a Cartesian arm, $q=(d_1, d_2, d_3)$

Configuration space

- A collision occurs when the robot contacts an obstacle in the workspace
  - We denote the subspace occupied by the robot at configuration *q* by *A*(q) where *A* is the homogeneous transformation
  - We denote O_i as the obstacles in the workspace
- The set of configurations which the robot collides is referred to as the configuration space obstacle defined by: $\mathcal{QO} = \{q \in \mathcal{Q} \mid \mathcal{A}(q) \cap \mathcal{O} \neq \emptyset\}$
- The set of collision-free configurations is defined by: $\mathcal{Q}_{\mathrm{free}} = \mathcal{Q} \setminus \mathcal{QO}$

Path planning methods

- Path planning problem: find a path from an initial configuration q_{init} to a final configuration q_{final} such that the robot does not collide with any obstacle.
  - A collision-free path from is a continuous map $\tau: [0,1] \rightarrow Q_{free}$ with $\tau(0) = q_{init}$ & $\tau(1) = q_{final}$

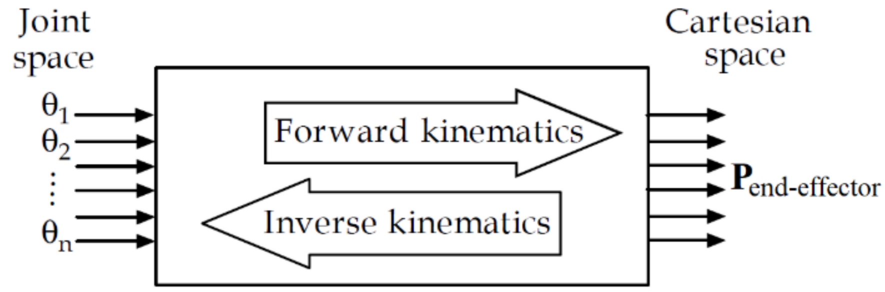
Trajectory planning

- A trajectory is a function of time $q(t)$ such that $q(\mathrm{t}_0)=q_{init}$ & $q(t_f)=q_{final}$.
  - Path τ as a time variable is a special case of a trajectory that will be executed in 1 unit of time.

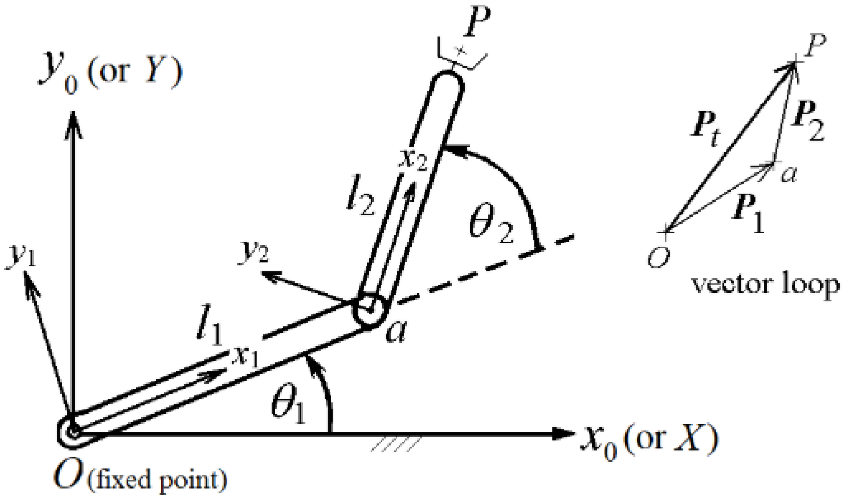
DYNAMICS

Revisiting Kinematics…

- **Objective**:
  - Develop a systematic modelling to describe geometric relationship between rigid bodies & the kinematics of a point of concern in a rigid body using coordinate frames & transformation.
- **Outcome**:
  - Forward/Inverse Kinematics, Jacobian & DH



Forward Kinematics



Position

$$
{}^0\mathbf{P}_t = \begin{Bmatrix} P_{tX} \\ P_{tY} \\ P_{tZ} \end{Bmatrix} = \begin{Bmatrix} l_1 c\theta_1 \\ l_1 s\theta_1 \\ 0 \end{Bmatrix} + \begin{Bmatrix} l_2 (c\theta_1 c\theta_2 - s\theta_1 s\theta_2) \\ l_2 (s\theta_1 c\theta_2 + c\theta_1 s\theta_2) \\ 0 \end{Bmatrix} = \begin{Bmatrix} l_1 c\theta_1 + l_2 c\theta_{12} \\ l_1 s\theta_1 + l_2 s\theta_{12} \\ 0 \end{Bmatrix}
$$

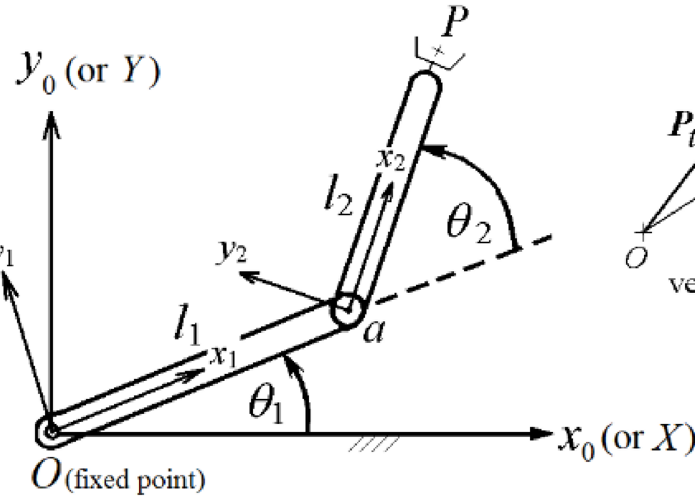
Velocity

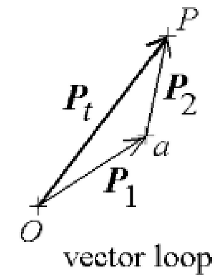
$$
{}^0\dot{\mathbf{P}}_t = \begin{Bmatrix} \dot{P}_{tX} \\ \dot{P}_{tY} \\ \dot{P}_{tZ} \end{Bmatrix} = \begin{Bmatrix} -l_1 \dot{\theta}_1 s\theta_1 - l_2 \dot{\theta}_{12} s\theta_{12} \\ l_1 \dot{\theta}_1 c\theta_1 + l_2 \dot{\theta}_{12} c\theta_{12} \\ 0 \end{Bmatrix}
$$

Acceleration

$$
{}^0\ddot{\mathbf{P}}_t = \begin{Bmatrix} \ddot{P}_{tX} \\ \ddot{P}_{tY} \\ \ddot{P}_{tZ} \end{Bmatrix} = \begin{Bmatrix} -l_1 \ddot{\theta}_1 s\theta_1 - l_1 \dot{\theta}_1^2 c\theta_1 - l_2 \ddot{\theta}_{12} s\theta_{12} - l_2 \dot{\theta}_{12}^2 c\theta_{12} \\ l_1 \ddot{\theta}_1 c\theta_1 - l_1 \dot{\theta}_1^2 s\theta_1 + l_2 \ddot{\theta}_{12} c\theta_{12} - l_2 \dot{\theta}_{12}^2 s\theta_{12} \\ 0 \end{Bmatrix}
$$

Jacobian



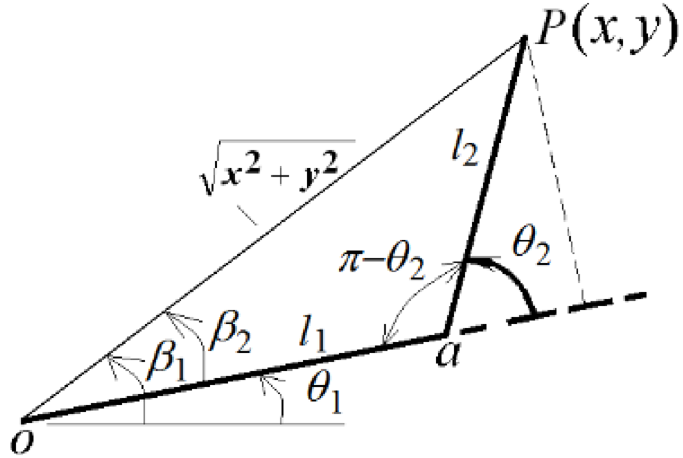


$$
{}^0\dot{\mathbf{P}}_t = \begin{Bmatrix} \dot{P}_{tX} \\ \dot{P}_{tY} \end{Bmatrix} = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix}
$$

$$
{}^0\ddot{\mathbf{P}}_t = \begin{Bmatrix} \dot{P}_{tX} \\ \dot{P}_{tY} \end{Bmatrix} = \begin{bmatrix} -l_1\dot{\theta}_1\cos\theta_1 - l_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2) & -l_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2) \\ -l_1\dot{\theta}_1\sin\theta_1 - l_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2) & -l_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2) \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix}
$$

$$
+ \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix}
$$

Inverse Kinematics



Joint Variables from intended end-effector workspace coordinates:

$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta_2}{l_1 + l_2\cos\theta_2}\right) \quad \text{or} \quad \theta_1 = \beta_1 - \beta_2 \qquad \theta_2 = \cos^{-1}\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)$$

1st Derivatives of Joint Variables intended speed of end-effector:

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \frac{1}{l_1 l_2 \sin\theta_2} \begin{bmatrix} l_2\cos(\theta_1+\theta_2) & l_2 \sin(\theta_1+\theta_2) \\ -l_1\cos\theta_1 - l_2\cos(\theta_1+\theta_2) & -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix}$$

2nd Derivatives of Joint Variables intended speed & acceleration of end-effector:

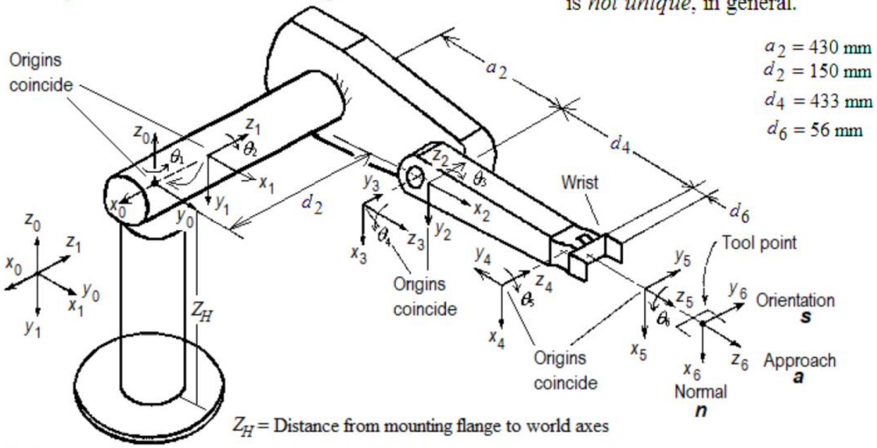
$$\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} = [B]\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + [\dot{B}]\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix}$$

$$[B] = [J]^{-1} = \frac{1}{l_1 l_2 \sin\theta_2} \begin{bmatrix} l_2 \cos(\theta_1+\theta_2) & l_2 \sin(\theta_1+\theta_2) \\ -l_1 \cos\theta_1 - l_2 \cos(\theta_1+\theta_2) & -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) \end{bmatrix}$$

DH Forward Kinematics

The robot is placed in a configuration with joitn variables of 0 or 90 degree.

* assignment of coordinate systems is *not unique*, in general.



NOTE: The above drawing is not to scale and is for illustrative purposes only.

D-H Parameters of PUMA Robot Arm (at position shown)									
Joint i	θ_i (deg)	d_i (mm)	a_i (mm)	α_i (deg)	$\cos\alpha_i$	$\sin\alpha_i$	Joint variables	Joint range (deg)	
1	90	0	0	-90	0	-1	θ_1	-160 to 160	← z_{i-1}, z_i axes (*intersect*)
2	0	d_2	a_2	0	1	0	θ_2	-225 to 45	← z_{i-1}, z_i axes (*parallel*)
3	90	0	0	90	0	1	θ_3	-45 to 225	← z_{i-1}, z_i axes (*intersect*)
4	0	d_4	0	-90	0	-1	θ_4	-110 to 170	← z_{i-1}, z_i axes (*intersect*)
5	0	0	0	90	0	1	θ_5	-100 to 100	← z_{i-1}, z_i axes (*intersect*)
6	0	d_6	0	0	1	0	θ_6	-266 to 266	← z_{i-1}, z_i axes (*collinear*)

Note: plus-minus of the D-H parameters depending on the rotation-direction/distance from axis x_{i-1} to axis x_i (based on the right-hand coordinate system)

Homogeneous Transformation Matrix

Generation from D-H Parameters

$$[H_{(i-1),i}] = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

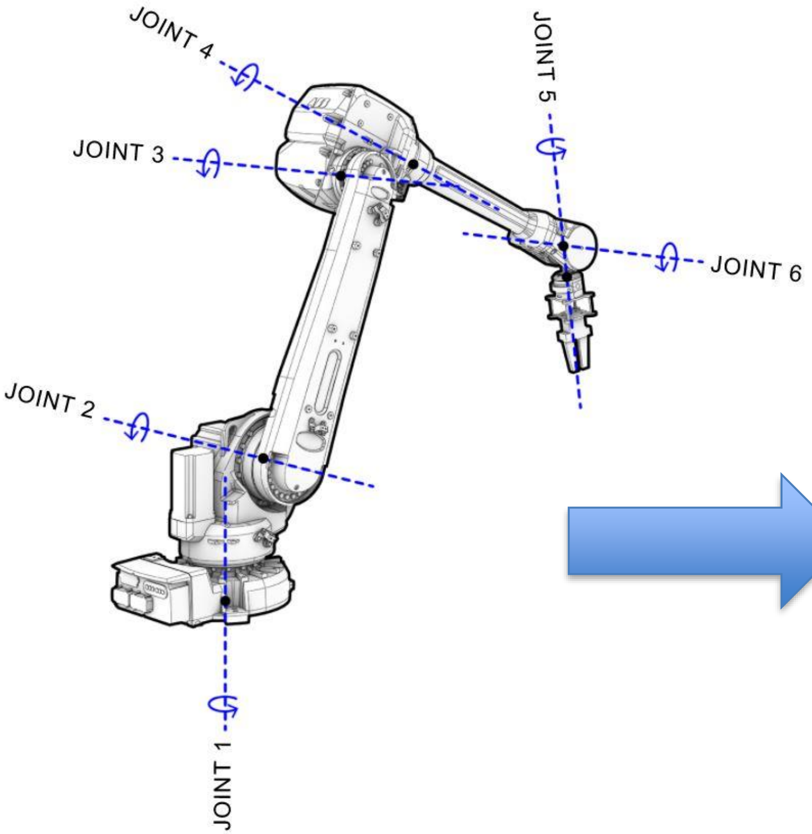
Geometric Interpretation

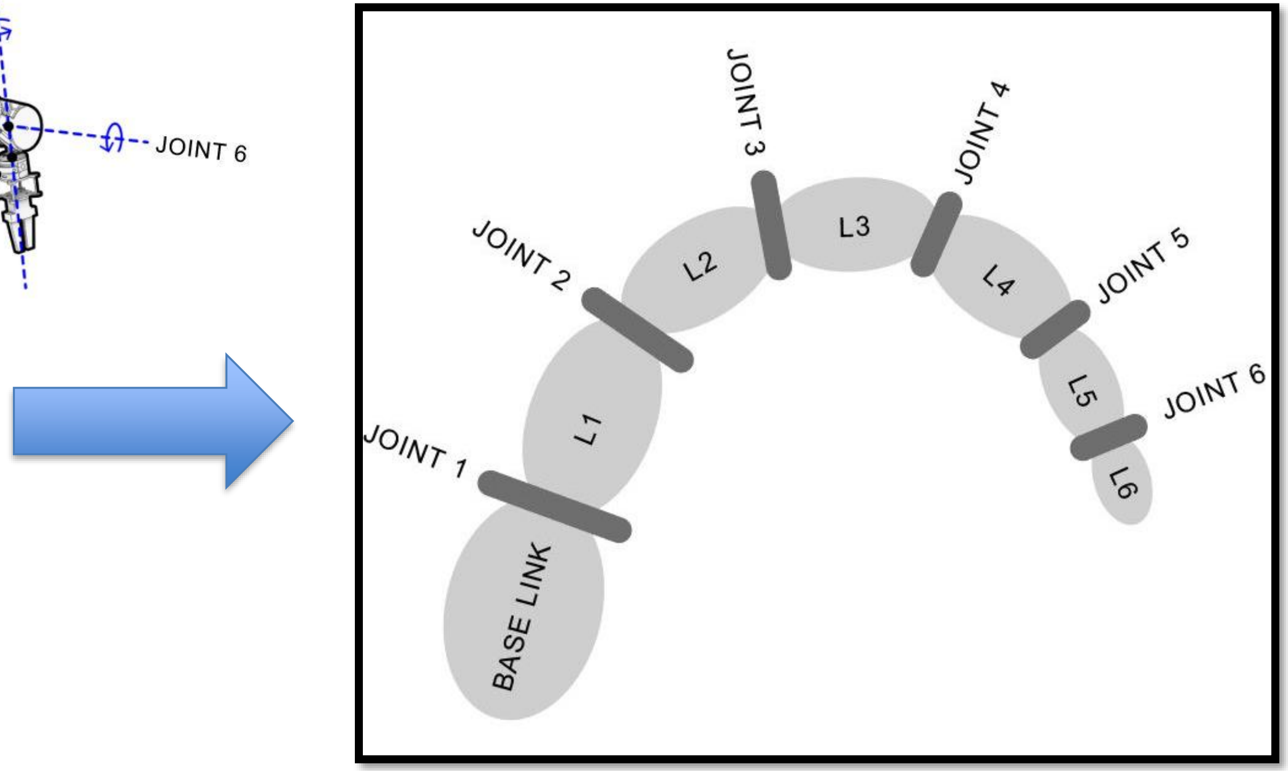
$$\mathbf{H} = \mathbf{H}_{06} = \mathbf{H}_F \mathbf{H}_L = \begin{bmatrix} n_X & s_X & a_X & p_X \\ n_Y & s_Y & a_Y & p_Y \\ n_Z & s_Z & a_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Kinematics:**
  - Kinematics is purely geometric relationship
  - Does not account for the mechanics of the robotic manipulators i.e.
    - Actuators provide torques at the joints
    - Links have inertia
- Dynamics :
  - Describes why & how motion occurs when forces & moments are applied on bodies

Why the need for Kinematics?

- Robot manipulator is a multi-rigid body motion
  - Open kinematic chain
  - Multiple links connected by multiple joints providing actuation
  - End-effector is at the very tip of the chain

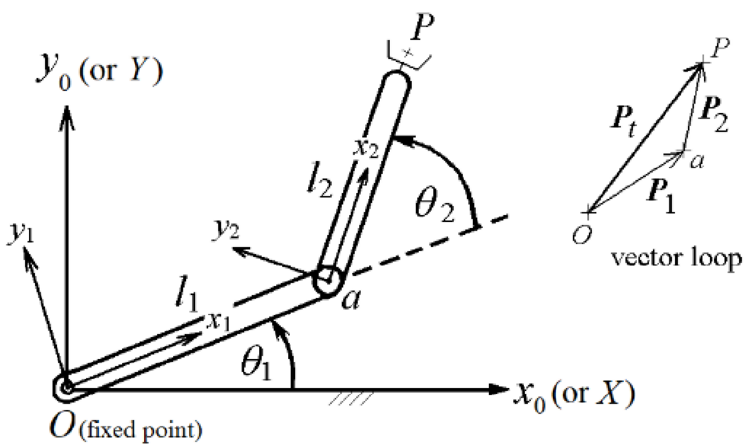


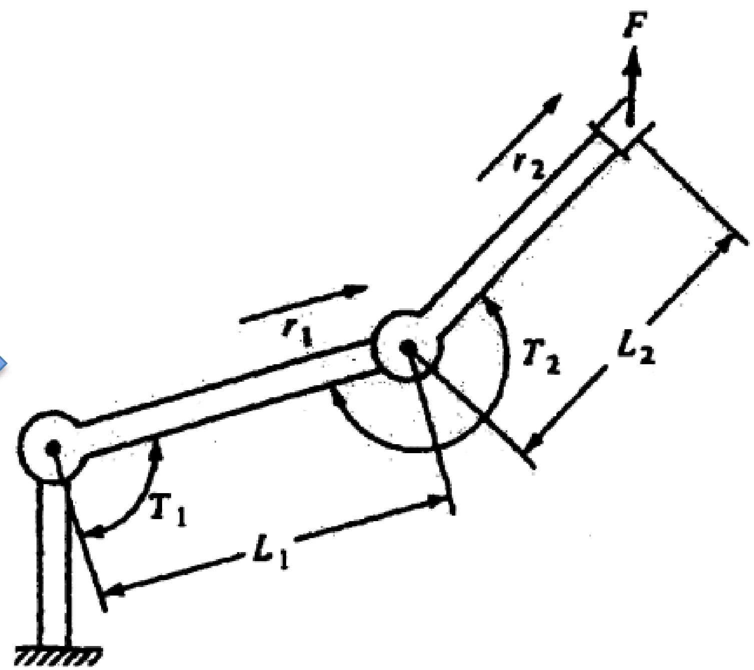


Dynamics

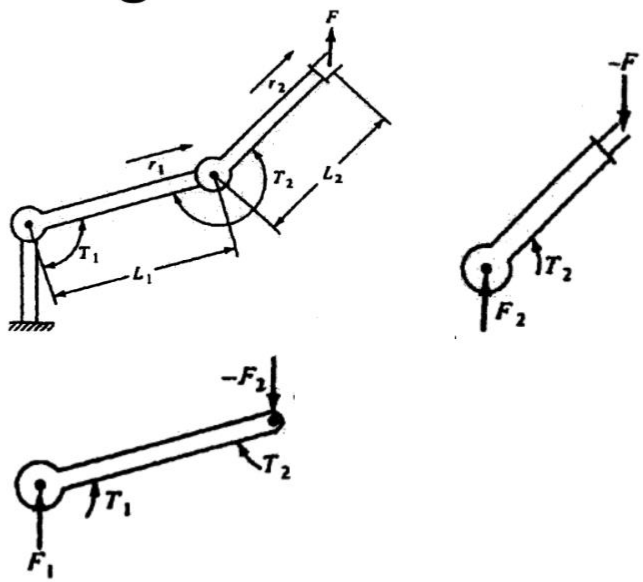
- **Objective**:
  - To find the necessary forces and torques required to induce desired accelerations in the robot's joints & links.
  - Sufficient acceleration & velocity required to achieve required trajectory and avoid positional error.
- **Outcome:**
  - Direct dynamics (predict motion for a given set of initial conditions & torques at joints)
  - **Inverse dynamics** (compute the forces & torques necessary to generate the prescribed trajectory for a given set of positions, velocities & acceleration)
  - Langrangian analysis

- Force analysis for 2-link manipulator
  - Newton-Euler analysis





- Force analysis for 2-link manipulator
  - Static Analysis – force required to maintain specific pose/configuration without motion



Forces:

$$\boldsymbol{F}_1 - \boldsymbol{F}_2 = 0$$
$$\boldsymbol{F}_2 - \boldsymbol{F} = 0$$
$$\therefore \boldsymbol{F}_2 = \boldsymbol{F}_1 = \boldsymbol{F}$$

Torques:

$$\boldsymbol{T}_1 = \boldsymbol{T}_2 + \boldsymbol{r}_1 \times \boldsymbol{F}$$
$$\boldsymbol{T}_2 = \boldsymbol{r}_2 \times \boldsymbol{F}$$
$$\therefore \boldsymbol{T}_1 = (\boldsymbol{r}_1 + \boldsymbol{r}_2) \times \boldsymbol{F}$$

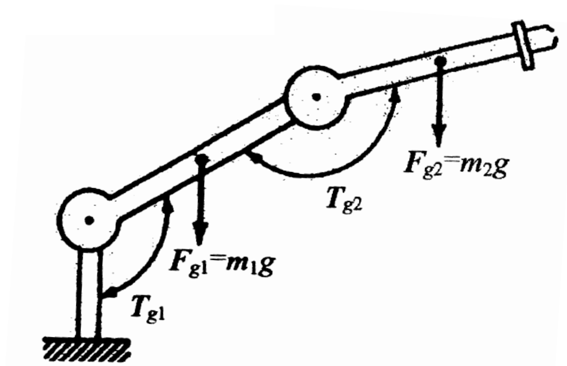
$$\boldsymbol{T}_1 = F_{\boldsymbol{y}}[L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2)] - F_{\boldsymbol{x}}[L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2)]$$

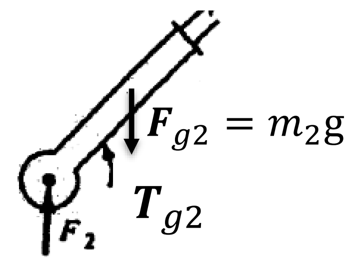
$$\boldsymbol{T}_2 = F_{\boldsymbol{y}}[L_2 \cos(\theta_1+\theta_2)] - F_{\boldsymbol{x}}[L_2 \sin(\theta_1+\theta_2)]$$

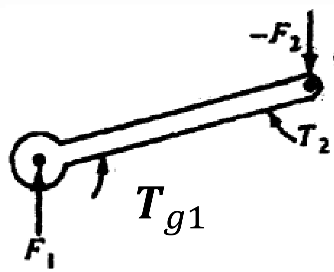
$$F_{\boldsymbol{x}} = \frac{T_1 L_2 \cos(\theta_1+\theta_2) - T_2[L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2)]}{L_1 L_2 \sin\theta_2}$$

$$F_{\boldsymbol{y}} = \frac{T_2 {L_2}^{\boldsymbol{2}} L_1 \sin\theta_2 + T_1 L_1 \cos(\theta_1+\theta_2)\sin(\theta_1+\theta_2) - T_2[L_1 L_2 \cos\theta_1 \sin(\theta_1+\theta_2) + L_2^2 \cos(\theta_1+\theta_2)\sin(\theta_1+\theta_2)]}{{L_2}^{\boldsymbol{2}} L_1 \sin\theta_2 \cos(\theta_1+\theta_2)}$$

- Force analysis for 2-link manipulator
  - Compensating for gravity







Forces:

$$F_2 = m_2 g$$

$$F_1 = F_2 + m_1 g$$

Torques:

$$\mathbf{T}_{g2} = \frac{-m_2 r_2 \times \mathbf{g}}{2} = \frac{g[m_2 L_2 \cos(\theta_1 + \theta_2)]}{2}$$

$$\mathbf{T}_{g1} = g\left[\left(\frac{m_1}{2} + m_2\right) L_1 \cos\theta_1 + \frac{m_2 L_2 \cos(\theta_1 + \theta_2)}{2}\right]$$

- **Limitations**:
  - Difficulty to apply Newtonian mechanics for robotic manipulator due to the following:
    - Multiple degree of freedom
    - Distributed mass
    - 3D nature of robots

- Langrangian mechanics
  - Based on energy terms with respect to system's variables and time

Lagrangian equation

$$L = K - P$$

L – Langrangian, K – Kinetic energy, P – Potential energy

Equation of motion

$$Q_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \left(\frac{\partial L}{\partial q_k}\right), \qquad k = 1 \text{ to } p.$$

Q_k- generalized forces (acting externally) of the system,
q_k- generalized coordinates (or variables) of the system

- Derivation of <u>equation of motion</u>

$$Q_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \left(\frac{\partial L}{\partial q_k}\right), \qquad k = 1 \text{ to } p.$$

$$F_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}, \qquad i = 1 \text{ to } m,$$

$$T_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_j}\right) - \frac{\partial L}{\partial \theta_j}, \qquad j = 1 \text{ to } n,$$

Q_k:	generalized forces (acting externally) of the system
q_k:	generalized coordinates (or variables) of the system
p:	total number of system variables, $p = m+n$
F_i:	the summation of all external forces for a linear motion
x_i (or s_i):	translational variables of the system
m:	total number of translational variables
T_j:	the summation of all torques in a rotational motion
θ_j:	rotational variables of the system
n:	total number of rotational variables
L:	Lagrangian of the system, $L = K - P$
K:	kinetic energy of the system
P:	potential energy of the system

• Derivation of kinetic energy

(ii) Kinetic Energy of a Rigid Body

For a rigid body (formed by an infinity of particles) experiencing a general motion, the kinetic energy is given by

$$K = \frac{1}{2}\int v^2 \,\mathrm{d}m \qquad (8.7)$$

If the centroid of the rigid body can be found, the form can be reduced to

For 3D motion

$$K = \frac{1}{2}\int v^2 \mathrm{d}m = \frac{1}{2} m \mathbf{v}_c{}^T \mathbf{v}_c + \frac{1}{2}\boldsymbol{\omega}_c{}^T \mathbf{I}_c \boldsymbol{\omega}_c = K_{translation} + K_{rotation} \qquad \textbf{(8.8)}$$

For a rigid body experiencing planar motion only, the expression kinetic energy is reduced to

For planar motion

$$K = \frac{1}{2} m v_c{}^2 + \frac{1}{2} I_c \omega_c{}^2 \qquad (8.9)$$

If A is a fixed point of the rotating link, we then have

Note: MI is for point A i.e. fixed point of rotating link

$$K = \frac{1}{2} I_A \omega_A{}^2 \text{ (pure rotation)}$$

Details of Eq. (8.8)

$$K = \frac{1}{2}\int v^2 \mathrm{d}m = \frac{1}{2} m \mathbf{v}_c{}^T \mathbf{v}_c + \frac{1}{2}\boldsymbol{\omega}_c{}^T \mathbf{I}_c \boldsymbol{\omega}_c$$

where

$$\mathbf{v}_c{}^T \mathbf{v}_c = \{v_{cx} \quad v_{cy} \quad v_{cz}\}\begin{Bmatrix} v_{cx} \\ v_{cy} \\ v_{cz} \end{Bmatrix} = \left(v_{cx}{}^2 + v_{cy}{}^2 + v_{cz}{}^2\right)$$

$$\boldsymbol{\omega}_c{}^T \mathbf{I}_c \boldsymbol{\omega}_c = \{\omega_{cx} \quad \omega_{cy} \quad \omega_{cz}\}\begin{bmatrix} I_{cxx} & I_{cxy} & I_{cxz} \\ I_{cyx} & I_{cyy} & I_{cyz} \\ I_{czx} & I_{czy} & I_{czz} \end{bmatrix}\begin{Bmatrix} \omega_{cx} \\ \omega_{cy} \\ \omega_{cz} \end{Bmatrix}$$

K is the kinetic energy of the rigid body with respect to ground;
m is the mass of the rigid body;
$\boldsymbol{v}_c$ is the velocity vector of the centroid (center of mass) of the rigid body (point c) with respect to ground;
$\boldsymbol{I}_c$ is the matrix of moment of inertia (or inertia matrix) of the rigid body about an axis passing through the centroid c;
$\boldsymbol{\omega}_c$ is the angular velocity vector of the rigid body with respect to ground.

v_c is the velocity (resultant and *absolute*) of the centroid of the rigid body (point c) with respect to ground;
I_c is the moment of inertia of the rigid body about an axis passing through the centroid c, and perpendicular to the plane of motion;
ω_c is the angular velocity (*absolute*) of the rigid body with respect to ground.

• Derivation of potential energy

There are two forms of potential energy to be considered in this chapter: gravitational potential energy and elastic potential energy (by springs). Gravitational potential energy is the energy stored in an object as the result of its vertical position (or height):

$$\boxed{P = mgh_v} \tag{8.10}$$

where h_v is the vertical height with respect to a datum. On the other hand, the elastic potential energy is the energy stored in elastic materials as the result of the stretching or compressing of a spring,

$$\boxed{P = \frac{1}{2}ks^2} \tag{8.11}$$

in which k is the spring constant and s is the stretched (extended) or compressed (retracted) length of the spring with respect to its equilibrium position.

Note that the kinetic energy is the function of the mass and speed of the object of concerned (absolute velocity), while the potential energy is the function of the mass and height (change of height for the potential due to *gravity*) of the object of concerned.

Attention should be paid to:

Datum selection for gravitational potential energy definition

Equilibrium position for elastic potential energy

- E.g. 1

Find the equations of motion for the two-degree-of-freedom system in Figure 8.2. *x and θ*

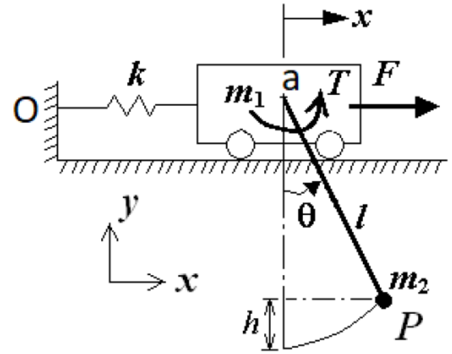


Figure 8.2 Schematic of a cart-pendulum system.

Solution:

There are two degree-of-freedom for the system shown in Figure 8.2, i.e. two coordinates x and θ. Therefore, there will be two equations of motion: one for the linear motion of the system and one for rotation of the pendulum.

2 equations of motion:

$$F = \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) - \frac{dL}{dx}$$

$$T = \frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) - \frac{dL}{d\theta}$$

The kinetic energy of the system is comprised of the kinetic energy of the cart and of the pendulum. Notice that the velocity of the pendulum at P is the summation of the velocity of the cart and of the pendulum relative to the cart, or

Velocity vector of *P*

$$\boldsymbol{v}_p = \boldsymbol{v}_c + \boldsymbol{v}_{p/c} = \dot{x}\hat{\mathbf{i}} + l\dot{\theta}\cos\theta\,\hat{\mathbf{i}} + l\dot{\theta}\sin\theta\,\hat{\mathbf{j}} = \left(\dot{x} + l\dot{\theta}\cos\theta\right)\hat{\mathbf{i}} + l\dot{\theta}\sin\theta\,\hat{\mathbf{j}}$$

Position vector of *P*

as $$\boldsymbol{P}_p = x\hat{\mathbf{i}} + l\sin\theta\,\hat{\mathbf{i}} - l\cos\theta\,\hat{\mathbf{j}} \qquad \text{(a)}$$

Therefore, $v_p^2 = v_x^2 + v_y^2 = \left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \left(l\dot{\theta}\sin\theta\right)^2$ and

$$K = \sum_{p=1}^{2} K_p = K_1 + K_2 = K_{\text{cart}} + K_{\text{pendulum}}, \qquad \text{(b)}$$

where

$$K_{\text{cart}} = \frac{1}{2}m_1\dot{x}^2 \quad \text{and} \quad K_{\text{pendulum}} = \frac{1}{2}m_2\left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}m_2\left(l\dot{\theta}\sin\theta\right)^2,$$

Derivation of total KE

which yields $$K = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{x}\cos\theta\right).$$

Derivation of total PE

Likewise, the potential energy is the summation of the potential energy in the spring and in the pendulum (due to gravity),

$$P = \frac{1}{2}kx^2 + m_2 gh = \frac{1}{2}kx^2 + m_2 gl(1-\cos\theta). \qquad \text{(c)}$$

Choice of datum for PE

Note that the zero-potential-energy-line (datum) is chosen at $\theta = 0^0$ for the pendulum, a vertical resting position. The Lagrangian is

Derivation of total Lagrangian

$$L = K - P$$
$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{x}\cos\theta\right) - \frac{1}{2}kx^2 - m_2 gl(1-\cos\theta) \qquad \text{(d)}$$

The derivatives and equations of motion for the linear motion by using Eq. (8.3) or (8.4), $q_1 = x$, $Q_1 = F$, are

$$\frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 l\dot{\theta}\cos\theta, \quad \text{(the only variable is } \dot{x}\text{)}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta\ ,$$
$$\frac{\partial L}{\partial x} = -kx; \quad \text{(the only variable is } x\text{)}$$

2 equations of motion:

$$F = \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) - \frac{dL}{dx}$$
$$T = \frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) - \frac{dL}{d\theta}$$

Equation of linear motion

$$\Rightarrow F = (m_1+m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta + kx\ . \qquad \text{(e)}$$

By using Eq. (8.3) or (8.5), $q_2 = \theta$, $Q_2 = T$, we have the results for the rotational motion

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 l^2\dot{\theta} + m_2 l\dot{x}\cos\theta\ ,$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_2 l^2\ddot{\theta} + m_2 l\ddot{x}\cos\theta - m_2 l\dot{x}\dot{\theta}\sin\theta,$$
$$\frac{\partial L}{\partial \theta} = -m_2 gl\sin\theta - m_2 l\dot{\theta}\dot{x}\sin\theta\ ;$$

Equation of rotational motion

$$\Rightarrow T = m_2 l^2\ddot{\theta} + m_2 l\ddot{x}\cos\theta + m_2 gl\sin\theta\ .$$

The two equations of motion are then

$$F = (m_1+m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta + kx\ ,$$
$$T = m_2 l^2\ddot{\theta} + m_2 l\ddot{x}\cos\theta + m_2 gl\sin\theta\ , \qquad \text{(f)}$$

which can be written in a matrix form

$$\begin{Bmatrix} F \\ T \end{Bmatrix} = \begin{bmatrix} m_1+m_2 & m_2 l\cos\theta \\ m_2 l\cos\theta & m_2 l^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & -m_2 l\sin\theta \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{x}^2 \\ \dot{\theta}^2 \end{Bmatrix} + \begin{Bmatrix} kx \\ m_2 gl\sin\theta \end{Bmatrix}.$$

Note that for this example of two-degree-of-freedom system:

$q_1 = x_1 = x$, $q_2 = \theta_1 = \theta$, $Q_1 = F_1 = F$, $Q_2 = T_1 = T$, $p = 2$, and $m = n = 1$.

- E.g. 2

Figure 8.3(a) shows an θ-s manipulator and Figure 8.4 shows the model schematically. The mass of the outer cylinder is assumed located at its center of mass, m_1, at a *constant* distance l_1 from the center of rotation. The telescoping radial arm and load are modeled as a mass m_2 at the distance s.

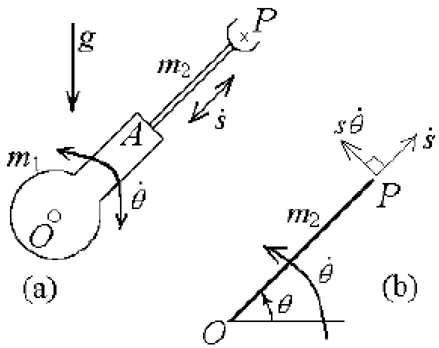


Figure 8.3 θ-s manipulator.

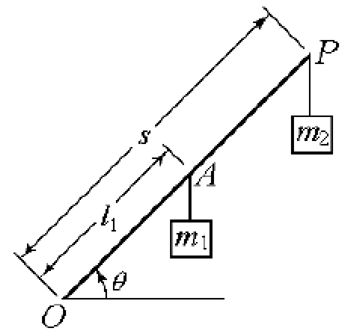


Figure 8.4 Schematic representation showing equivalent masses.

Solution:

We begin by deriving the kinetics by finding the kinetic energy of the two masses. Considering mass m_1 first, its Cartesian position is

Location of center of mass m_1

$$x_1 = l_1\cos\theta,\ y_1 = l_1\sin\theta \qquad \text{(a)}$$

which is differentiated with respect to time to obtain Cartesian velocities,

Velocity at center of mass m_1

$$\dot{x}_1 = -l_1\dot{\theta}\sin\theta,\ \dot{y}_1 = l_1\dot{\theta}\cos\theta \qquad \text{(b)}$$

The magnitude of the velocity vector is

$$\begin{aligned} v_1^2 &= (\dot{x}_1)^2 + (\dot{y}_1)^2 \\ v_1^2 &= l_1^2\dot{\theta}^2\sin^2\theta + l_1^2\dot{\theta}^2\cos^2\theta \\ &= l_1^2\dot{\theta}^2\left(\sin^2\theta + \cos^2\theta\right) = l_1^2\dot{\theta}^2 \end{aligned} \qquad \text{(c)}$$

The kinetic energy of a mass m moving is $K = \frac{1}{2}mv^2$

KE 1

so that $K_1 = \frac{1}{2}m_1\left(l_1^2\dot{\theta}^2\right)$ (d)

For the derivation of K_2, the change of both the joint angle θ and the sliding distance s need to be taken into account:

Location of center of mass m_2

$$x_2 = s\cos\theta,\ \ y_2 = s\sin\theta$$

$$\dot{x}_2 = \dot{s}\cos\theta - s\,\dot{\theta}\sin\theta,\ \dot{y}_2 = \dot{s}\sin\theta + s\,\dot{\theta}\cos\theta$$

Velocity at center of mass m_2

$$v_2^2 = (\dot{s}\cos\theta - s\,\dot{\theta}\sin\theta)^2 + (\dot{s}\sin\theta + s\,\dot{\theta}\cos\theta)^2 \qquad \text{(e)}$$

which simplifies to (see also Figure 8.3(b))

$$v_2^2 = \dot{s}^2 + s^2\dot{\theta}^2 \qquad \text{(f)}$$

KE 2

Therefore, $$K_2 = \frac{1}{2}m_2\left(\dot{s}^2 + s^2\dot{\theta}^2\right) \qquad \text{(g)}$$

Now, we find the potential energy by using

$$P = mgh$$

where h is the height and g the acceleration of gravity

Derivation of total PE

$$P_1 = m_1 g l_1 \sin\theta,\ \ P_2 = m_2 g s \sin\theta \qquad \text{(h)}$$

The total kinetic energy is thus

$$K = K_1 + K_2 = \frac{1}{2}m_1 l_1^2\dot{\theta}^2 + \frac{1}{2}m_2\dot{s}^2 + \frac{1}{2}m_2 s^2\dot{\theta}^2 \qquad \text{(i)}$$

and the total potential energy is

$$P = m_1 g l_1 \sin\theta + m_2 g s \sin\theta \qquad \text{(j)}$$

We can now find the system Lagrangian as

Derivation of total Lagrangian

$$L = \frac{1}{2}m_1 l_1^2\dot{\theta}^2 + \frac{1}{2}m_2\dot{s}^2 + \frac{1}{2}m_2 s^2\dot{\theta}^2 - m_1 g l_1\sin\theta - m_2 g s\sin\theta \qquad \text{(k)}$$

We will first find the torque about θ by using Eq. (8.5):

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 l_1^2 \dot{\theta} + m_2 s^2 \dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m_1 l_1^2 \ddot{\theta} + m_2 s^2 \ddot{\theta} + 2 m_2 s \dot{\theta} \dot{s}$$

$$\frac{\partial L}{\partial \theta} = -g \cos\theta (m_1 l_1 + m_2 s)$$

2 equations of motion:

$$F = \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) - \frac{dL}{dx}$$

$$T = \frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) - \frac{dL}{d\theta}$$

Equation of rotational motion

$$\Rightarrow \quad T_\theta = m_1 l_1^2 \ddot{\theta} + m_2 s^2 \ddot{\theta} + 2 m_2 s \dot{s} \dot{\theta} + g \cos\theta (m_1 l_1 + m_2 s) \qquad (l)$$

Equation of linear motion

$$\Rightarrow \quad F_s = m_2 \ddot{s} - m_2 s \dot{\theta}^2 + m_2 g \sin\theta \qquad (m)$$

Equations (l) and (m) together provide a complete description of the kinetics of this manipulator; that is, they provide a relationship between the torque T_θ or force F_s applied by an actuator and the resulting motions.

Note:

1. Rotational KE was not needed. Why?
2. Equation of motion for torque derived.

- E.g. 3

Using the Lagrangian method, derive the equations of motion for the two degrees-of-freedom robot arm, shown in Figure 8.5. The center of mass for each link is at the center of the link. The moments of inertia are I_1 and I_2.

Solution:

The two links in Figure 8.5 have distributed masses, requiring the use of moments of inertia (see Figure 8.6 for example) in the calculation of the kinetic energy. We will follow the same steps as before.

Velocity vector of *P*

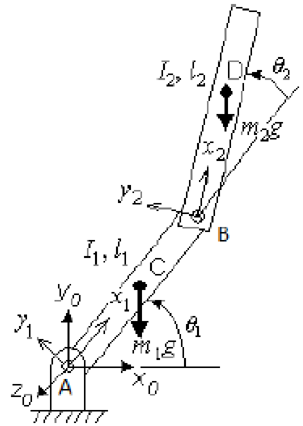


Figure 8.5 A two degree-of-freedom robot arm.

First, we calculate the velocity of the center of mass of link 2 by differentiating its position:

Position & velocity at center of mass for link 2

$$x_D = l_1C_1 + l_2C_{12}/2 \quad \rightarrow \quad \dot{x}_D = -l_1S_1\dot{\theta}_1 - l_2S_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)/2\,,$$
$$y_D = l_1S_1 + l_2S_{12}/2 \quad \rightarrow \quad \dot{y}_D = l_1C_1\dot{\theta}_1 + l_2C_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)/2\,, \qquad \text{(a)}$$

in which $S_1 = \sin\theta_1$, $C_1 = \cos\theta_1$, $S_{12} = \sin(\theta_1+\theta_2)$, and $C_{12} = \cos(\theta_1+\theta_2)$.

Therefore, the total velocity of the center of mass of link 2 is

$$v_D^2 = \dot{x}_D^2 + \dot{y}_D^2$$
$$= \dot{\theta}_1^2\left(l_1^2 + l_2^2/4 + l_1l_2C_2\right) + \dot{\theta}_2^2\left(l_2^2/4\right) + \dot{\theta}_1\dot{\theta}_2\left(l_2^2/2 + l_1l_2C_2\right). \qquad \text{(b)}$$

The kinetic energy of the total system is the sum of the kinetic energies of links 1 and 2. Remembering the formula for finding kinetic energy for a link rotating about a fixed axis (for link 1) and about the center of mass (for link 2), we have

Derivation of total KE (KE 1 derived for pure rotation of link 1 utilizing MI @A)

$$K=\sum_{p=1}^{2}K_p=K_1+K_2=\left[\frac{1}{2}I_A\dot{\theta}_1^2\right]+\left[\frac{1}{2}m_2v_D^2+\frac{1}{2}I_D\left(\dot{\theta}_1+\dot{\theta}_2\right)^2\right]$$
$$=\left[\frac{1}{2}\left(\frac{1}{3}m_1l_1^2\right)\dot{\theta}_1^2\right]+\left[\frac{1}{2}\left(\frac{1}{12}m_2l_2^2\right)\left(\dot{\theta}_1+\dot{\theta}_2\right)^2+\frac{1}{2}m_2v_D^2\right]. \tag{c}$$

Note that $I_D=\frac{1}{12}m_2l_2^2$ & $I_A=\frac{1}{3}m_1l_1^2$

KE1 is derived as pure rotation by leveraging I_A

Note that the first moment of inertia I_A for the slender rod is about the axis passing through point A, while the second inertia I_D is about point D, the mass center of link 2. Figure 8.6 lists the moments of inertia about the axes of some common homogenous shapes.

The system kinetic energy in Eq. (c) can also be obtained by

$$K=\sum_{p=1}^{2}K_p=K_1+K_2=\left(\frac{1}{2}m_1v_{C1}^2+\frac{1}{2}I_{C1}\omega_1^2\right)+\left(\frac{1}{2}m_2v_{C2}^2+\frac{1}{2}I_{C2}\omega_2^2\right)$$

Both rotational & linear KEs

To illustrate the concept of the moment of inertia with respect to an axis passing through different points, we consider the pure rotation of a single arm as shown in Figure 8.7. The kinetic energy of a rotating arm can be obtained either by point A,

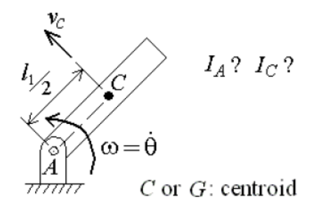


Figure 8.7 A rotating arm.

$$K=\frac{1}{2}I_A\omega^2=\frac{1}{2}I_A\dot{\theta}^2 \tag{d}$$

or via point C,

$$K=\frac{1}{2}mv_C^2+\frac{1}{2}I_C\omega^2=\frac{1}{2}m\left(\frac{l_1}{2}\dot{\theta}\right)^2+\frac{1}{2}I_C\dot{\theta}^2, \tag{e}$$

which will both eventually reach the same result. Note that $v_A = 0$.

Substituting Eq. (b) into Eq. (c) and re-grouping, we have

Alternative derivatior total KE (KE 1 & 2 witl MIs at center of mass link 1 &2)

$$K=\dot{\theta}_1^2\left(\frac{1}{6}m_1l_1^2+\frac{1}{6}m_2l_2^2+\frac{1}{2}m_2l_1^2+\frac{1}{2}m_2l_1l_2C_2\right)$$
$$+\dot{\theta}_2^2\left(\frac{1}{6}m_2l_2^2\right)+\dot{\theta}_1\dot{\theta}_2\left(\frac{1}{3}m_2l_2^2+\frac{1}{2}m_2l_1l_2C_2\right). \tag{f}$$

Derivation of total PE

The potential energy of the system is the sum of the potential energies:

$$P = \sum_{p=1}^{2} P_p = P_1 + P_2 = m_1 g \frac{l_1}{2} S_1 + m_2 g\left(l_1 S_1 + \frac{l_2}{2} S_{12} \right). \qquad (g)$$

Derivation of total Lagrangian

The Lagrangian for the two-link robot arm shown in Figure 8.5 is then

$$L = K\text{-}P = \dot{\theta}_1^2\left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) + \dot{\theta}_2^2\left(\frac{1}{6} m_2 l_2^2 \right)$$
$$+ \dot{\theta}_1 \dot{\theta}_2 \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) \text{-} m_1 g \frac{l_1}{2} S_1 \text{ - } m_1 g\left(l_1 S_1 + \frac{l_2}{2} S_{12} \right). \qquad (h)$$

Taking the derivatives of the Lagrangian and substituting the terms into Eq. (8.5) yields the following two equations of motion for the required torque at joints A and B:

Equation of rotational motion

$$T_1 = \left(\frac{1}{3} m_1 l_1^2 + m_2 l_1^2 + \frac{1}{3} m_2 l_2^2 + m_2 l_1 l_2 C_2 \right) \ddot{\theta}_1 + \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) \ddot{\theta}_2$$
$$\text{-} \left(m_2 l_1 l_2 S_2 \right) \dot{\theta}_1 \dot{\theta}_2 - \left(\frac{1}{2} m_2 l_1 l_2 S_2 \right) \dot{\theta}_2^2 + \left(\frac{1}{2} m_1 + m_2 \right) g l_1 C_1 + \frac{1}{2} m_2 g l_2 C_{12}, \qquad (i)$$

$$T_2 = \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) \ddot{\theta}_1 + \left(\frac{1}{3} m_2 l_2^2 \right) \ddot{\theta}_2 + \left(\frac{1}{2} m_2 l_1 l_2 S_2 \right) \dot{\theta}_1^2 + \frac{1}{2} m_2 g l_2 C_{12}. \qquad (j)$$

Equations (j) and (k) can also be written in a matrix form. Note that T_1 and T_2 are the required torque at the joint to drive link 1 and 2, respectively.

Note:
Unlike example 2, no equation of linear motion was derived. Why?

- # Lagrangian for 2-link manipulator

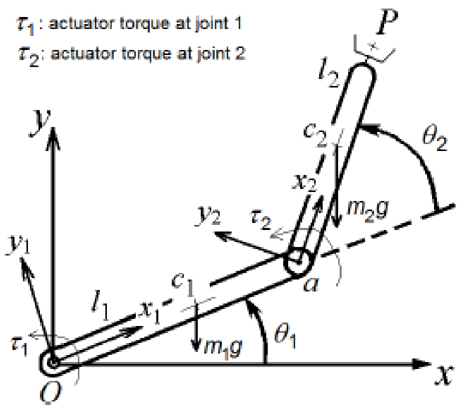


Figure 8.8 Link lengths and the centroids

Position vector of the centroid 1 (see Figure 8.8)

$$\{P_{c1}\} = [R_{01}]\{p_{c1}\}$$

$$= \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} l_{c1} \\ 0 \\ 0 \end{Bmatrix}$$

$$= \{l_{c1}c\theta_1 \quad l_{c1}s\theta_1 \quad 0\}^T \quad \text{as } c\theta_1 = \cos\theta_1 \text{ and } s\theta_1 = \sin\theta_1.$$

Position vector of the centroid 2

$$\{P_{c2}\} = [R_{01}][R_{12}]\{p_{c2}\} + [R_{01}]\{p_1\}$$

$$= \begin{bmatrix} c\theta_{12} & -s\theta_{12} & 0 \\ s\theta_{12} & c\theta_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} l_{c2} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} l_1 c\theta_1 \\ l_1 s\theta_1 \\ 0 \end{Bmatrix}$$

$$= \{l_1 c\theta_1 + l_{c2}c\theta_{12} \quad l_1 s\theta_1 + l_{c2}s\theta_{12} \quad 0\}^T, \quad \text{where } \theta_{12} = (\theta_1 + \theta_2)$$

Position vector of the arm's tip

$$\{P_t\} = [R_{01}]\{p_1\} + [R_{02}]\{p_2\} = \{l_1 c\theta_1 + l_2 c\theta_{12} \quad l_1 s\theta_1 + l_2 s\theta_{12} \quad 0\}^T \quad \text{(c)}$$

Velocity of the centroid 2

$$\{\dot{P}_{c2}\} = \begin{Bmatrix} -\dot{\theta}_1 l_1 s\theta_1 - \dot{\theta}_{12} l_{c2} s\theta_{12} \\ \dot{\theta}_1 l_1 c\theta_1 + \dot{\theta}_{12} l_{c2} c\theta_{12} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\dot{\theta}_1 (l_1 s\theta_1 + l_{c2} s\theta_{12}) - \dot{\theta}_2 l_{c2} s\theta_{12} \\ \dot{\theta}_1 (l_1 c\theta_1 + l_{c2} c\theta_{12}) + \dot{\theta}_2 l_{c2} c\theta_{12} \\ 0 \end{Bmatrix} \quad \text{(e)}$$

where $\dot{\theta}_{12} = (\dot{\theta}_1 + \dot{\theta}_2)$

Velocity of the arm's tip

$$\{\dot{P}_t\} = \begin{Bmatrix} -\dot{\theta}_1 (l_1 s\theta_1 + l_2 s\theta_{12}) - \dot{\theta}_2 l_2 s\theta_{12} \\ \dot{\theta}_1 (l_1 c\theta_1 + l_2 c\theta_{12}) + \dot{\theta}_2 l_2 c\theta_{12} \\ 0 \end{Bmatrix} \quad \text{(f)}$$

• Lagrangian for 2-link manipulator

System kinetic energy for links 1 and 2

$$K = \frac{1}{2}m_1\{\dot{P}_{c1}\}^T\{\dot{P}_{c1}\} + \frac{1}{2}m_2\{\dot{P}_{c2}\}^T\{\dot{P}_{c2}\} + \frac{1}{2}\{\omega_1\}^T[I_{c1}]\{\omega_1\} + \frac{1}{2}\{\omega_2\}^T[I_{c2}]\{\omega_2\} \quad \text{(g)}$$

where $\{\omega_1\}^T = \{0 \quad 0 \quad \dot{\theta}_1\}$, $\{\omega_2\}^T = \{0 \quad 0 \quad \dot{\theta}_1 + \dot{\theta}_2\}$

$$[I_{c1}] = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}, \qquad [I_{c2}] = \begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix}$$

$$K = \frac{1}{2}m_1\{-\dot{\theta}_1 l_{c1} s\theta_1 \quad \dot{\theta}_1 l_{c1} c\theta_1 \quad 0\}\begin{Bmatrix} -\dot{\theta}_1 l_{c1} s\theta_1 \\ \dot{\theta}_1 l_{c1} c\theta_1 \\ 0 \end{Bmatrix}$$

$$+\frac{1}{2}\{0 \quad 0 \quad \dot{\theta}_1\}\begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{Bmatrix}$$

$$+\frac{1}{2}m_2\begin{Bmatrix} -\dot{\theta}_1 l_1 s\theta_1 - \dot{\theta}_{12} l_{c2} s\theta_{12} \\ \dot{\theta}_1 l_1 c\theta_1 + \dot{\theta}_{12} l_{c2} c\theta_{12} \\ 0 \end{Bmatrix}^T \begin{Bmatrix} -\dot{\theta}_1 l_1 s\theta_1 - \dot{\theta}_{12} l_{c2} s\theta_{12} \\ \dot{\theta}_1 l_1 c\theta_1 + \dot{\theta}_{12} l_{c2} c\theta_{12} \\ 0 \end{Bmatrix}$$

$$+\frac{1}{2}\{0 \quad 0 \quad \dot{\theta}_1 + \dot{\theta}_2\}\begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{Bmatrix}$$

$$= \frac{1}{2}m_1\dot{\theta}_1^2 l_{c1}^2 + \frac{1}{2}I_{zz1}\dot{\theta}_1^2 + \frac{1}{2}I_{zz2}(\dot{\theta}_1 + \dot{\theta}_2)^2 \quad \text{(h)}$$

$$+\frac{1}{2}m_2\left[\dot{\theta}_1^2 l_1^2 + (\dot{\theta}_1 + \dot{\theta}_2)^2 l_{c2}^2 + 2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) l_1 l_{c2} c\theta_2\right]$$

System potential energy for links 1 and 2

$$P = m_1 g\Delta h_1 + m_2 g\Delta h_2 = m_1 g l_{c1}\sin\theta_1 + m_2 g(l_1\sin\theta_1 + l_{c2}\sin\theta_{12}) \quad \text{(i)}$$

Inverse Dynamics (for actuator torques, τ_1 and τ_2)

$$\tau_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}, \qquad k = 1,\ 2. \quad \text{(j)}$$

For $\theta_2 (k = 2)$, joint 2:

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2}$$
$$= I_{zz2}(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2\left[l_{c2}^2(\ddot{\theta}_1 + \ddot{\theta}_2) + \ddot{\theta}_1 l_1 l_{c2} c\theta_2\right] + m_2\dot{\theta}_1\dot{\theta}_2 l_1 l_{c2} s\theta_2 + m_2 g l_{c2} c\theta_{12} \quad \text{(k)}$$

For $\theta_1 (k = 1)$, joint 1:

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1}$$
$$= (m_1 l_{c1}^2 + I_{zz1})\ddot{\theta}_1 + I_{zz2}(\ddot{\theta}_1 + \ddot{\theta}_2) \quad \text{(l)}$$
$$+ m_2\left[\ddot{\theta}_1 l_1^2 + (\ddot{\theta}_1 + \ddot{\theta}_2) l_{c2}^2 + (2\ddot{\theta}_1 + \ddot{\theta}_2) l_1 l_{c2} c\theta_2 - (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) l_1 l_{c2} s\theta_2\right]$$
$$+ m_1 g l_{c1} c\theta_1 + m_2 g(l_1 c\theta_1 + l_{c2} c\theta_{12})$$

Summary:

(i) System K & P, associated with each centroid; (ii) Respective velocities (linear and angular; absolute); (iii) Equations of motion associated with the respective joint variables (by Lagrangian equation).

For practice

8.9 A uniform rod of length l and mass m hangs at the edge of a vertical wheel of radius r, as shown in Figure P8.9. The rod can be taken a gondola car on a ferris wheel. The wheel rotates with a constant angular velocity ω about a horizontal axis through the midpoint O. Note that the rod is rotating at an angle β with respect to the vertical axis x.

a. Assuming the mass of wheel is negligible, find the kinetic and potential energies of the system. Note that I_{zz} is the moment of inertia of the rod about the z-axis passing through its centroid C. For the slender rod, $I_{zz} = ml^2/12$.

b. Find the equation of motion of the rod associated to the joint acceleration, $\ddot{\beta}$.

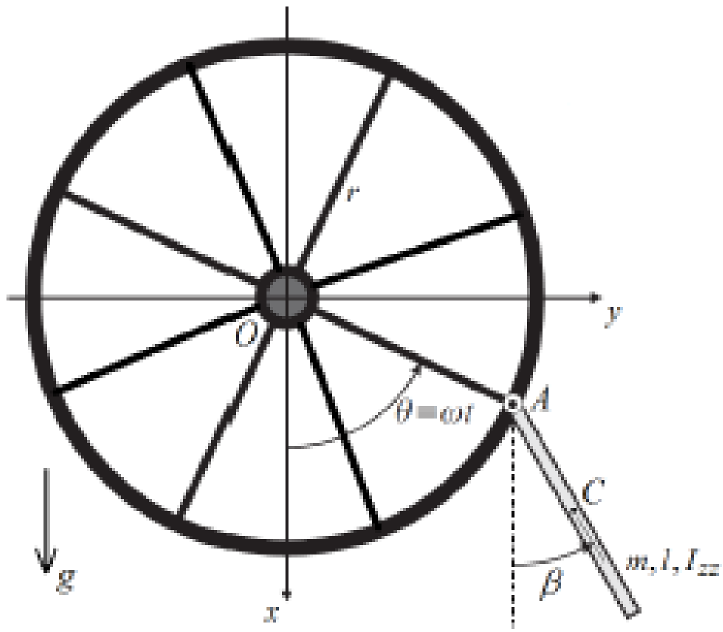


Figure P8.9

Thank you!