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Module 5

MOTION PLANNING & INVERSE KINEMATICS

What is a robot?

- A robot is a machine in a form of mechanically constructed system.
- Programmable by a computer.
- Robots can be articulated, autonomous or semi-autonomous depending on the required application
  - Motion constitutes an intrinsic element towards autonomy

What is a robot?

- Fundamentals of robotics includes:
  - Kinematics
  - Dynamics
  - Motion planning
  - Computer vision
  - Control

Scope

- An industrial robot manipulator is a feedback controlled, reprogrammable, multipurpose system.
  - It is reprogrammable in three or more degrees of freedom.
  - Robot manipulators are used in processes of industrial automation (ISO 8373 standard)

Scenario

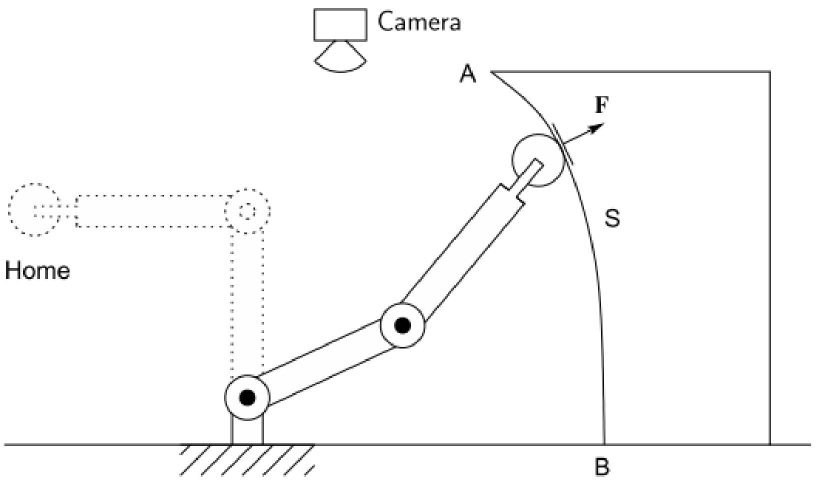
5 typical jobs in manufacturing:

1. Deformation (extrusion, rolling)
2. Material removal (grinding)
3. Solidification (moulding)
4. Assembly
5. Material handling

Scenario

- Material removal (Grinding)
  - Task
    - to make use of tool with edge in to remove materials from workpiece to form desired surface profiles
  - Action
    - To attach & rotate tool, to fix workpiece to expose surface, to move and rotate tool until desired surface profile is obtained
  - Motion
    - Sequence of motion descriptions in the form of script languages or library functions
  - Path
    - Set of equations of motion without time constraints
  - Trajectory
    - Set of equations of trajectory with time constraints

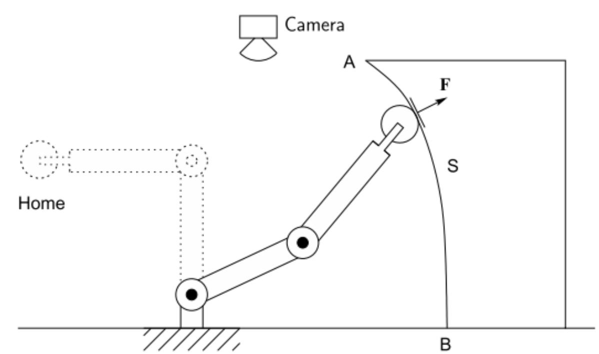
Scenario



- Simple 2-link planar manipulator with grinding tool to remove metal from a surface.
- *What are the issues to be resolved & what conceptualizations are required to enable programming a robot to perform the tasks?*

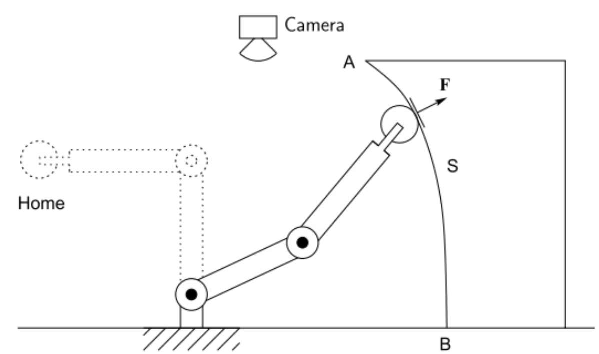
Scenario

- **Forward kinematics**:
  - Describe both position of tool & location A & B w.r.t. common coordinate system
  - Manipulator can measure joint variables θ_1 & θ_2 with encoders
  - What is needed is to express positions A & B in terms of these joint angles
  - **Homogeneous coordinates/ transformation DHS**



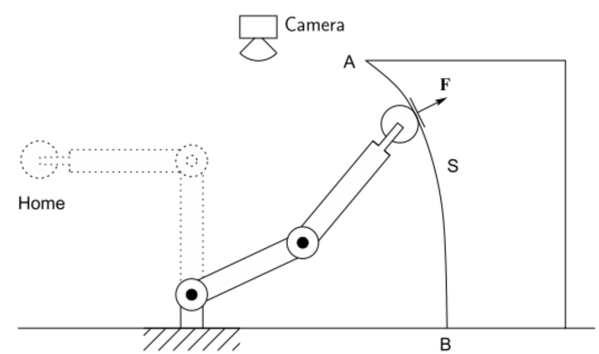
Scenario

- **Inverse kinematics**:
  - To command robot to move to location *A* we need the joint variables θ_1 & θ_2 in terms of the *x* & *y* coordinates.
  - There may be 0 to infinite solutions



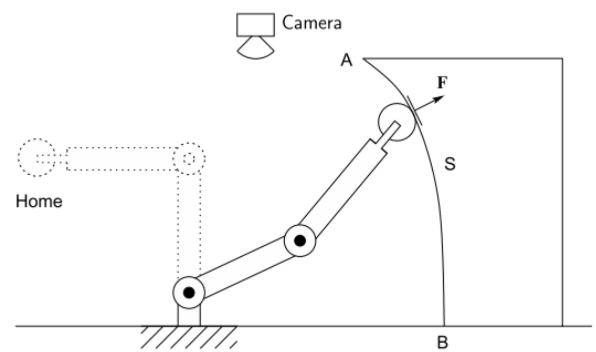
Scenario

- **Velocity kinematics**:
  - To maintain constant chosen velocity, the relationship between the tool & joint velocities must be established
  - To achieve required trajectory.
  - **Jacobian**



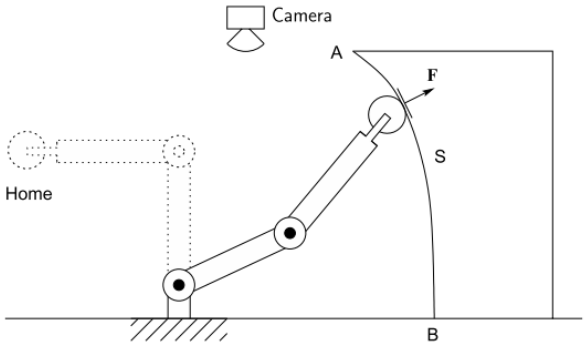
Scenario

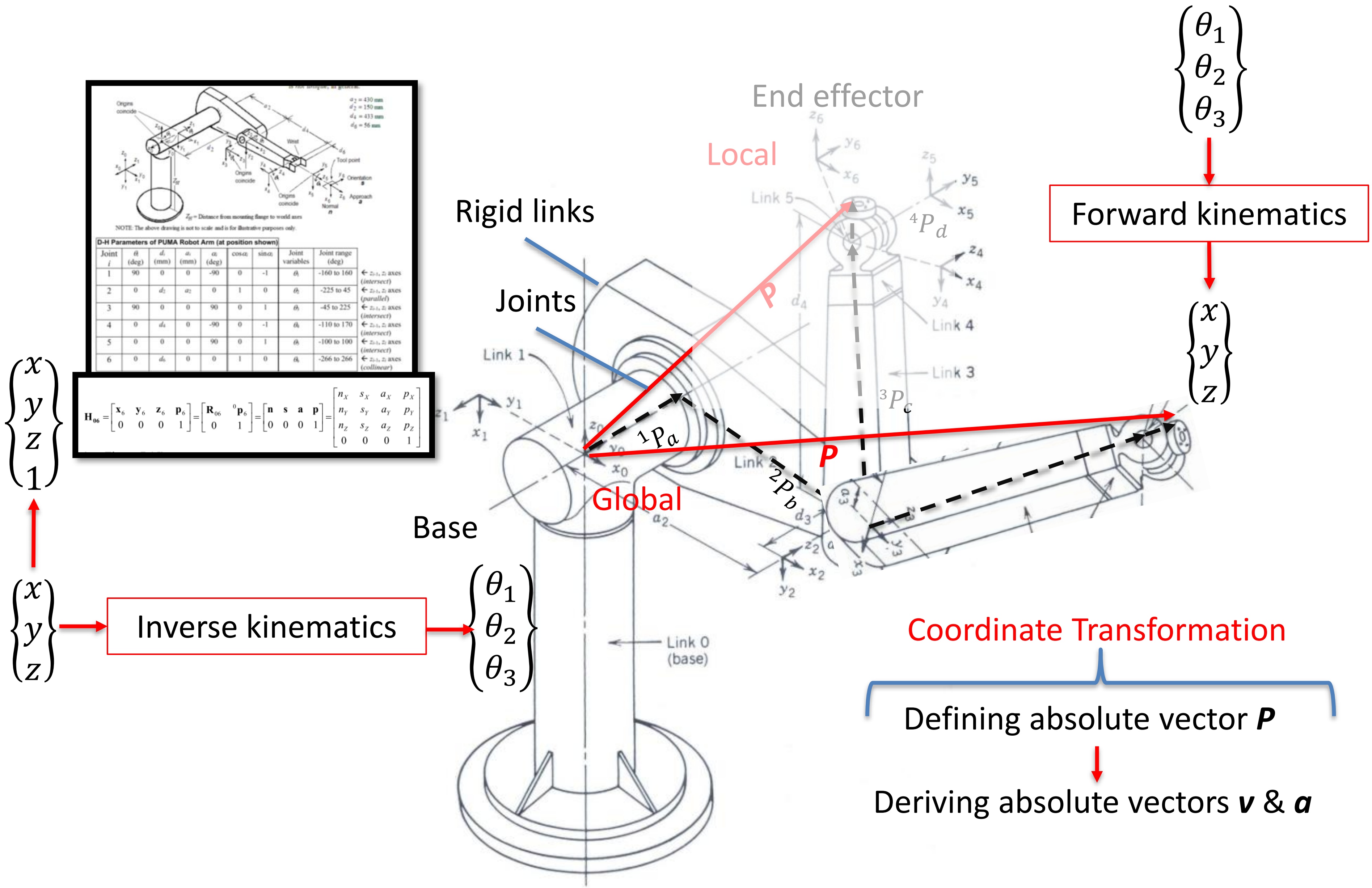
- **Path planning & Trajectory Generation**:
  - Path planning: enable movement of robot to goal position while avoiding collisions with objects in its workspace
    - Path encode position & orientation information without time considerations.
  - Trajectory planning: generate reference trajectories that determine the time history of the manipulator along a given path



Scenario

- **Dynamics**
- **Position control**
- **Force control**
- **Vision**
- **Vision-based control**





End effector
Local
Forward kinematics
Rigid links
Joints
Link 1
Link 2
Link 3
Link 4
Link 5
Link 0 (base)
Global
Base
P
Inverse kinematics
Coordinate Transformation
Defining absolute vector P
Deriving absolute vectors v & a
D-H Parameters of PUMA Robot Arm (at position shown)

Course Outline

- Rigid body motion
- Coordinate transformation
- General motion: Forward & Inverse Kinematics
- Homogeneous Transformation & Matrix
- D-H method (parameters & matrix)
- Path planning & Trajectory

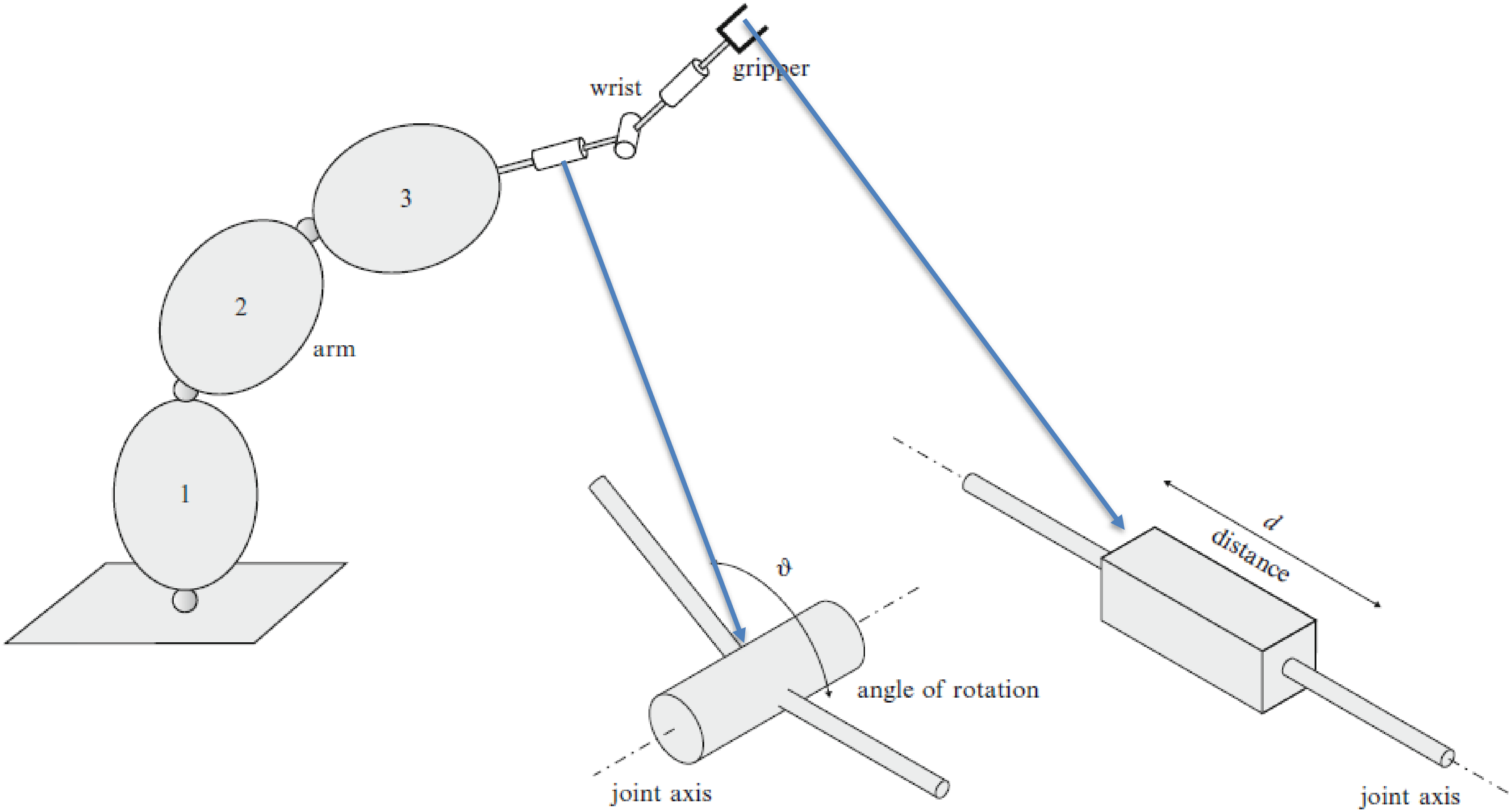
Definitions & Preliminaries

- **Rigid Body**:
  - Solid body with negligible deformation.
  - Distance between 2 points remains constant.
  - Rigid Body Motion: 6 dofs – 3 translations & 3 rotations.
    - 3 translations: Position of the body
    - 3 rotations: Orientation of the body

- **Kinematics**:
  - Describes motion of points, bodies and systems of bodies without considering the forces that cause them to move.
    - Trajectories & velocities can be measured by joint sensors described through joint variables
    - Position and orientation of end-effectors is described through external coordinates

- **Objective**:
  - Develop a systematic modelling to describe geometric relationship between rigid bodies & the kinematics of a point of concern in a rigid body using coordinate frames & transformation.
- **Overarching assumptions** (…*rigid bodies*):
  - Rigid links/parts
  - Rotational & translational motion

- **Conceptualization** (…*point of concern*):
  - End gripper/link described thru position vector ***P***
  - Derivation of velocity ***V*** & acceleration ***A***
- **Tools** (…*coordinate frames & transformation*):
  - Defining appropriate global and local coordinate systems
  - Leveraging transformation of coordinate systems to attain full modelling



- **Open kinematic chain**

Mathematical background

Scalars, Vectors and Matrices

A *scalar* quantity is expressible as a single, real number.

A quantity having direction as well as magnitude is called a *vector*. In addition, vectors must have certain transformation properties. For example, vector magnitudes are unchanged after a rotation of axes.

A *matrix* quantity is expressible as a two-dimensional array of numbers.

Scalar	*Vector*	*Matrix*
M---	F----	I------
E-----	M-----	*xxx*------
T----------	V-------	
T---	A-----------	

Scalar (or dot) product

Given $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}, \quad \boldsymbol{b} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}$

$$
\begin{aligned}
\boldsymbol{a}\cdot\boldsymbol{b} &= a_x b_x + a_y b_y + a_z b_z \\
&\quad + (\,\ldots.)\,\boldsymbol{i}\cdot\boldsymbol{j} + (\,\ldots.)\,\boldsymbol{i}\cdot\boldsymbol{k} + (\,\ldots.)\,\boldsymbol{j}\cdot\boldsymbol{k} \\
&= a_x b_x + a_y b_y + a_z b_z
\end{aligned}
$$

Vector (or cross) product

$$
\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \quad \text{or} \quad
\begin{aligned}
&(\,a_y b_z - a_z b_y)\,\boldsymbol{i} \\
&+ (\,a_z b_x - a_x b_z)\,\boldsymbol{j} \\
&+ (\,a_x b_y - a_y b_x)\,\boldsymbol{k}
\end{aligned}
$$

Matrix

A matrix is a set of numbers which are arranged in rows and columns.

A *rectangular* matrix of order $m \times n$ has m rows and n columns and is written in the form

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & - & - & A_{1n} \\ A_{21} & A_{22} & - & - & A_{2n} \\ \vdots & \vdots & - & - & \vdots \\ \vdots & \vdots & - & - & \vdots \\ A_{m1} & A_{m2} & - & - & A_{mn} \end{bmatrix}$$

diagonal matrix:

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix}; \quad \begin{array}{c} D_{ij} = 0, \text{if } \ i \neq j; \\ i, j = 1 \ \text{ to } \ 3 \end{array}$$

unit (or identity) matrix:

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

symmetric matrix: a square matrix for which

$$\boldsymbol{A} = \boldsymbol{A}^T \quad (\text{or } A_{ij} = A_{ji})$$

skew-symmetric matrix (anti-symmetric): a square matrix for $A_{ij} = - A_{ji}$. This requires that the elements on the principal diagonal be zero.

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 0 & -A_{21} & -A_{31} \\ A_{21} & 0 & -A_{32} \\ A_{31} & A_{32} & 0 \end{bmatrix}$$

Inverse of matrix

$$\mathbf{A}^{-1} = \frac{\text{adjoint matrix of } \mathbf{A}}{|\mathbf{A}|}$$

Note that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad (\text{Identity Matrix})$$

addition: $C_{ij} = A_{ij} + B_{ij}$ (or $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$)

subtraction: $C_{ij} = A_{ij} - B_{ij}$

multiplication: $\underset{m \times q}{C_{ij}} = \sum_{k=1}^{n} \underset{m \times \boldsymbol{n}}{A_{i\boldsymbol{k}}} \; \underset{\boldsymbol{n} \times q}{B_{\boldsymbol{k}j}}$

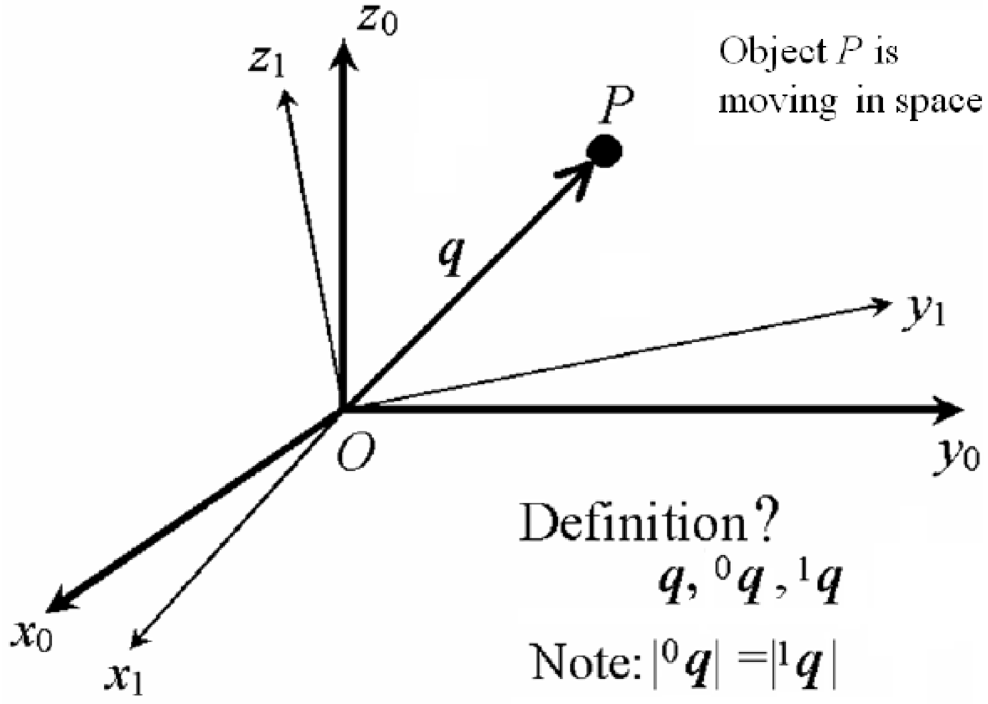
The multiplication is defined only when the two matrices are *conformable*, i.e. the number of columns in the first matrix is equal to the number of rows in the second, i.e. $\boldsymbol{k}=\boldsymbol{k}$.

$\boldsymbol{C} = \boldsymbol{AB}$; $\boldsymbol{B}$ is *pre*-multiplied by $\boldsymbol{A}$, or $\boldsymbol{A}$ is *post*-multiplied by $\boldsymbol{B}$.

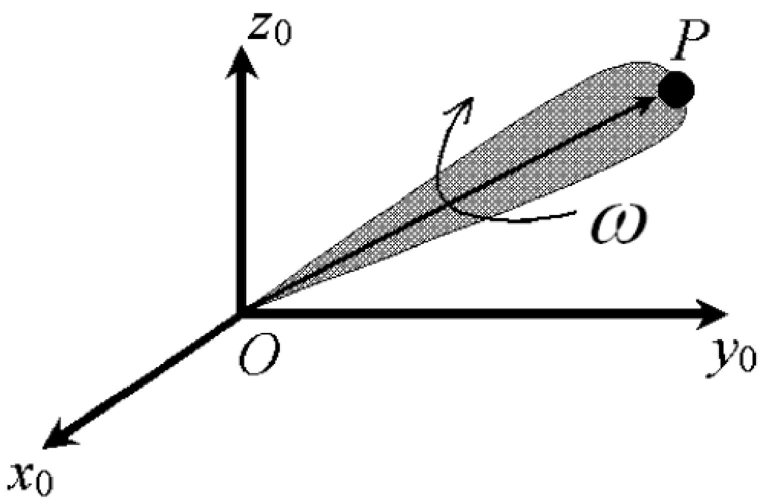
Note: $\boldsymbol{AB} \neq \boldsymbol{BA}$ [*row*1 x *col*1] ≠ [*row*2 x *col*2]

Pure Rotation & Coordinate Systems with Shared Origin

Pure Rotation: an object in two Cartesian coordinate systems *with a common origin*

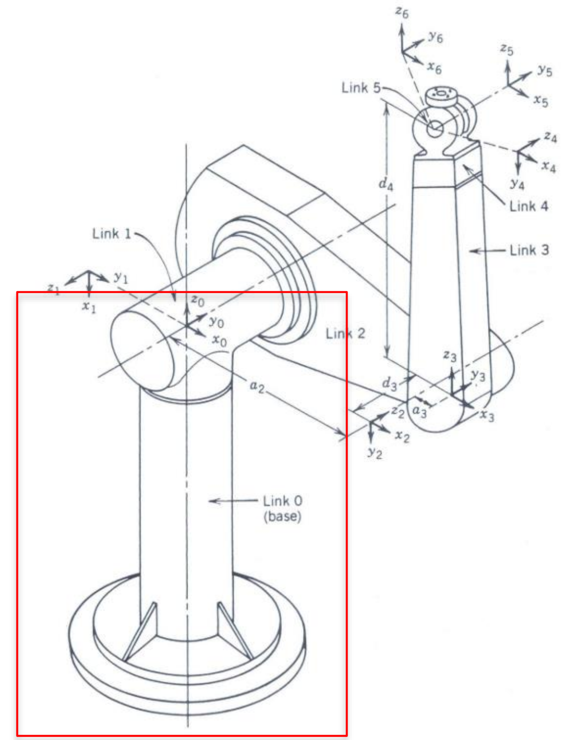


A Single-Link System

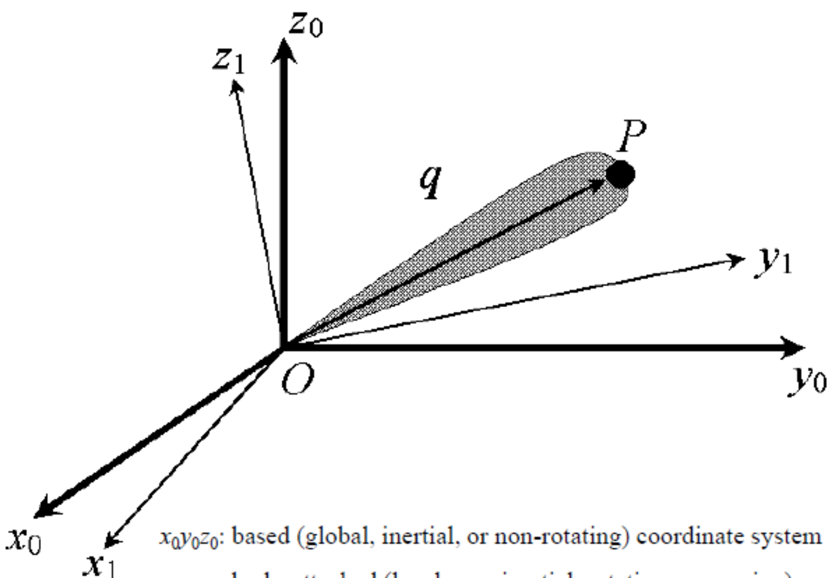


Rigid link OP rotates at ω about point O

$x_0y_0z_0$: Inertial coordinate system

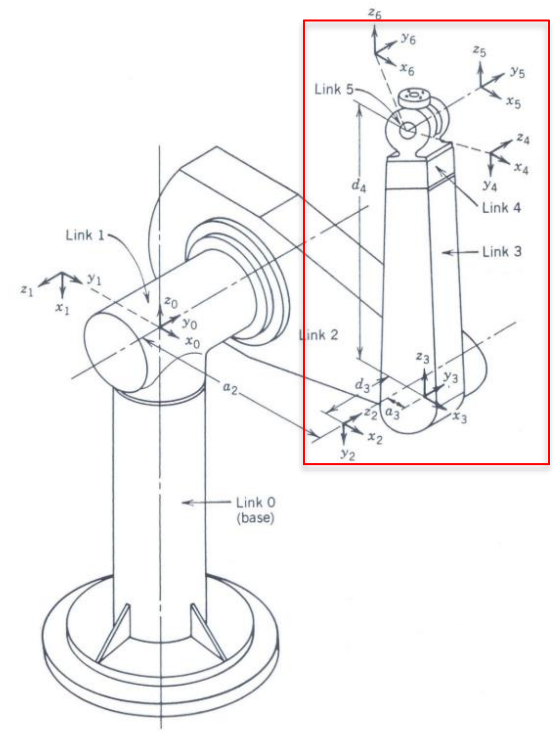


A Single-Link System



$x_0y_0z_0$: based (global, inertial, or non-rotating) coordinate system

$x_1y_1z_1$: body-attached (local, non-inertial, rotating, or moving) coordinate system

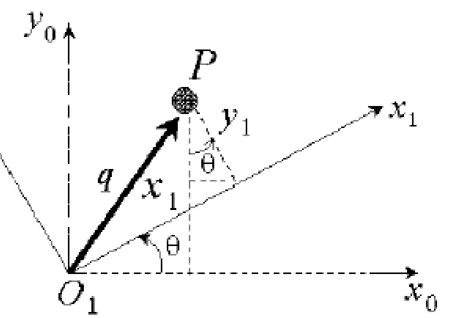


Coordinate Transformation via Rotation Matrix

Pure Rotation

Two-dimensional problem

O_1P rotates about O_1



$$x_0 = x_1 \cos\theta - y_1 \sin\theta \quad \text{——— (1)}$$
$$y_0 = x_1 \sin\theta + y_1 \cos\theta \quad \text{——— (2)}$$

or expressed in vector-matrix or matrix form

$$\begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} \quad \begin{matrix} \leftarrow \text{Eq. (1)} \\ \leftarrow \text{Eq. (2)} \end{matrix}$$

where $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{2\text{x}2} \Leftarrow$ rotation matrix, $\mathbf{R}_{01}$

Representation in 3D problem (same origin)

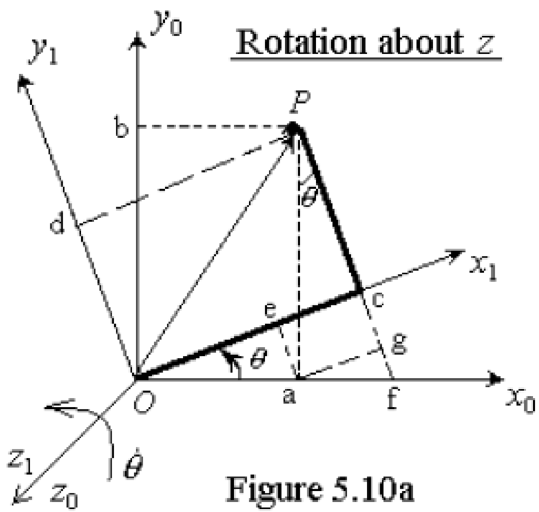


Figure 5.10a

$$x_0 = x_1 \cos\theta - y_1 \sin\theta$$
$$y_0 = x_1 \sin\theta + y_1 \cos\theta$$
$$z_0 = z_1$$

in matrix form

$$\begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}$$

or

$$x_1 = x_0 \cos\theta + y_0 \sin\theta$$
$$y_1 = -x_0 \sin\theta + y_0 \cos\theta$$
$$z_1 = z_0$$

in matrix form

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix}$$

Transformation of Co-ordinates (Chapter 5)

The relations between the components of a vector $\boldsymbol{q}$ (OP in the figure) in various coordinate systems are very useful to many problems of dynamics, especially those in robotics.

Express the vector $\boldsymbol{q}$ in terms of components along two Cartesian sets of axes $\{x_0, y_0, z_0\}$ and $\{x_1, y_1, z_1\}$.

The unit vectors along these axes are denoted by

$\{\boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0\}$ and $\{\boldsymbol{i}_1, \boldsymbol{j}_1, \boldsymbol{k}_1\}$, respectively.

Expression of $\boldsymbol{q}$ in the respective frame

$$\mathbf{q} \Rightarrow \quad {}^0\mathbf{q} = x_0\mathbf{i}_0 + y_0\mathbf{j}_0 + z_0\mathbf{k}_0$$

$$\text{or} \quad {}^1\mathbf{q} = x_1\mathbf{i}_1 + y_1\mathbf{j}_1 + z_1\mathbf{k}_1$$

Note: $\sqrt{x_0^2 + y_0^2 + z_0^2} = \sqrt{x_1^2 + y_1^2 + z_1^2}$ i.e. constant length OP

Rotation matrix

relation between x_0, x_1; y_0, y_1; and z_0, z_1.

$$\begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}$$

$\boldsymbol{R}_{01}$

or $\;{}^0\mathbf{q} = \mathbf{R}_{01}\,{}^1\mathbf{q}$

$3\times1 \quad 3\times3 \quad 3\times1$

${}^0\mathbf{q} = x_0\mathbf{i}_0 + y_0\mathbf{j}_0 + z_0\mathbf{k}_0$

${}^1\mathbf{q} = x_1\mathbf{i}_1 + y_1\mathbf{j}_1 + z_1\mathbf{k}_1$

$\boldsymbol{R}_{01}$: transformation (or rotation) matrix from

System 1 $\{x_1, y_1, z_1\}$ to System 0 $\{x_0, y_0, z_0\}$

Find ${}^1\boldsymbol{q}$, given ${}^0\boldsymbol{q}$ $\qquad {}^1\mathbf{q} = \mathbf{R}_{10}\,{}^0\mathbf{q} = \mathbf{R}_{01}^{-1}\;{}^0\mathbf{q} = \mathbf{R}_{01}^{T}\;{}^0\mathbf{q}$

Orthogonal property: $\qquad \Rightarrow \mathbf{R}^{-1} = \mathbf{R}^{T}$

Coordinate transformation of *vectors* by making use of $\boldsymbol{R}$

$${}^1\mathbf{q} = \mathbf{R}_{10}\,{}^0\mathbf{q} \quad \text{or} \quad {}^0\mathbf{q} = \mathbf{R}_{10}^{T}\;{}^1\mathbf{q} \quad \text{since} \quad \mathbf{R}_{10}^{-1} = \mathbf{R}_{10}^{T}$$

where

${}^1\mathbf{q}$ $\quad 3\times1$ vector $\mathbf{q}$ expressed in coordinate $\{x_1 y_1 z_1\}$

${}^0\mathbf{q}$ $\quad 3\times1$ vector $\mathbf{q}$ expressed in coordinate $\{x_0 y_0 z_0\}$

Recall ….

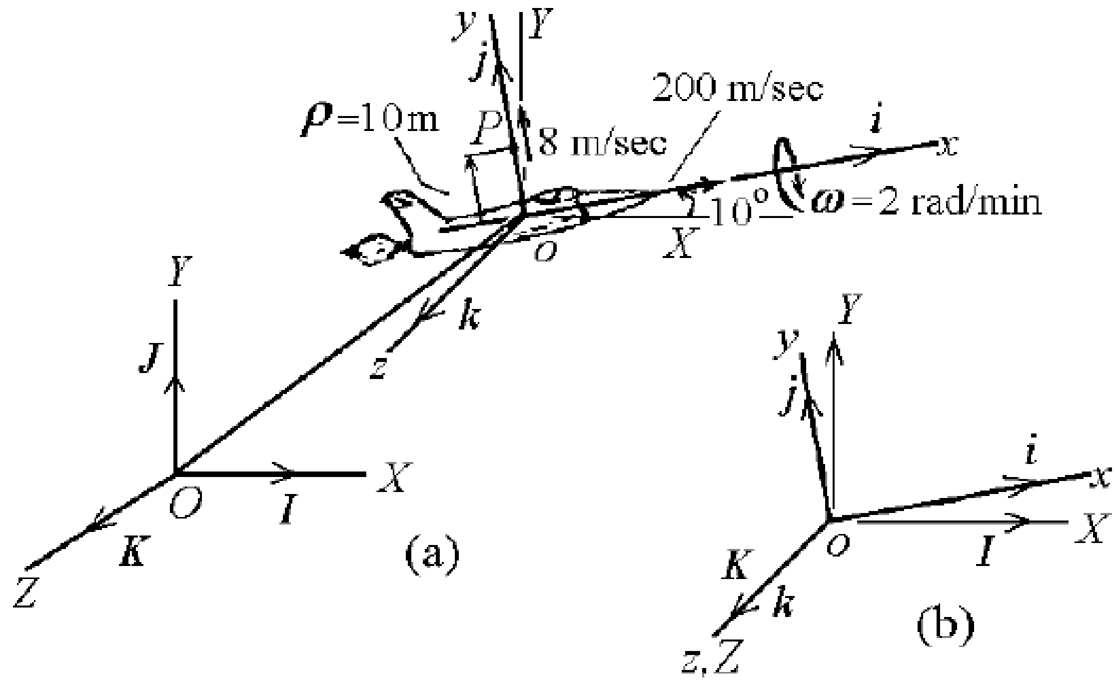
$$\begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}$$

Direction cosine

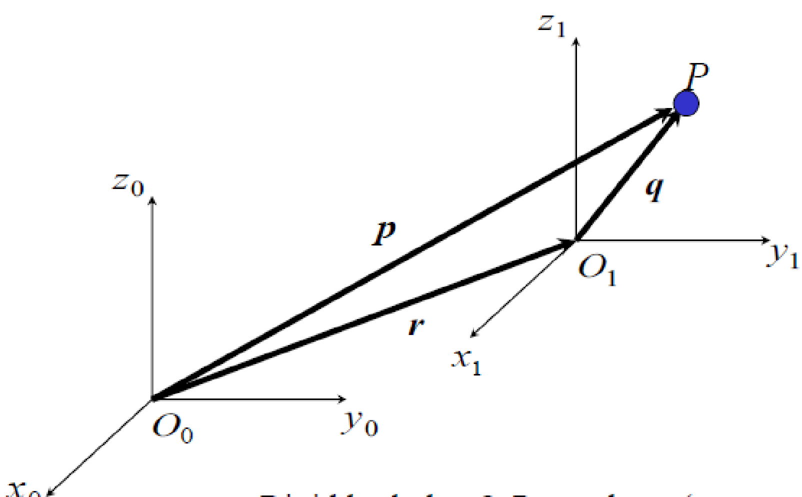
or $$\underset{3\times 1}{{}^{0}\mathbf{q}} = \underset{3\times 3}{\mathbf{R}_{01}} \; \underset{3\times 1}{{}^{1}\mathbf{q}}$$

Rotation matrix

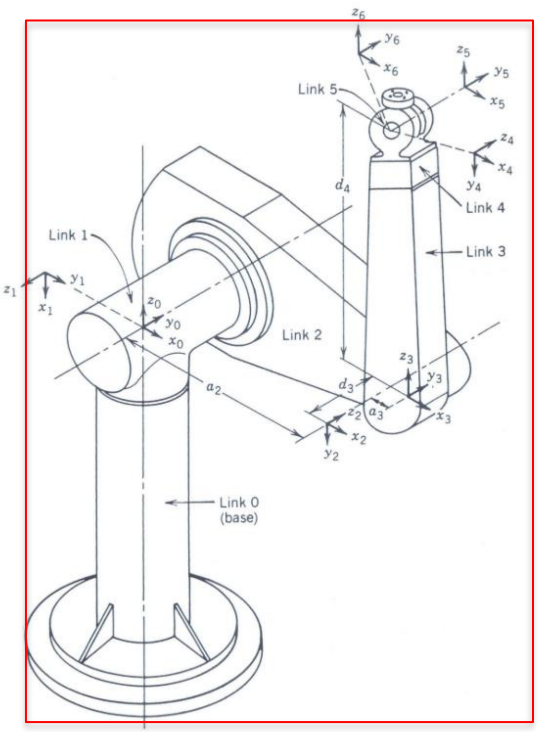
General Motion (Rotation & Translation) & Coordinate Systems with Different Origins



Coordinate transformation:
pure translation of system* 1 *(w.r.t. system* 0*)



Rigid body by O_1P translates (no rotation) from O_0 to O_1 along $\boldsymbol{r}$



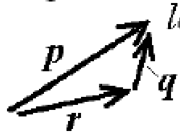
Coordinate transformation: combined translation and rotation

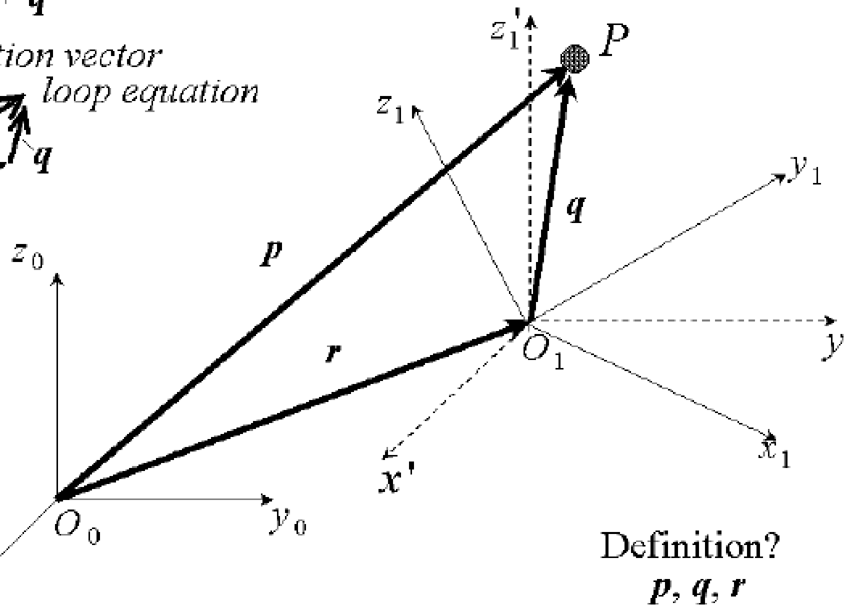
First, rigid body O_1P translates (no rotation) from O_0 to O_1. Next, O_1P rotates about O_1.

Position of point P:

$\boldsymbol{p} = \boldsymbol{r} + \boldsymbol{q}$

position vector loop equation

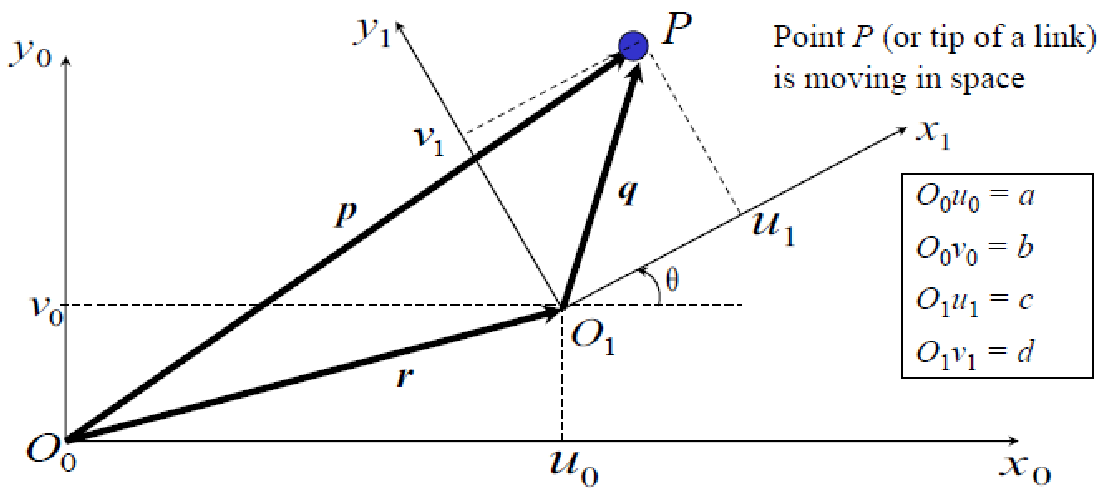




Definition?

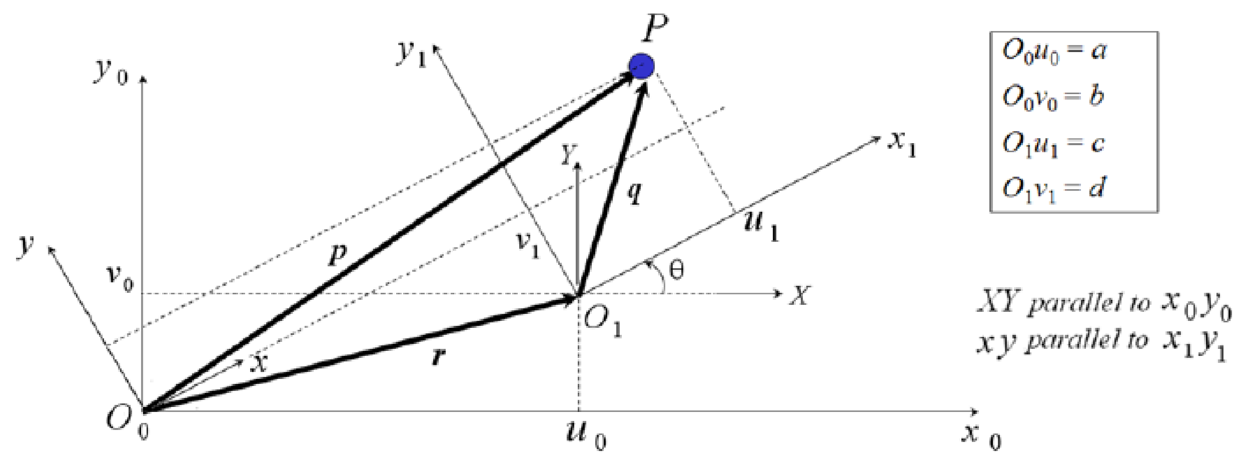
$\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$

***Explanation*: Position Vector Loop**



Position vector of point P (vector loop equation):

$\boldsymbol{p} = \boldsymbol{r} + \boldsymbol{q}$ (Generic; independent of coordinate systems)



${}^0\boldsymbol{p}$: position vector of Point P expressed in terms of system 0

${}^1\boldsymbol{p}$: position vector of Point P expressed in terms of system 1 (system xy)

Note: ${}^1\boldsymbol{p}$ is the position vector of point P with respect to the fixed point O_0, in which the vector is expressed in terms of coordinate system xy (parallel to x_1y_1 in all times)

Translation and Rotation about axis z

$\boldsymbol{p} = \boldsymbol{r} + \boldsymbol{q}$ (generic)

${}^0\boldsymbol{p} = {}^0\boldsymbol{r} + {}^0\boldsymbol{q} = {}^0\boldsymbol{r} + \boldsymbol{R}_{01}\,{}^1\boldsymbol{q}$

Since ${}^0\boldsymbol{p} = p_{x0}\,\hat{\imath}_0 + p_{y0}\,\hat{\jmath}_0$,

$$p_{x0} = a + c\cos\theta - d\sin\theta$$
$$p_{y_0} = b + c\sin\theta + d\cos\theta$$

or

$$\begin{Bmatrix} p_{x0} \\ p_{y_0} \end{Bmatrix} = \begin{Bmatrix} a \\ b \end{Bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} c \\ d \end{Bmatrix}$$

where

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Leftarrow \text{rotation matrix, } \mathbf{R}_{01}$$

Definition:

$\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, {}^0\boldsymbol{p}, {}^1\boldsymbol{p}, {}^0\boldsymbol{q}$, etc.?

How about

${}^1\boldsymbol{p} = {}^1\boldsymbol{r} + {}^1\boldsymbol{q}$?

For example:

${}^0\boldsymbol{p}$: position vector of Point P expressed in terms of system 0

${}^1\boldsymbol{p}$: position vector of Point P expressed in terms of (virtual) system 1

Why expressed in system 1?

Transport Theorem

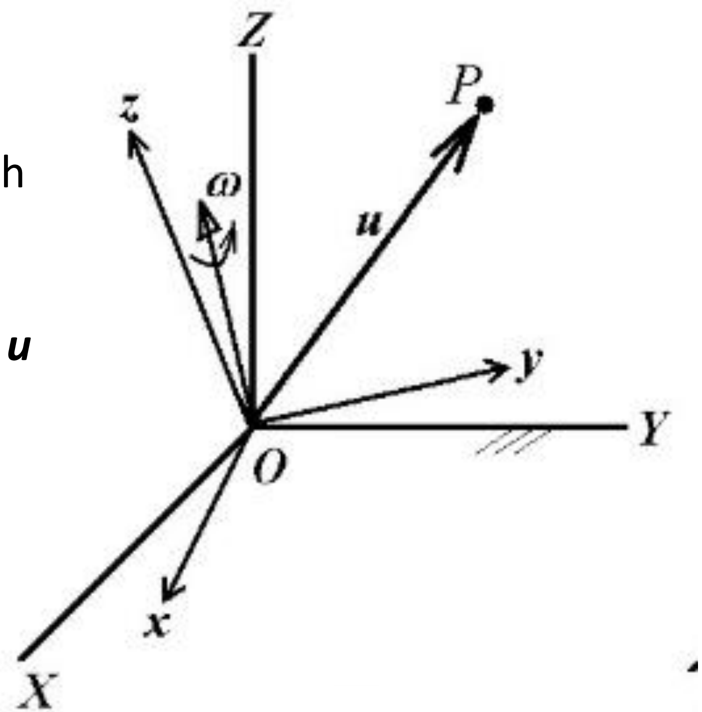
- Formal definition for <u>rate of change of vector</u> expressed in <u>observed reference frame</u> to the <u>desired reference frame</u>.

$$^{A}\left[\frac{d\boldsymbol{u}}{dt}\right] = {}^{B}\left[\frac{d\boldsymbol{u}}{dt}\right] + {}^{A}\boldsymbol{\omega}_{B} \times \boldsymbol{u}$$

- For vector defined in frame B, the rate of change of said vector defined in frame A is equivalent to summation of rate of change of the vector defined in frame B and the vector’s angular velocity

- **Point in rotating frame**

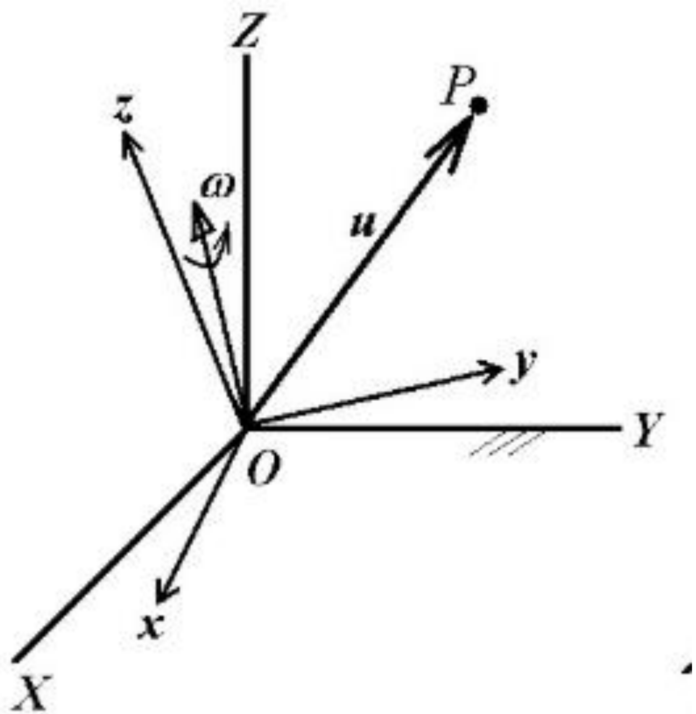
1. Global Frame XYZ
2. Local Frame xyz rotating with absolute angular velocity $\boldsymbol{\omega}$
3. Both frames share origin *O*
4. Point *P* with position vector $\boldsymbol{u}$
5. Point *P* is stationary



- Define absolute position vector $\boldsymbol{u}$

$$\boldsymbol{u} = u_x\,\boldsymbol{i} + u_y\,\boldsymbol{j} + u_z\,\boldsymbol{k}$$

1. Global Frame XYZ
2. Local Frame xyz rotating with $\boldsymbol{\omega}$
3. Both frames share origin *O*
4. Point *P* defined $\boldsymbol{u}$
5. Point *P* is stationary



- Define absolute velocity vector $\dot{\boldsymbol{u}}$

$$\dot{\boldsymbol{u}} = (\dot{u}_x \boldsymbol{i} + \dot{u}_y \boldsymbol{j} + \dot{u}_z \boldsymbol{k}) + (u_x \dot{\boldsymbol{i}} + u_y \dot{\boldsymbol{j}} + u_z \dot{\boldsymbol{k}})$$

$$\dot{\boldsymbol{u}} = (\dot{u}_x \boldsymbol{i} + \dot{u}_y \boldsymbol{j} + \dot{u}_z \boldsymbol{k}) + (u_x \dot{\boldsymbol{i}} + u_y \dot{\boldsymbol{j}} + u_z \dot{\boldsymbol{k}})$$

$$\left(\frac{d\boldsymbol{u}}{dt}\right)_{\text{rel}} = (\dot{u}_x \boldsymbol{i} + \dot{u}_y \boldsymbol{j} + \dot{u}_z \boldsymbol{k}) = (\dot{\boldsymbol{u}})_r$$

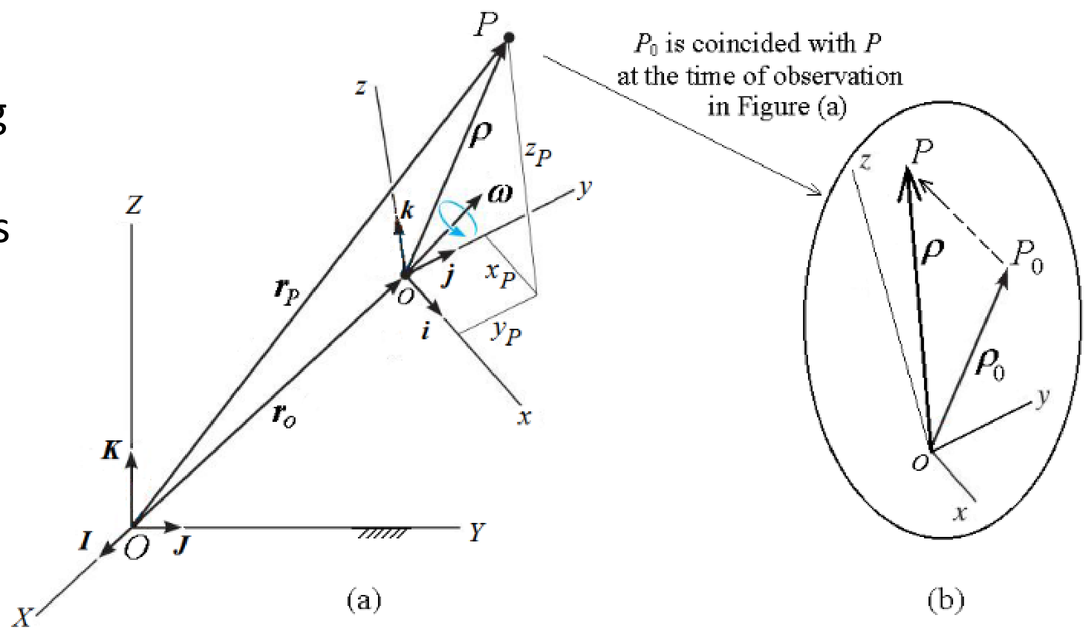
$$\dot{\boldsymbol{i}} = \boldsymbol{\omega} \times \boldsymbol{i}, \quad \dot{\boldsymbol{j}} = \boldsymbol{\omega} \times \boldsymbol{j}, \quad \dot{\boldsymbol{k}} = \boldsymbol{\omega} \times \boldsymbol{k}, \quad \boxed{\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}}$$

$$\begin{aligned}(u_x \dot{\boldsymbol{i}} + u_y \dot{\boldsymbol{j}} + u_z \dot{\boldsymbol{k}}) &= u_x(\boldsymbol{\omega} \times \boldsymbol{i}) + u_y(\boldsymbol{\omega} \times \boldsymbol{j}) + u_z(\boldsymbol{\omega} \times \boldsymbol{k}) \\ &= \boldsymbol{\omega} \times (u_x \boldsymbol{i} + u_y \boldsymbol{j} + u_z \boldsymbol{k}) = \boldsymbol{\omega} \times \boldsymbol{u}\end{aligned}$$

$$\dot{\boldsymbol{u}} = \frac{d\boldsymbol{u}}{dt} = \left(\frac{d\boldsymbol{u}}{dt}\right)_{\text{rel}} + \boldsymbol{\omega} \times \boldsymbol{u} = (\dot{\boldsymbol{u}})_r + \boldsymbol{\omega} \times \boldsymbol{u}$$

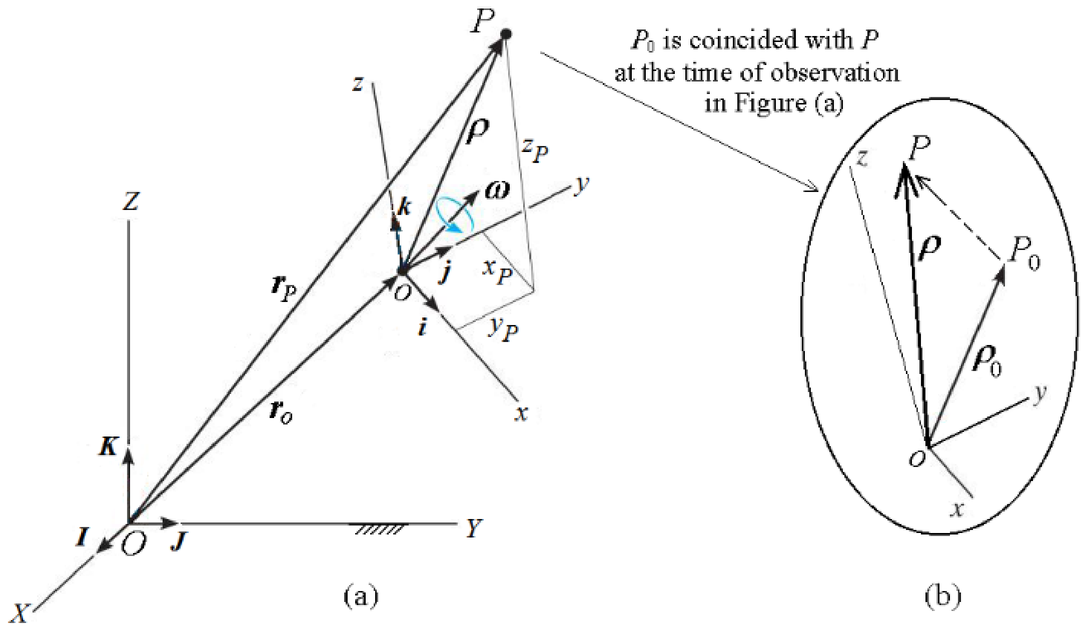
Frame in general motion

1. Global Frame XYZ
2. Local Frame xyz rotating with $\boldsymbol{\omega}$
3. Origin of local frame *o* is offset from *O* by $\boldsymbol{r}_o$
4. Point *P* defined $\boldsymbol{\rho}$



- Define absolute position vector $\boldsymbol{r}_p$

$$\boldsymbol{r}_P = \boldsymbol{r}_o + \boldsymbol{\rho} \qquad \boldsymbol{\rho} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$



(a) (b)

- Define general absolute velocity vector ***v***

$$\boldsymbol{v} = \dot{\boldsymbol{r}}_P = \dot{\boldsymbol{r}}_O + \dot{\boldsymbol{\rho}}$$

$$\boldsymbol{v} = \dot{\boldsymbol{r}}_P = \dot{\boldsymbol{r}}_o + \dot{\boldsymbol{\rho}} \qquad \boldsymbol{\rho} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\dot{\boldsymbol{\rho}} = (\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}) + (x\dot{\boldsymbol{i}} + y\dot{\boldsymbol{j}} + z\dot{\boldsymbol{k}})$$

$$\dot{\boldsymbol{\rho}} = (\dot{\boldsymbol{\rho}})_r + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

$$\boldsymbol{v} = \dot{\boldsymbol{r}}_P = \dot{\boldsymbol{r}}_o + \boldsymbol{\omega} \times \boldsymbol{\rho} + (\dot{\boldsymbol{\rho}})_r$$

$\dot{\boldsymbol{r}}_P$ = velocity (absolute) of point P relative to fixed point O (origin of XYZ-system; inertial system)

$\dot{\boldsymbol{r}}_o$ = velocity (absolute) of point o (origin of xyz-system) relative to fixed point O (origin of XYZ-system; inertial system)

$\boldsymbol{\omega}$ = angular velocity (absolute) of xyz-system relative to fixed point O (origin of XYZ-system; inertial system)

$(\dot{\boldsymbol{\rho}})_r$ = velocity of point P relative to point o (origin of xyz-system)

- Define general absolute acceleration vector ***a***

$$\boldsymbol{a} = \frac{d}{dt}(\dot{\boldsymbol{r}}_P) = \frac{d}{dt}[\dot{\boldsymbol{r}}_O + \boldsymbol{\omega} \times \boldsymbol{\rho} + (\dot{\boldsymbol{\rho}})_r]$$

$$\frac{d}{dt}[(\dot{\boldsymbol{\rho}})_r] = (\ddot{\boldsymbol{\rho}})_r + \boldsymbol{\omega} \times (\dot{\boldsymbol{\rho}})_r$$

$$\dot{\boldsymbol{\rho}} = (\dot{\boldsymbol{\rho}})_r + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

$$\frac{d}{dt}(\dot{\boldsymbol{r}}_O) = \ddot{\boldsymbol{r}}_O$$

$$\frac{d}{dt}(\boldsymbol{\omega} \times \boldsymbol{\rho}) = \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}$$

$$\boldsymbol{a} = \ddot{\boldsymbol{r}}_P = \ddot{\boldsymbol{r}}_O + \underbrace{\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}}_{\text{Tangential acceleration}} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})}_{\text{Centripetal acceleration}} + (\ddot{\boldsymbol{\rho}})_r + \underbrace{2\boldsymbol{\omega} \times (\dot{\boldsymbol{\rho}})_r}_{\text{Coriolis}}$$

$$\boldsymbol{a} = \ddot{\boldsymbol{r}}_P = \ddot{\boldsymbol{r}}_o + \overbrace{\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}}^{\text{Tangential acceleration}} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})}_{\text{Centripetal acceleration}} + (\ddot{\boldsymbol{\rho}})_r + \overbrace{2\boldsymbol{\omega} \times (\dot{\boldsymbol{\rho}})_r}^{\text{Coriolis}}$$

Note that

$\ddot{\boldsymbol{r}}_P$ = acceleration (absolute) of point P relative to fixed point O (origin of XYZ-system; inertial system)

in which

$\ddot{\boldsymbol{r}}_o$ = acceleration (absolute) of point o (origin of xyz-system) relative to fixed point O (origin of XYZ-system; inertial system)

$\dot{\boldsymbol{\omega}}$ (or $\boldsymbol{\alpha}$) = angular acceleration (absolute) of xyz-system relative to fixed point O (origin of XYZ-system; inertial system)

$(\ddot{\boldsymbol{\rho}})_r$ = acceleration of point P relative to point o (origin of xyz-system)

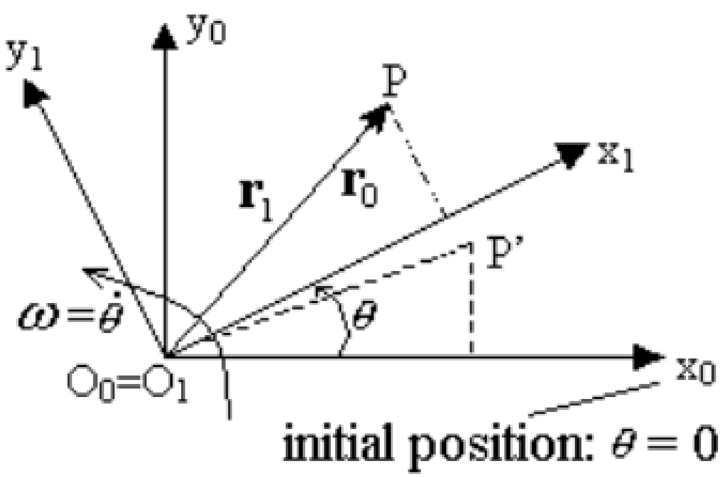
Note that $\boldsymbol{a} = \ddot{\boldsymbol{r}}_o + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$

if $(\dot{\boldsymbol{\rho}})_r = (\ddot{\boldsymbol{\rho}})_r = 0$.

Coordinate transform due to rotation

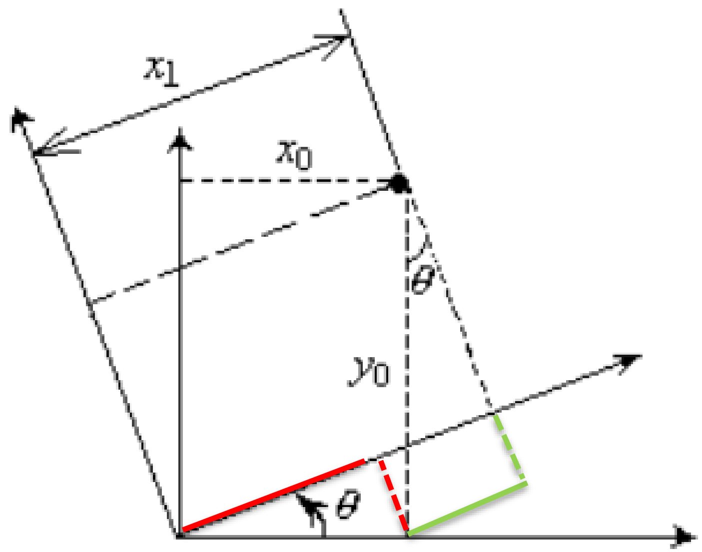
- General motion in rigid body motion consists of translation and rotation.
- Consider and account for rotation.

1. Global Frame $x_o y_o z_o$
2. Local Frame $x_1 y_1 z_1$ rotating with $\boldsymbol{\omega}=\dot{\boldsymbol{\theta}}$ (Right hand rule)
3. Both frames share origin *O*
4. Original point *P'* moves to *P* due to rotation
5. Point *P* defined by $\boldsymbol{r}_1$ and or $\boldsymbol{r}_0$.

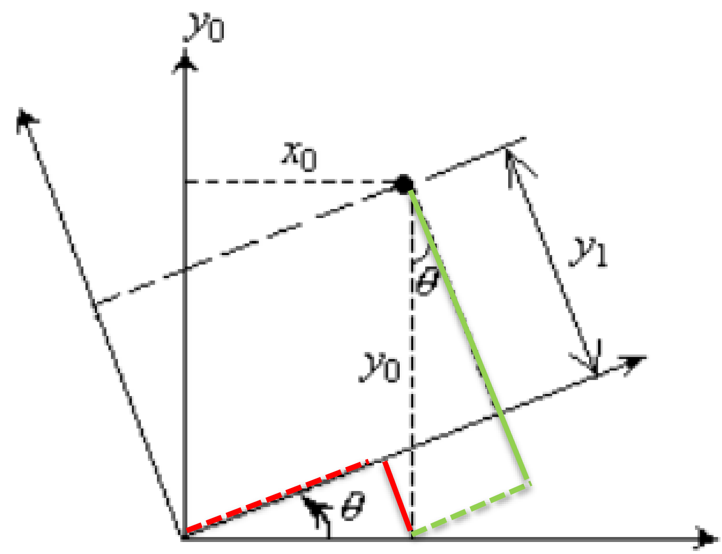


$$r_1 = r_0 \text{ or } x_1\hat{i}_1 + y_1\hat{j}_1 = x_0\hat{i}_0 + y_0\hat{j}_0$$

- **Point in rotating frame**



$$x_1 = x_0 \cos\theta + y_0 \sin\theta$$



$$y_1 = -x_0 \sin\theta + y_0 \cos\theta$$

$$x_1\hat{\boldsymbol{i}}_1 + y_1\hat{\boldsymbol{j}}_1 = (x_0 \cos\theta + y_0 \sin\theta)\hat{\boldsymbol{i}}_1 + (-x_0 \sin\theta + y_0 \cos\theta)\hat{\boldsymbol{j}}_1$$

$$x_1\hat{\boldsymbol{i}}_1 + y_1\hat{\boldsymbol{j}}_1 = (x_0 \cos\theta + y_0 \sin\theta)\hat{\boldsymbol{i}}_1 + (-x_0 \sin\theta + y_0 \cos\theta)\hat{\boldsymbol{j}}_1$$

Rotation matrix

$$\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix}$$

$$\boxed{\begin{Bmatrix} \hat{\boldsymbol{i}}_1 \\ \hat{\boldsymbol{j}}_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{i}}_0 \\ \hat{\boldsymbol{j}}_0 \end{Bmatrix}}$$

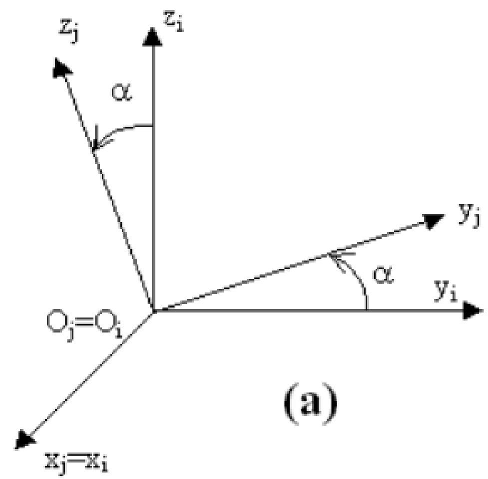
$$\begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}$$

Inverse rotation matrix

$$\boxed{\begin{Bmatrix} \hat{\boldsymbol{i}}_0 \\ \hat{\boldsymbol{j}}_0 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{i}}_1 \\ \hat{\boldsymbol{j}}_1 \end{Bmatrix}}$$

$$\boldsymbol{R}_{ij}(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

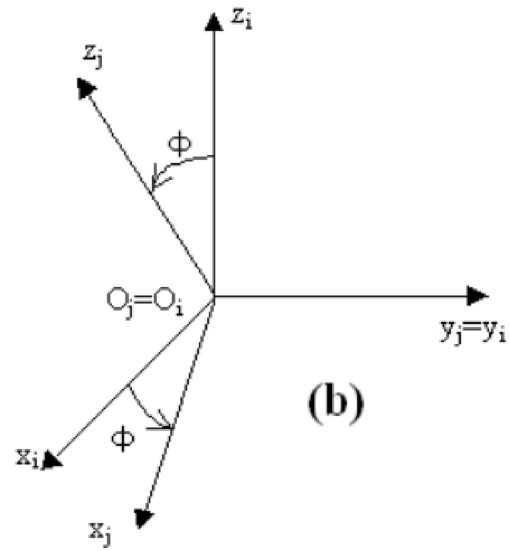
- **Basic Rotation Matrix**
  - about x-axis



(a)

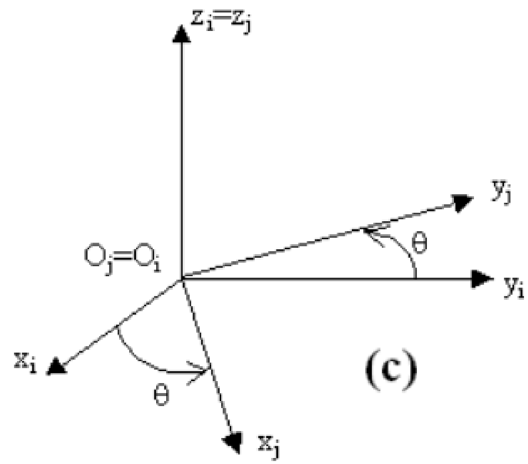
$$\boldsymbol{R}_{ij}(x,\alpha)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix}, \quad i<j$$

- **Basic Rotation Matrix**
  - about y-axis



$$\boldsymbol{R}_{ij}(y,\phi)=\begin{bmatrix}\cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi\end{bmatrix},\quad i<j$$

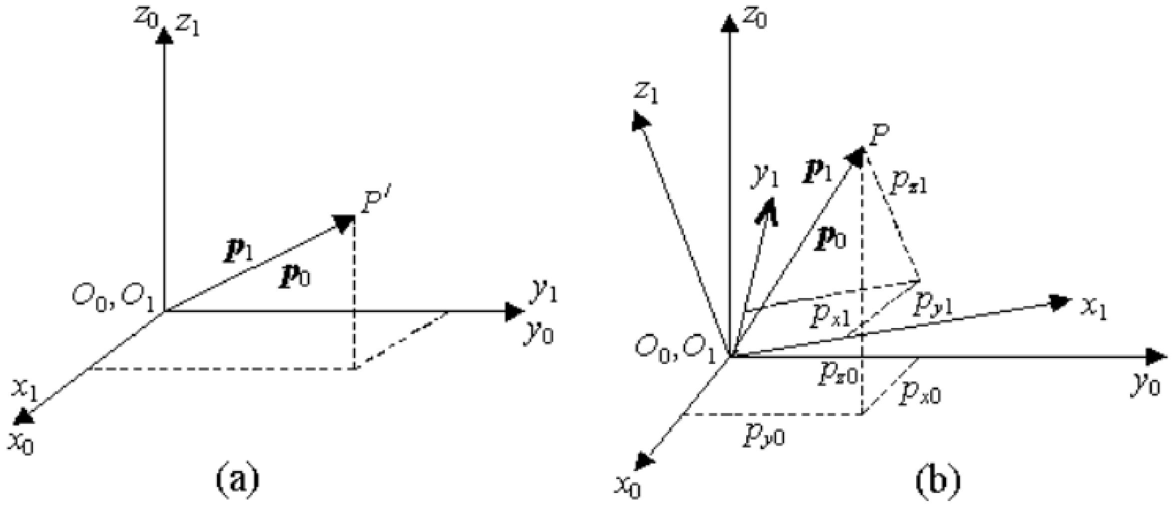
- **Basic Rotation Matrix**
  - about z-axis



$$\boldsymbol{R}_{ij}(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad i < j$$

General rotation

- Generalization of the rotation matrix.
- Introduction of *direction cosines* (dot product)



$$p_0 = p_{x0}\, i_0 + p_{y0}\, j_0 + p_{z0}\, k_0$$

$$p_1 = p_{x1}\, i_1 + p_{y1}\, j_1 + p_{z1}\, k_1$$

$$\boldsymbol{p}_0 = p_{x0}\,\boldsymbol{i}_0 + p_{y0}\,\boldsymbol{j}_0 + p_{z0}\,\boldsymbol{k}_0$$

$$\boldsymbol{p}_1 = p_{x1}\,\boldsymbol{i}_1 + p_{y1}\,\boldsymbol{j}_1 + p_{z1}\,\boldsymbol{k}_1$$

- What is the difference

$$\begin{aligned} p_{x0} &= \boldsymbol{p}_0 \cdot \boldsymbol{i}_0 = \boldsymbol{p}_1 \cdot \boldsymbol{i}_0 \text{ as } \boldsymbol{p}_0 = \boldsymbol{p}_1 \\ &= \left(p_{x1}\,\boldsymbol{i}_1 + p_{y1}\,\boldsymbol{j}_1 + p_{z1}\,\boldsymbol{k}_1\right)\cdot \boldsymbol{i}_0 \\ &= (\boldsymbol{i}_0 \cdot \boldsymbol{i}_1)\,p_{x1} + (\boldsymbol{i}_0 \cdot \boldsymbol{j}_1)\,p_{y1} + (\boldsymbol{i}_0 \cdot \boldsymbol{k}_1)\,p_{z1} \end{aligned}$$

$$\begin{aligned} p_{y0} &= \boldsymbol{p}_0 \cdot \boldsymbol{j}_0 = \boldsymbol{p}_1 \cdot \boldsymbol{j}_0 \\ &= \left(p_{x1}\,\boldsymbol{i}_1 + p_{y1}\,\boldsymbol{j}_1 + p_{z1}\,\boldsymbol{k}_1\right)\cdot \boldsymbol{j}_0 \\ &= (\boldsymbol{j}_0 \cdot \boldsymbol{i}_1)\,p_{x1} + (\boldsymbol{j}_0 \cdot \boldsymbol{j}_1)\,p_{y1} + (\boldsymbol{j}_0 \cdot \boldsymbol{k}_1)\,p_{z1} \end{aligned}$$

$$\begin{aligned} p_{z0} &= \boldsymbol{p}_0 \cdot \boldsymbol{k}_0 = \boldsymbol{p}_1 \cdot \boldsymbol{k}_0 \\ &= \left(p_{x1}\boldsymbol{i}_1 + p_{y1}\boldsymbol{j}_1 + p_{z1}\boldsymbol{k}_1\right)\cdot \boldsymbol{k}_0 \\ &= (\boldsymbol{k}_0 \cdot \boldsymbol{i}_1)\,p_{x1} + (\boldsymbol{k}_0 \cdot \boldsymbol{j}_1)\,p_{y1} + (\boldsymbol{k}_0 \cdot \boldsymbol{k}_1)\,p_{z1} \end{aligned}$$

$$\begin{Bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix} \begin{Bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{Bmatrix}$$

$$\begin{Bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix} \begin{Bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{Bmatrix}$$

Define $$\boldsymbol{R}_{01} = \begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix}$$

$$\boldsymbol{i}_0 \cdot \boldsymbol{i}_1 = \cos\theta_{Xx} \qquad \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 = \cos\theta_{Xy}, \quad \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 = \cos\theta_{Xz}$$

$$\boldsymbol{R}_{01} = \begin{bmatrix} \cos\theta_{Xx} & \cos\theta_{Xy} & \cos\theta_{Xz} \\ \cos\theta_{Yx} & \cos\theta_{Yy} & \cos\theta_{Yz} \\ \cos\theta_{Zx} & \cos\theta_{Zy} & \cos\theta_{Zz} \end{bmatrix} = \begin{bmatrix} R_{Xx} & R_{Xy} & R_{Xz} \\ R_{Yx} & R_{Yy} & R_{Yz} \\ R_{Zx} & R_{Zy} & R_{Zz} \end{bmatrix}$$

$$\boldsymbol{p}_0 = \boldsymbol{R}_{01}\boldsymbol{p}_1$$

- Generalized rotation matrix

$$\boldsymbol{p}_1 = \boldsymbol{R}_{10}\boldsymbol{p}_0?$$

$$\boldsymbol{R}_{10} = \begin{bmatrix} \boldsymbol{i}_1 \cdot \boldsymbol{i}_0 & \boldsymbol{i}_1 \cdot \boldsymbol{j}_0 & \boldsymbol{i}_1 \cdot \boldsymbol{k}_0 \\ \boldsymbol{j}_1 \cdot \boldsymbol{i}_0 & \boldsymbol{j}_1 \cdot \boldsymbol{j}_0 & \boldsymbol{j}_1 \cdot \boldsymbol{k}_0 \\ \boldsymbol{k}_1 \cdot \boldsymbol{i}_0 & \boldsymbol{k}_1 \cdot \boldsymbol{j}_0 & \boldsymbol{k}_1 \cdot \boldsymbol{k}_0 \end{bmatrix} = \begin{bmatrix} R_{xX} & R_{xY} & R_{xZ} \\ R_{yX} & R_{yY} & R_{yZ} \\ R_{zX} & R_{zY} & R_{zZ} \end{bmatrix}$$

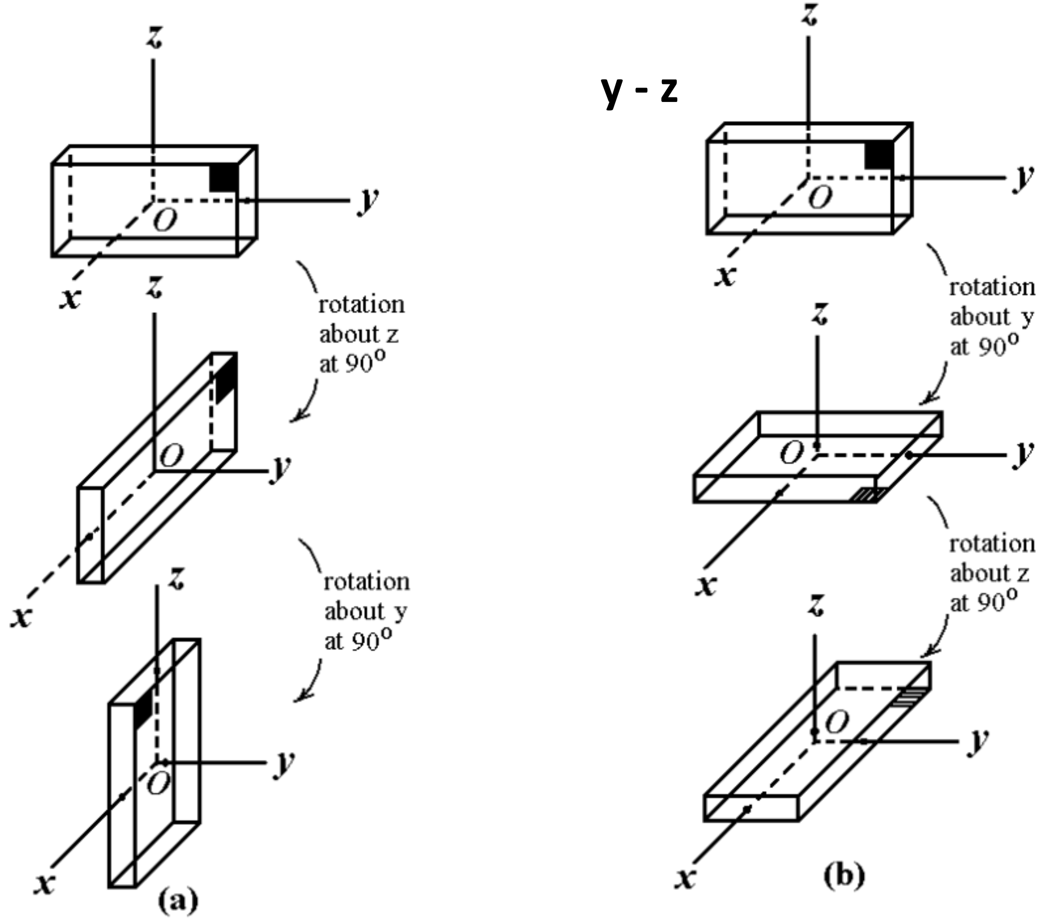
$\boldsymbol{R_{01}}$

$$\boxed{\begin{bmatrix} \boldsymbol{i}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{i}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{j}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{j}_0 \cdot \boldsymbol{k}_1 \\ \boldsymbol{k}_0 \cdot \boldsymbol{i}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{j}_1 & \boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \end{bmatrix}}$$

- Sequence

z - y

y - z



- General rotation : Successive Independent Rotations (i.e. sequences)

x-y-x, x-y-z, x-z-x, x-z-y, y-x-y, y-x-z,
y-z-x, y-z-y, z-x-y, z-x-z, z-y-x, z-y-z.

Group 1

xyz , xzy, yxz, yzx, zxy, zyx

Group 2

xyx , xzx, yxy, yzy, zxz, zyz

Cardan Angles (roll, pitch, yaw)

Euler Angles (precision, nutation, spin)

Homogeneous Coordinates

- **Background:**
  - Purpose: relating 2 coordinate systems with <u>different origins.</u>
  - Homogenous matrix able to define <u>translation & orientation</u>.
  - 3D rigid body motion is represented in $\mathbb{R}^4$

$$\underset{\text{physical } \mathbf{R}^3}{\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}} \Leftrightarrow \underset{\text{homogeneous } \mathbf{R}^4}{\begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}} \text{ Scaling factor}$$

Homogeneous Coordinates

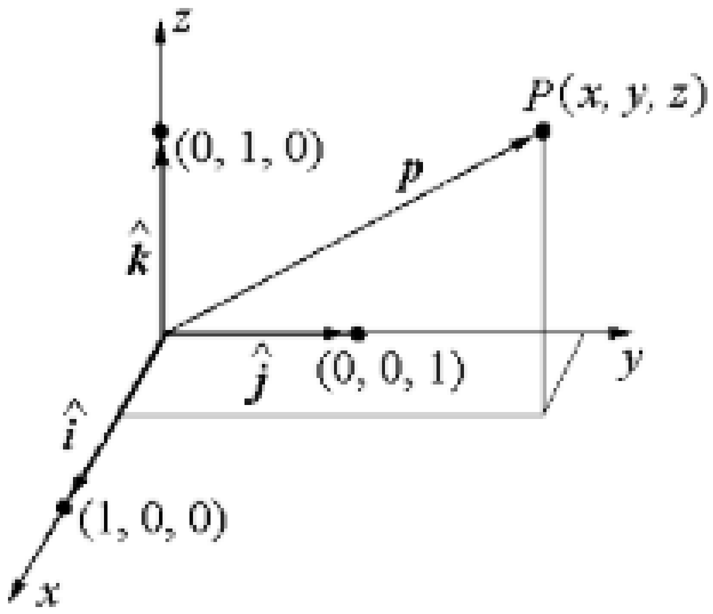
- **Definition:**

Let $\boldsymbol{x} = \{x_1, \ldots x_n\}$ be the Cartesian coordinates of a point in $\boldsymbol{R}^n$. Then any set of (n+1) numbers (y_1, y_2,…, y_n, y_{n+1}) where $y_{n+1} \neq 0$ for which

Scaling factor $$\frac{y_1}{y_{n+1}} = x_1, \frac{y_2}{y_{n+1}} = x_2, \ldots \frac{y_n}{y_{n+1}} = x_n$$

is called a set of homogeneous coordinates for the point x.

- **Vector representation**



$$\boldsymbol{p} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$$ Cartesian coordinates

$$\hat{\boldsymbol{p}} = \{x, y, z, 1\}^T = \{wx, wy, wz, w\}^T = \{\hat{x}, \hat{y}, \hat{z}, w\}^T$$

Homogeneous coordinates

$$\hat{\boldsymbol{p}} = \{x, y, z, 1\}^T = \{wx, wy, wz, w\}^T = \{\hat{x}, \hat{y}, \hat{z}, w\}^T$$

Scaling factor

- **Scaling factor** – a factor that scales the size of the vector which may take any number.
    - *w*<1: coordinates scales up, *w*>1: coordinates scales down
    - w=1: coordinate remains, default for application in robotics
- **Direction** – when *w* is set to 0 then the vector in homogeneous coordinates represents a direction.

$$\hat{\boldsymbol{p}} = \begin{Bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ w \end{Bmatrix} \Rightarrow \quad \hat{\boldsymbol{i}} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \hat{\boldsymbol{j}} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad \hat{\boldsymbol{k}} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

- **Origin of coordinate frame –** the null vector representing a point at the origin of a reference frame is defined as $\{0\ 0\ 0\ w\}^T$ where the scaling factor w can be any number other than zero.

For the general homogeneous coordinate representation, there are a number of unintended/unrequired consequences:

- Non-unique representation of vector:

$$p = xi + yj + zk = 3i + 4j + 5k$$

Homogeneous representations:

$$\hat{p} = \begin{Bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ w \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4 \\ 5 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 8 \\ 10 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 15 \\ 20 \\ 25 \\ 5 \end{Bmatrix} = \begin{Bmatrix} -3 \\ -4 \\ -5 \\ -1 \end{Bmatrix} \Leftarrow \begin{Bmatrix} wx \\ wy \\ wz \\ w \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

- Representation of direction with *w*=0

$$\hat{p} = \begin{Bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ w \end{Bmatrix} \Rightarrow \quad \hat{i} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \hat{j} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad \hat{k} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

Consequently, for application in DH representation, we let *w*=1 permanently

Homogeneous Transformation

- **Homogeneous transformation matrix *H***
  - 4x4 matrix
  - 4 sub-matrices

$$\boldsymbol{H} = \left[\begin{array}{c|c} \boldsymbol{R}_{3\times 3} & \boldsymbol{p}_{3\times 1} \\ \hline \boldsymbol{\beta}_{1\times 3} & s_{1\times 1} \end{array}\right] = \left[\begin{array}{c|c} \text{rotation matrix} & \text{position vector} \\ \hline \text{perspective transformation} & \text{scaling factor} \end{array}\right]$$

- **Principle diagonal**
  - 1st 3 elements *a b c :* Local scaling

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix} = \begin{Bmatrix} ax \\ by \\ cz \\ 1 \end{Bmatrix}$$

  - 4th element *w=s* : Global scaling

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \\ s \end{Bmatrix} \quad \Leftarrow \begin{Bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ w \end{Bmatrix}$$

$$\hat{x} = \frac{x}{s}, \quad \hat{y} = \frac{y}{s}, \quad \hat{z} = \frac{z}{s}, \quad w = \frac{s}{s} = 1$$

1<*s* – global reduction
0<*s*<1 – global enlargement

- **Basic Homogeneous Transformation**

$$\boldsymbol{H}_{\text{rot}}(x,\alpha) = \boldsymbol{H}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{H}_{\text{rot}}(y,\phi) = \boldsymbol{H}_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{H}_{\text{rot}}(z,\theta) = \boldsymbol{H}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Upper left 3x3 submatrix rotates while the upper right 3x1 submatrix translates

$$\boldsymbol{H}_{\text{trans}} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Represents a translated coordinate system *oxyz* with axes parallel to the reference coordinate system *OXYZ* but origin at d_x d_y d_z

- Geometric Interpretation of Homogeneous Transformation Matrices

$$\boldsymbol{H}_{12} = \begin{bmatrix} n_X & s_X & a_X & p_X \\ n_Y & s_Y & a_Y & p_Y \\ n_Z & s_Z & a_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{a} & \boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or

$$\boldsymbol{H}_{12} = \begin{bmatrix} & \boldsymbol{R}_{12} & & {}^{1}\boldsymbol{p}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{H}_{12} = \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{Bmatrix} = \begin{array}{c} \begin{array}{cccc} x_2 & y_2 & z_2 & \\ \downarrow & \downarrow & \downarrow & \end{array} \\ \begin{bmatrix} n_X & s_X & a_X & p_X \\ n_Y & s_Y & a_Y & p_Y \\ n_Z & s_Z & a_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{Bmatrix} \begin{array}{l} \longleftarrow x_1 \\ \longleftarrow y_1 \\ \longleftarrow z_1 \\ \end{array}$$

- Each of the 3 column vectors ***n***, ***s*** & ***a*** represent the direction vector of coordinate system 2
- The 4th column vector represents the location of coordinate system 2 origin

- Inverse of the homogeneous rotation matrix

$$
H^{-1} = \begin{bmatrix} n_X & n_Y & n_Z & -n^T p \\ s_X & s_Y & s_Z & -s^T p \\ a_X & a_Y & a_Z & -a^T p \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & n^T & & -n^T p \\ & s^T & & -s^T p \\ & a^T & & -a^T p \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & -n^T p \\ & R_{3\times 3}^T & & -s^T p \\ & & & -a^T p \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Homogeneous Rigid Body Motion

- Pure Translation

$$\hat{\mathbf{a}}_{XYZ} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 4(1)+1(5) \\ 3(1) \\ 2(1)+1(-3) \\ 1(1) \end{Bmatrix} = \begin{Bmatrix} 9 \\ 3 \\ -1 \\ 1 \end{Bmatrix}$$

$$\hat{\mathbf{b}}_{XYZ} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 5 \\ 2 \\ 4 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 2 \\ 1 \\ 1 \end{Bmatrix}$$

- Pure rotation

$$\mathbf{H}_{01} = \begin{bmatrix} a_{x1} & b_{x1} & c_{x1} & 0 \\ a_{y1} & b_{y1} & c_{y1} & 0 \\ a_{z1} & b_{z1} & c_{z1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with the result $\boldsymbol{r}_0 = \boldsymbol{H}_{01}\boldsymbol{r}_1$ or in matrix form

$$\begin{Bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{Bmatrix} = \begin{bmatrix} a_{x1} & b_{x1} & c_{x1} & 0 \\ a_{y1} & b_{y1} & c_{y1} & 0 \\ a_{z1} & b_{z1} & c_{z1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} x_1 a_{x1} + y_1 b_{x1} + z_1 c_{x1} \\ x_1 a_{y1} + y_1 b_{y1} + z_1 c_{y1} \\ x_1 a_{z1} + y_1 b_{z1} + z_1 c_{z1} \\ 1 \end{Bmatrix}$$

Homogeneous Transformation Matrix

$$\boldsymbol{H} = \left[\begin{array}{c|c} \boldsymbol{R}_{3\times3} & \boldsymbol{p}_{3\times1} \\ \hline \boldsymbol{\beta}_{1\times3} & s_{1\times1} \end{array}\right] = \left[\begin{array}{c|c} \text{rotation matrix} & \text{position vector} \\ \hline \text{perspective transformation} & \text{scaling factor} \end{array}\right]$$

Rigid body motion

Rotation

$$\boldsymbol{H}_{\text{rot}}(z,\theta) = \boldsymbol{H}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation

$$\boldsymbol{H}_{\text{trans}} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Geometric interpretation

$$\boldsymbol{H}_{12} = \begin{bmatrix} n_X & s_X & a_X & p_X \\ n_Y & s_Y & a_Y & p_Y \\ n_Z & s_Z & a_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{a} & \boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or

$$\boldsymbol{H}_{12} = \begin{bmatrix} \boldsymbol{R}_{12} & {}^{1}\boldsymbol{p}_2 \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} n_X \\ n_Y \\ n_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad y_1 = x_2 \qquad \begin{Bmatrix} s_X \\ s_Y \\ s_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad x_1 = -y_2$$

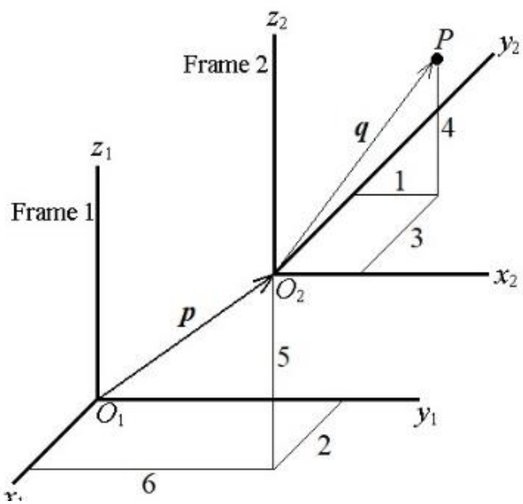
$$\begin{Bmatrix} a_X \\ a_Y \\ a_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \quad \text{or} \quad z_1 = z_2 \qquad \begin{Bmatrix} p_X \\ p_Y \\ p_Z \\ 1 \end{Bmatrix} = \begin{Bmatrix} a \\ b \\ c \\ 1 \end{Bmatrix} \quad \text{or} \quad \begin{cases} p_x = a \\ p_y = b \\ p_z = c \end{cases}$$

O_2

- Rigid body motion

$$\boldsymbol{H}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{H}_{\text{trans}} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$\boldsymbol{R}_{3x3}$:rotation

$\boldsymbol{P}_{3x1}$: translation

$$^1\boldsymbol{q} = \boldsymbol{H}_{12}\,{}^2\boldsymbol{q}$$

$$\begin{Bmatrix} q_{x1} \\ q_{y1} \\ q_{z1} \\ 1 \end{Bmatrix} = [H_{12}] \begin{Bmatrix} q_{x2} \\ q_{y2} \\ q_{z2} \\ 1 \end{Bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 3 \\ 4 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 7 \\ 9 \\ 1 \end{Bmatrix} \Longrightarrow \mathbf{H}_{0n} = \prod_{i=1}^{n} \mathbf{H}_{(i-1),i}$$

which implies: $^1\boldsymbol{q} = -1\,\boldsymbol{i}_1 + 7\,\boldsymbol{j}_1 + 9\,\boldsymbol{k}_1$.

Using rotational matrix and physical coordinates:

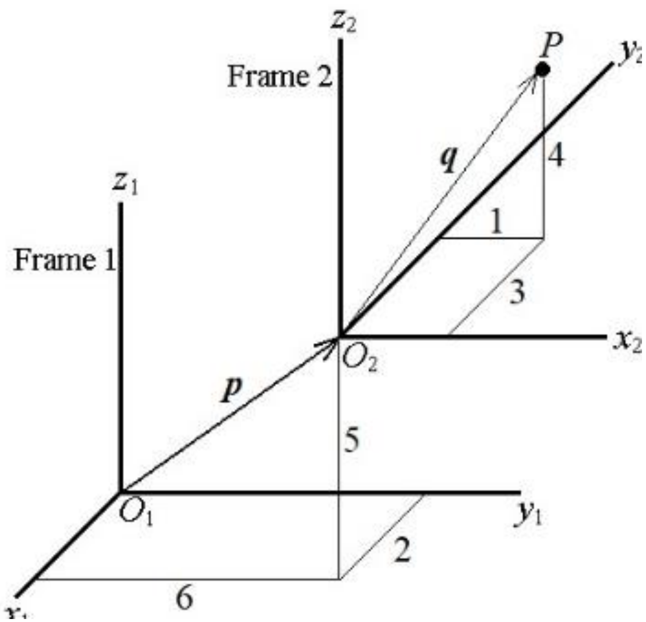
$$^1\overrightarrow{O_1P} = \boldsymbol{p} + \boldsymbol{q} = \boldsymbol{R}_{12}\ {}^2\boldsymbol{p} + \boldsymbol{R}_{12}\ {}^2\boldsymbol{q} \Longrightarrow {}^i\boldsymbol{p}_j = \sum_{k=i}^{j} \boldsymbol{R}_{ij}\ {}^j\boldsymbol{p}_j$$

Vector loop: translation

Rotation matrix: rotation

- Geometrical Interpretation

$$\boldsymbol{H}_{12} = \begin{bmatrix} n_X & s_X & a_X & p_X \\ n_Y & s_Y & a_Y & p_Y \\ n_Z & s_Z & a_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{Bmatrix} n_X \\ n_Y \\ n_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad y_1 = x_2$$

$$\begin{Bmatrix} s_X \\ s_Y \\ s_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad x_1 = -y_2$$

$$\begin{Bmatrix} a_X \\ a_Y \\ a_Z \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \quad \text{or} \quad z_1 = z_2$$

$$\begin{Bmatrix} p_X \\ p_Y \\ p_Z \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 6 \\ 5 \\ 1 \end{Bmatrix} O_2$$

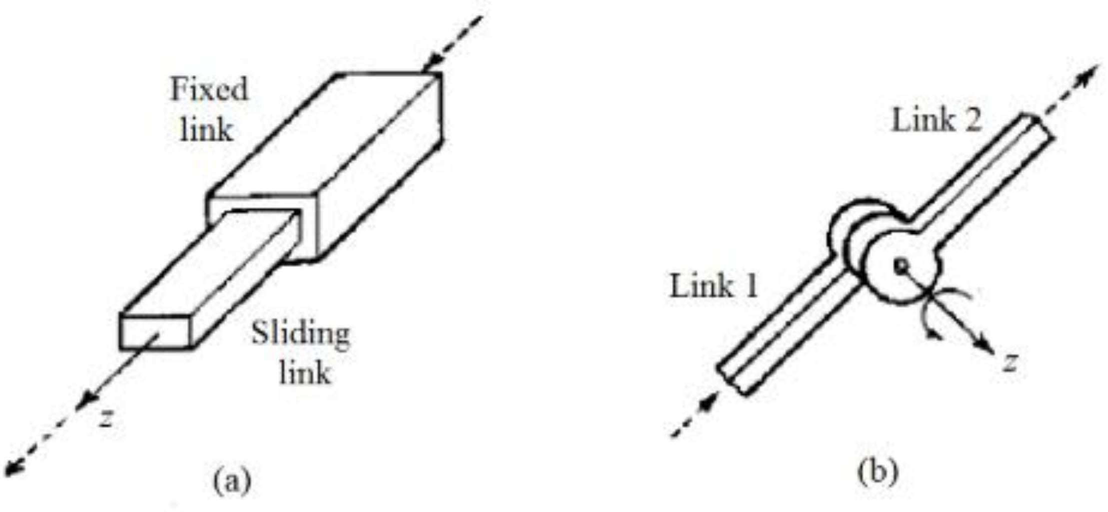
$${}^{1}\boldsymbol{p} = \begin{Bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \\ 1 \end{Bmatrix} = [H_{12}] \begin{Bmatrix} O_{2x2} \\ O_{2y2} \\ O_{2y2} \\ 1 \end{Bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 6 \\ 5 \\ 1 \end{Bmatrix}$$

Denavit-Hartenberg (D-H) Representation

- Developed in 1955 as method for **kinematic modelling** of *lower pair mechanisms* based on matrices.
- Technique applied in 1981 as a standard way of **representing robots** & **modelling their motion**.
  - Use of Joint/link classification

Adoption of this convention provide a systematic procedure which standardize and simplify

- Defining Links, Joints and their Parameters
  - Links (0 to n+1) – series of rigid bodies connected by joints
  - Joints (1 to n) – either revolute or prismatic (associated with an actuator)



Joints (a) Prismatic joint, and (b) revolute joint

- Robot with *n* joints will have *n*+1 links
  - Joint *i* connects link *i*-1 to link *i*
  - Actuation of joint *i* moves link *i*
  - Link 0 is the base/fixed and does not move

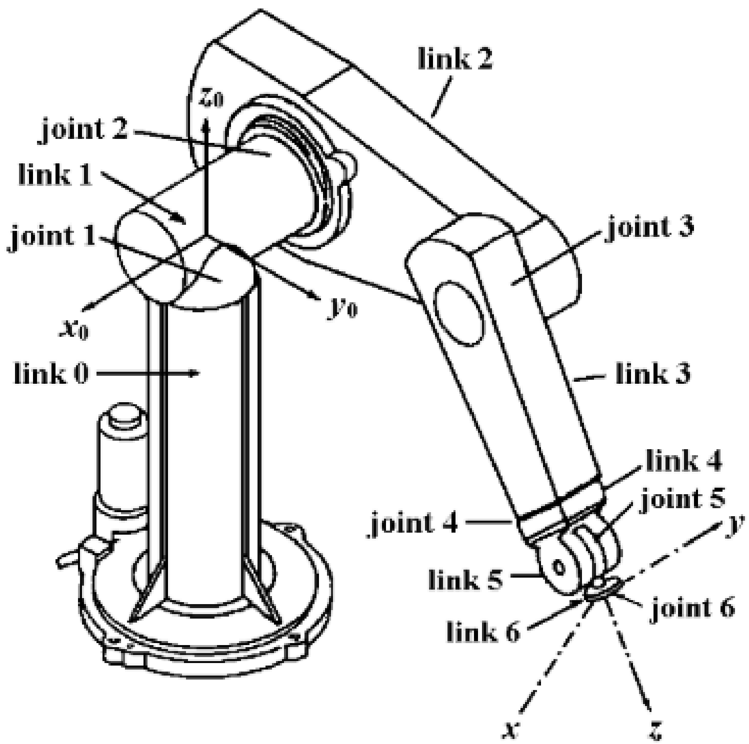


Figure 7.6 PUMA robot arm with joints (1 to 6) and links (0, 1 to 6; 0 is the base).

- Robot arms are modelled as open kinematic chain
  - A single DOF of the robot is associated/effected by a single joint/actuation

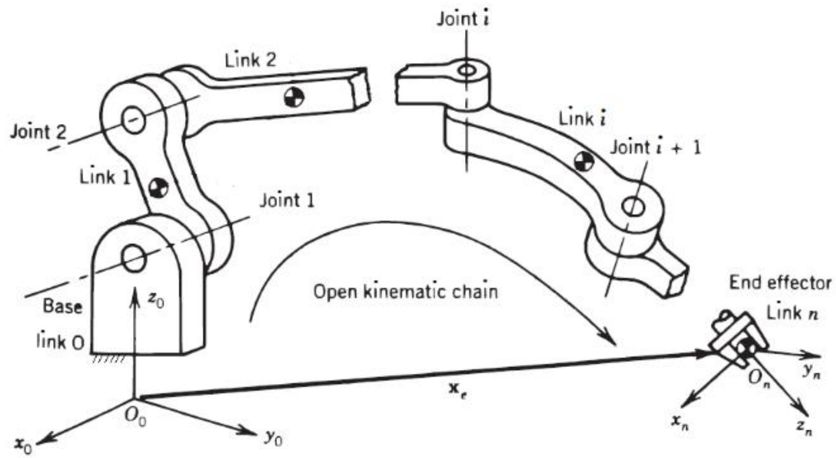


Figure 7.7b Robotic Arm formed by open kinematic chain.

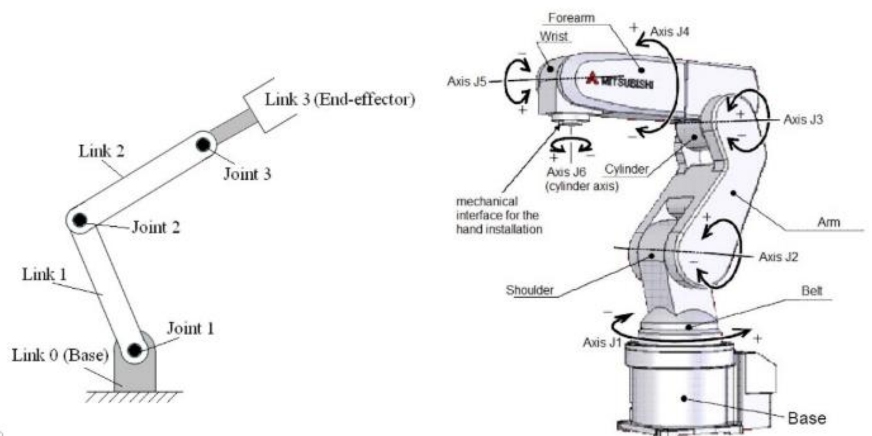


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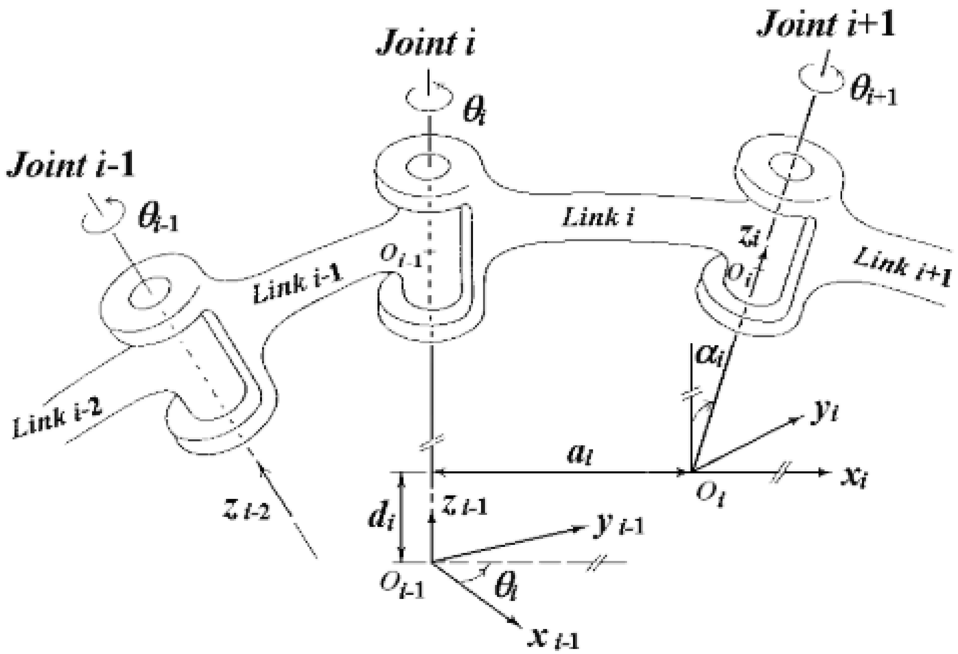
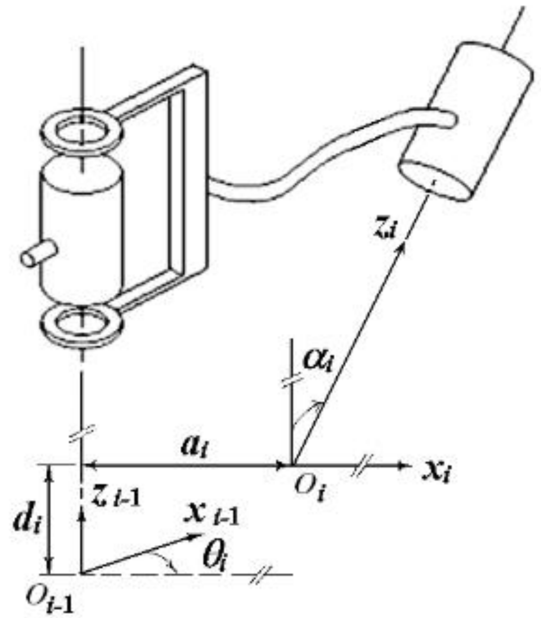
Figure 7.7c Robots with joints and links.

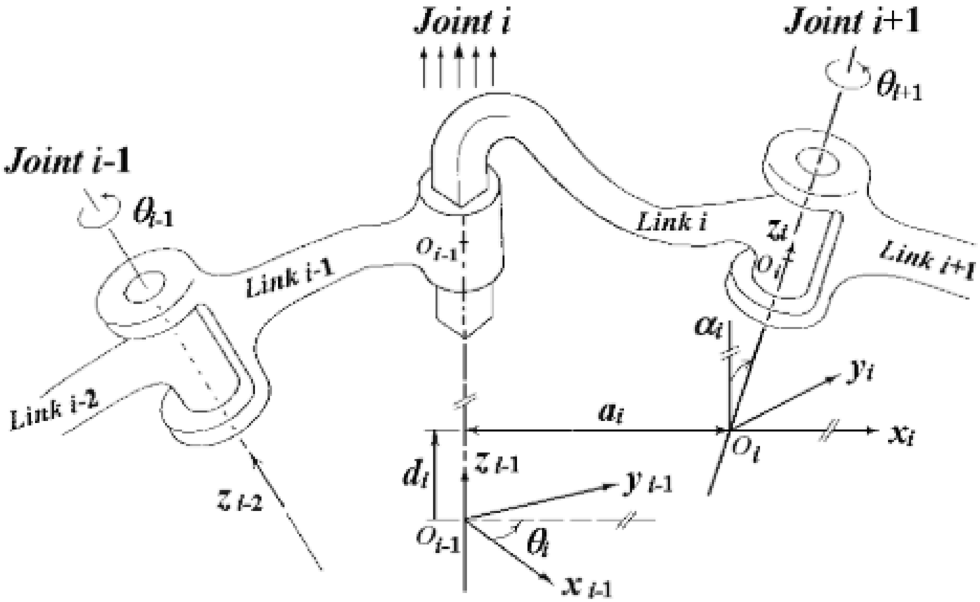
- DH Parameters for joints and links
  - Each joint has a joint axis z_i and 2 normal x_{i-1} and x_i with respect to each link connected
  - d_i and θ_i associated with distance and angle between adjacent links (i.e. relative position)
    - d_i : distance measured along the joint axis between normals x_{i-1} and x_i
    - θ_i : joint angle measured between the normal x_{i-1} and x_i in a plane normal to the joint axis



- DH Parameters for joints and links
  - Each link maintains a fixed configuration between their joints based of the *length* and *twist angle* of link *i*
  - a_i and α_i associated with distance and angle between adjacent links
    - a_i : shortest distance measured along the common normal between joint axes z_{i-1} and z_i
    - α_i : angle measured between the joint axes z_{i-1} and z_i perpendicular to a_i



- DH Parameters for joints and links
  - Link parameters a_i and α_i determine the structure of the link
  - Joint parameters d_i and θ_i determine the relative position of neighbouring links

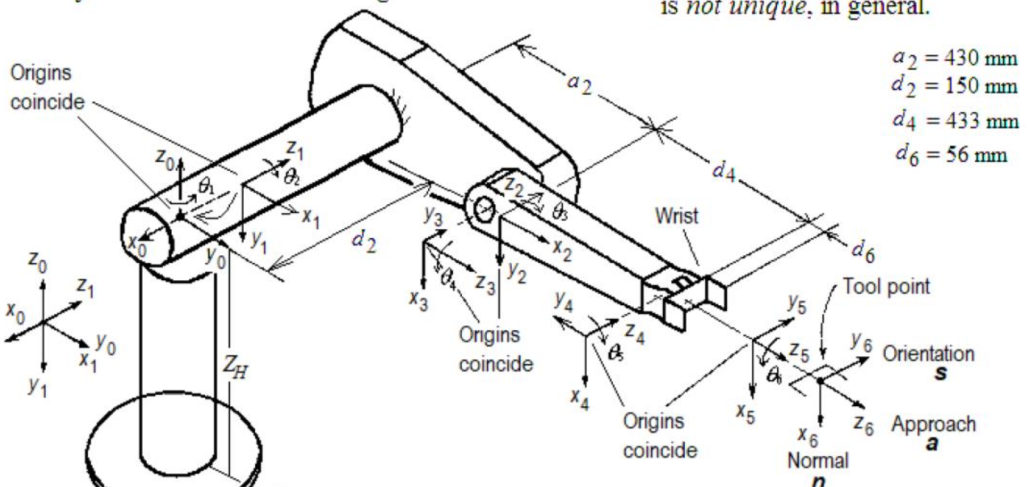


(b)

• Assigning D-H Coordinate Systems

The robot is placed in a configuration with joitn variables of 0 or 90 degree.

* assignment of coordinate systems is *not unique*, in general.



Z_H = Distance from mounting flange to world axes

NOTE: The above drawing is not to scale and is for illustrative purposes only.

1. Establishing Link Coordinate Systems

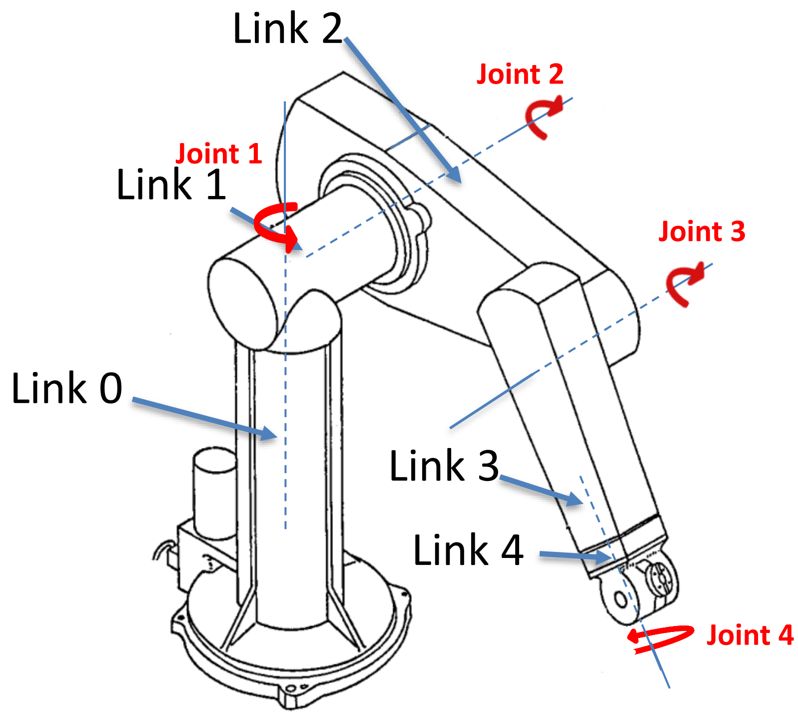
2. Define the D-H Parameters

D-H Parameters of PUMA Robot Arm (at position shown)

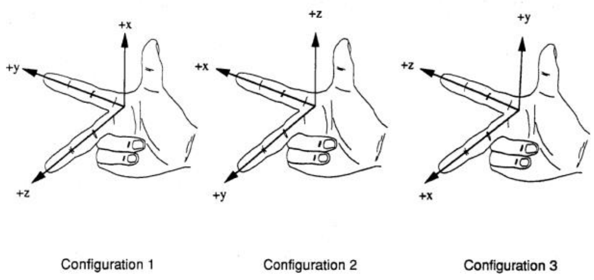
Joint i	θ_i (deg)	d_i (mm)	a_i (mm)	α_i (deg)	$\cos\alpha_i$	$\sin\alpha_i$	Joint variables	Joint range (deg)	
1	90	0	0	-90	0	-1	θ_1	-160 to 160	← z_{i-1}, z_i axes (*intersect*)
2	0	d_2	a_2	0	1	0	θ_2	-225 to 45	← z_{i-1}, z_i axes (*parallel*)
3	90	0	0	90	0	1	θ_3	-45 to 225	← z_{i-1}, z_i axes (*intersect*)
4	0	d_4	0	-90	0	-1	θ_4	-110 to 170	← z_{i-1}, z_i axes (*intersect*)
5	0	0	0	90	0	1	θ_5	-100 to 100	← z_{i-1}, z_i axes (*intersect*)
6	0	d_6	0	0	1	0	θ_6	-266 to 266	← z_{i-1}, z_i axes (*collinear*)

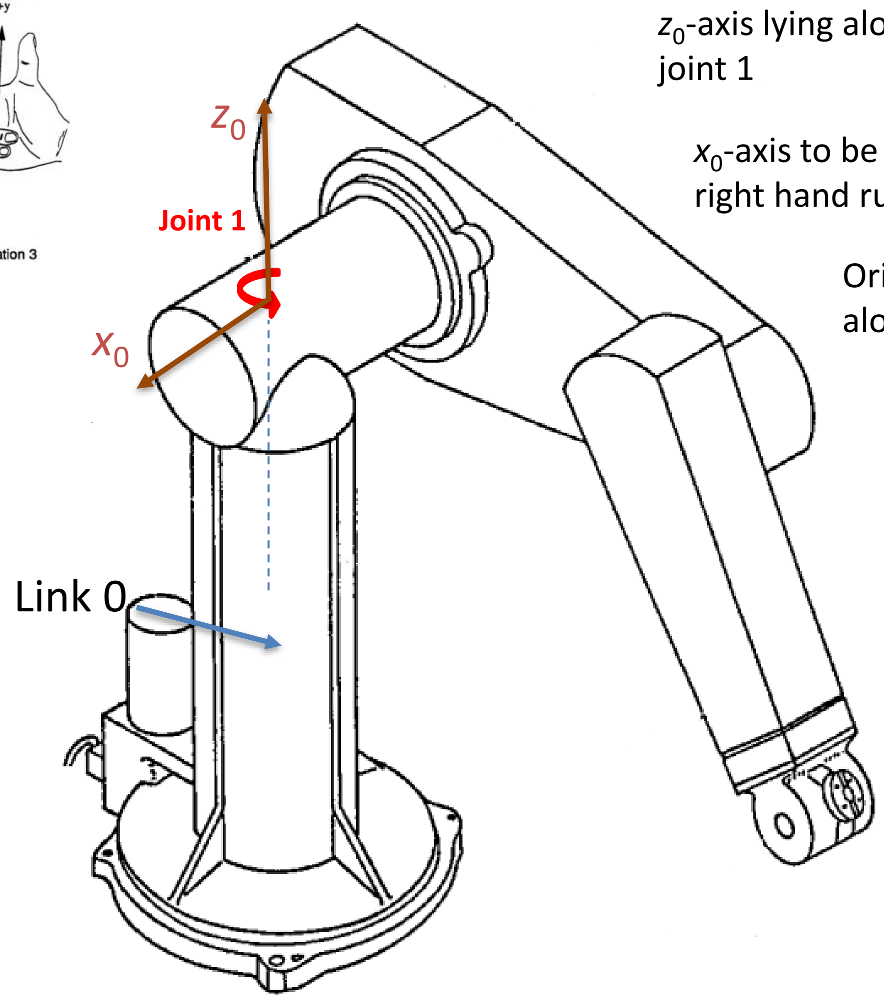
Note: plus-minus of the D-H parameters depending on the rotation-direction/distance from axis x_{i-1} to axis x_i (based on the right-hand coordinate system)

- Establishing Link Coordinate Systems
  - To assign Cartesian coordinate frames for the links in a serial link *n*-joints manipulator.
    - **Each joint represents 1 DOF**
    - **Each joint is either revolute or prismatic.**
  - Manipulator will have *n*+1 links.



- Establishing Link Coordinate Systems
  - Establish base coordinate system *XYZ* to the base of the manipulator for 1st link



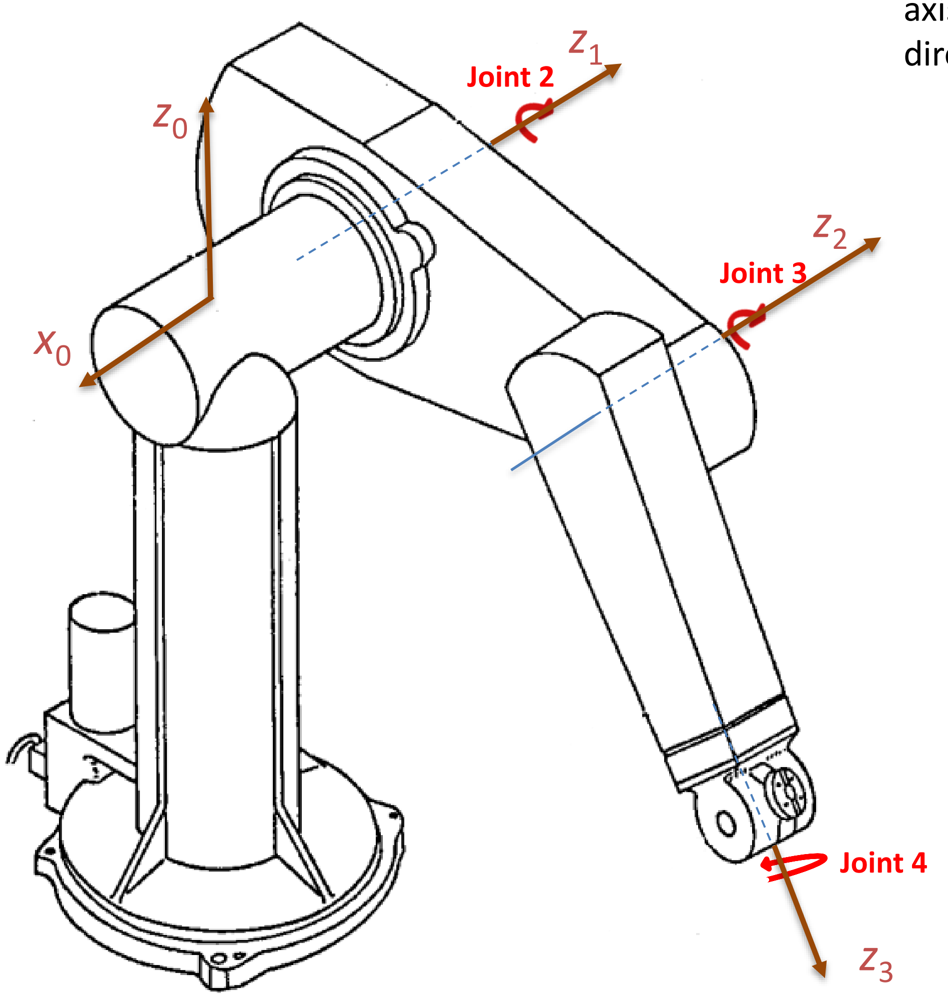


z_0-axis lying along axis of the motion of joint 1

x_0-axis to be conveniently established by right hand rule

Origin placed anywhere along z_0-axis

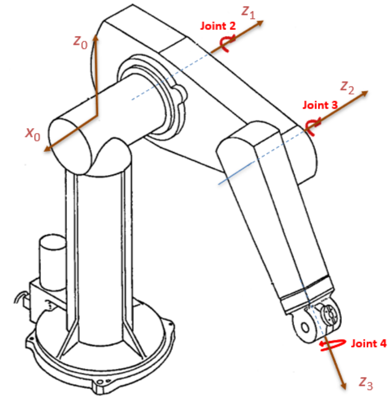
- Establishing Link Coordinate Systems
  - Establish link coordinates 1 to 4



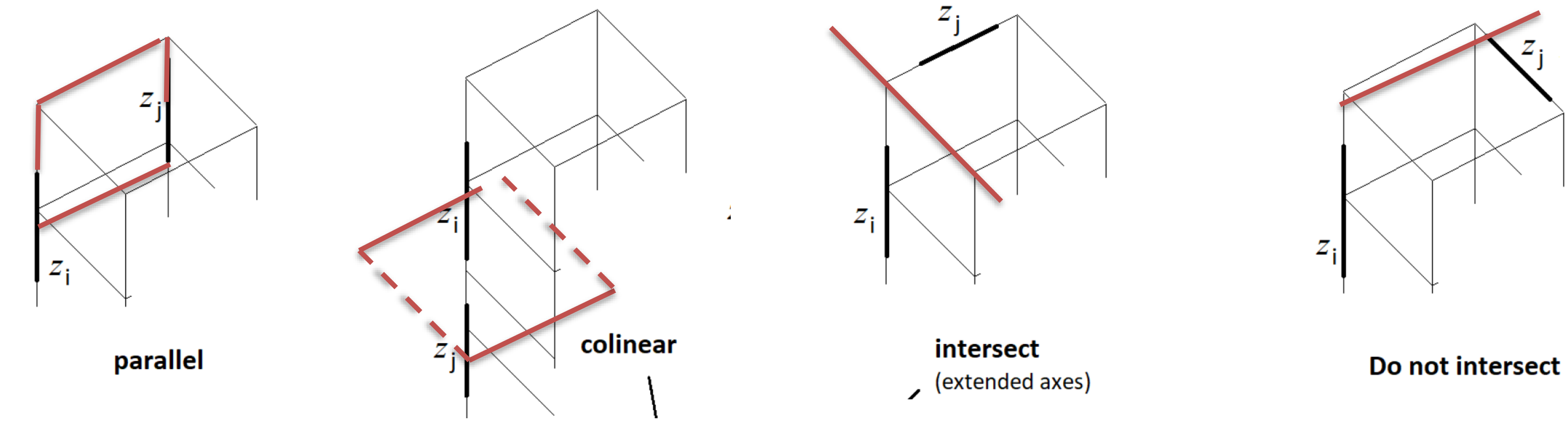
Align z_{i-1}-axis about the joint i axis i.e. axis of revolution for revolute and direction of linear motion for prismatic

Establish the **origin** of the ith coordinate: a) @ intersection of the z_{i-1}-axis and the common normal or b) @intersection of z_{i-1} & z_i axes

- Establishing Link Coordinate Systems
  - Establish link coordinates 1 to 4
    - Establish x_i-axis is normal to the z_{i-1} axis
    - The x_i-axis is normal to both z_i & z_{i-1} axes



Relationship between two z axes



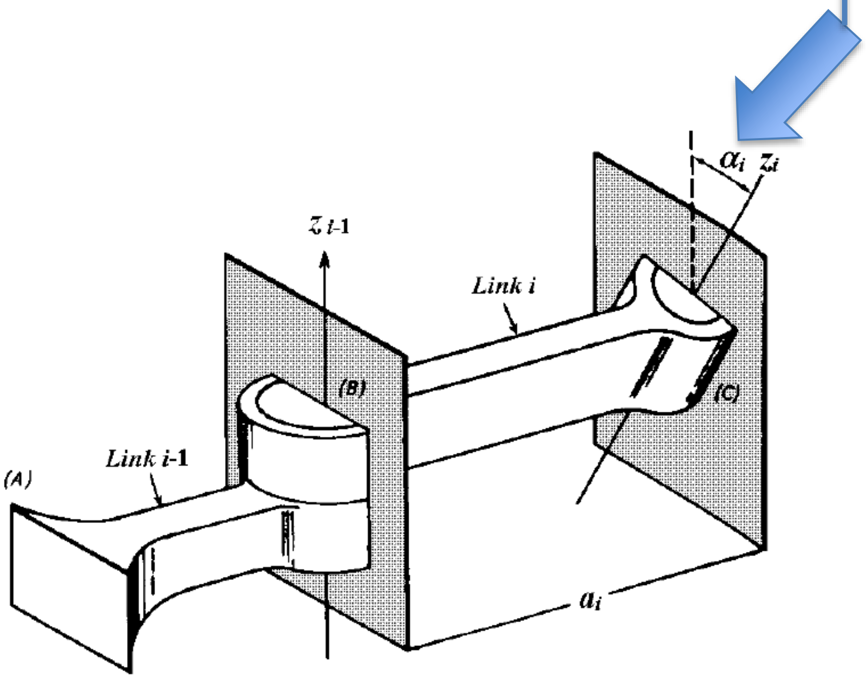
- Infinite no. of common normal
- Direction is defined from z_{i-1} towards z_i
- x_i will be the common normal passing thru O_i

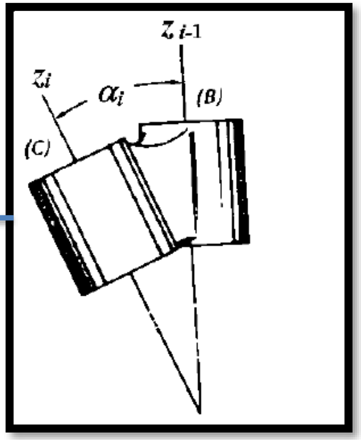
- x_i can be chosen anywhere in the plane perpendicular to z_i and z_{i-1}

- No common normal connecting z_i and z_{i-1}
- x_i can be chosen along either of 2 opposite directions perpendicular to z_i and z_{i-1}

- x_i is along the common normal to z_i and z_{i-1}s
- Direction is defined from z_{i-1} towards z_i

- Establishing Link Coordinate Systems
  - Establish link coordinates 1 to 4
    - Establish x_i-axis



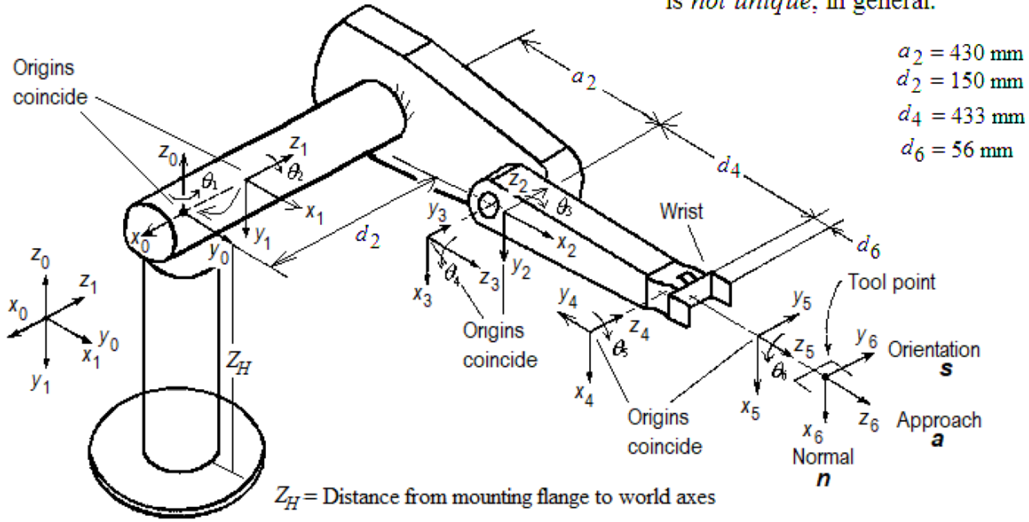


The robot is placed in a configuration with joitn variables of 0 or 90 degree.

* assignment of coordinate systems is *not unique*, in general.

z-axes (extended, if required)

skewlines: $\alpha_i \neq 0$ and $a_i \neq 0$
parallel: $\alpha_i = 0$ and $a_i \neq 0$
collinear: $\alpha_i = 0$ and $a_i = 0$
intersecting: $\alpha_i \neq 0$ and $a_i = 0$



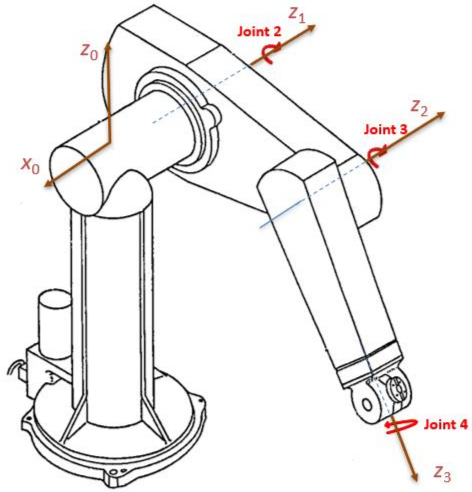
Z_H = Distance from mounting flange to world axes

NOTE: The above drawing is not to scale and is for illustrative purposes only.

D-H Parameters of PUMA Robot Arm (at position shown)

Joint i	θ_i (deg)	d_i (mm)	a_i (mm)	α_i (deg)	$\cos\alpha_i$	$\sin\alpha_i$	Joint variables	Joint range (deg)	
1	90	0	0	-90	0	-1	θ_1	-160 to 160	← z_{i-1}, z_i axes (*intersect*)
2	0	d_2	a_2	0	1	0	θ_2	-225 to 45	← z_{i-1}, z_i axes (*parallel*)
3	90	0	0	90	0	1	θ_3	-45 to 225	← z_{i-1}, z_i axes (*intersect*)
4	0	d_4	0	-90	0	-1	θ_4	-110 to 170	← z_{i-1}, z_i axes (*intersect*)
5	0	0	0	90	0	1	θ_5	-100 to 100	← z_{i-1}, z_i axes (*intersect*)
6	0	d_6	0	0	1	0	θ_6	-266 to 266	← z_{i-1}, z_i axes (*collinear*)

Note: plus-minus of the D-H parameters depending on the rotation-direction/distance from axis x_{i-1} to axis x_i (based on the right-hand coordinate system)



The robot is placed in a configuration
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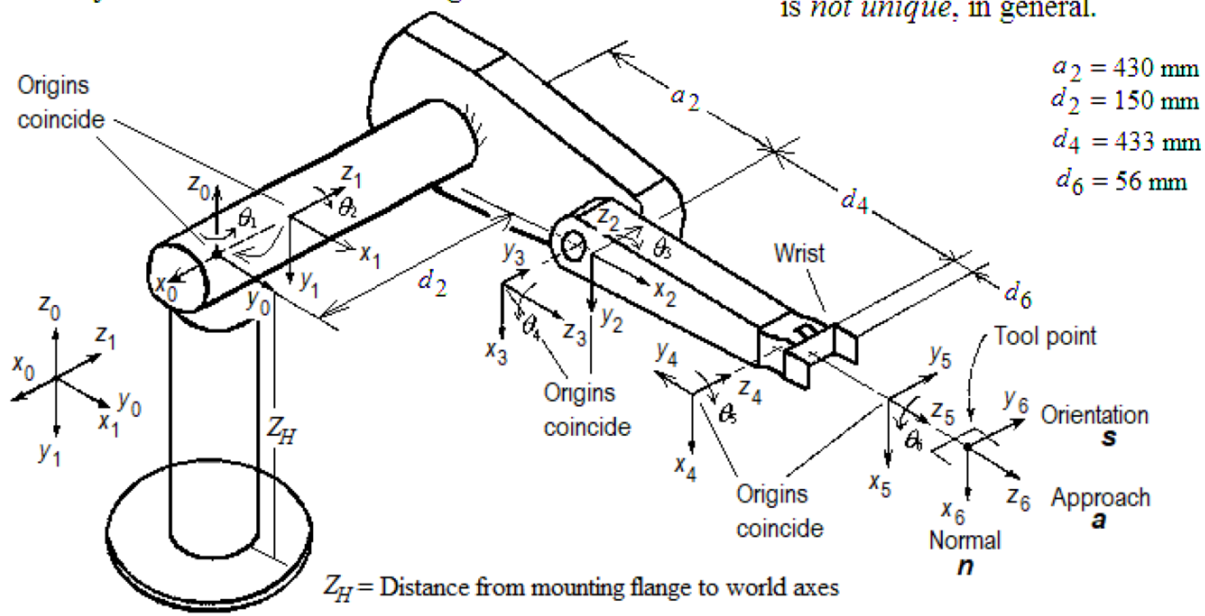

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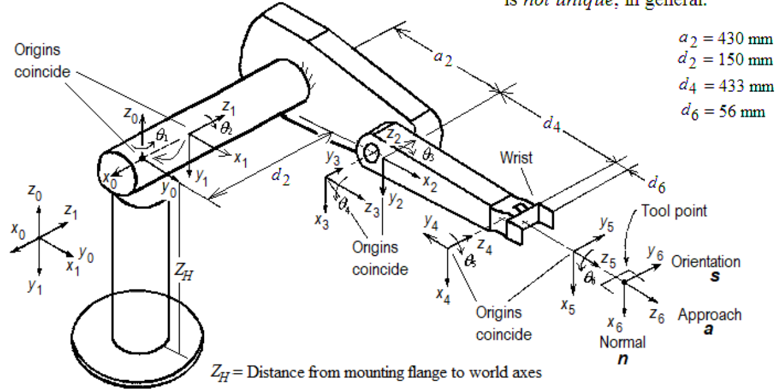

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D-H Parameters of PUMA Robot Arm (at position shown)

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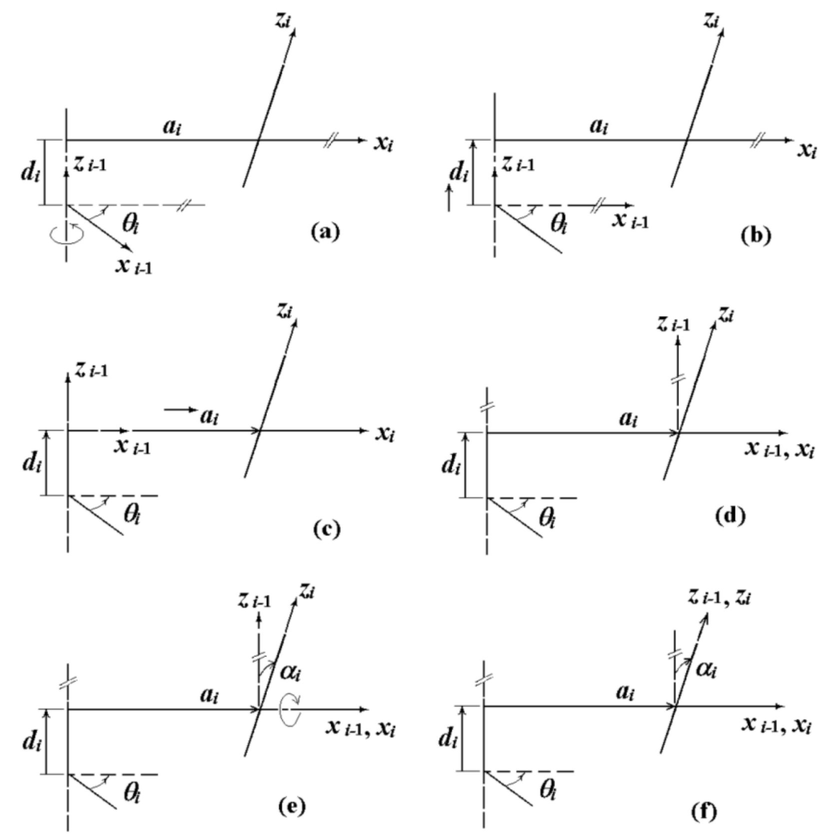
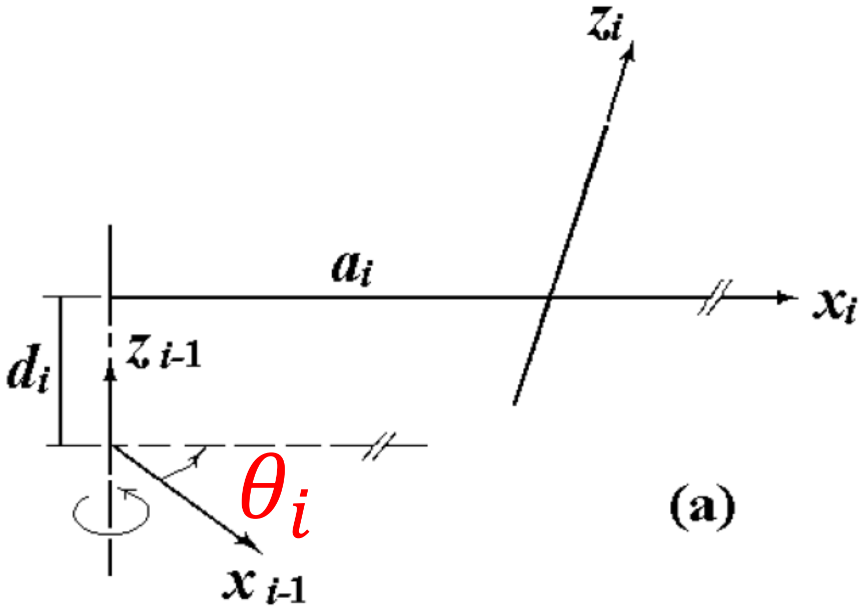
3	90	0	0	90	0	1	θ_3	-45 to 225	← z_{i-1}, z_i axes (*intersect*)
4	0	d_4	0	-90	0	-1	θ_4	-110 to 170	← z_{i-1}, z_i axes (*intersect*)
5	0	0	0	90	0	1	θ_5	-100 to 100	← z_{i-1}, z_i axes (*intersect*)
6	0	d_6	0	0	1	0	θ_6	-266 to 266	← z_{i-1}, z_i axes (*collinear*)

Note: plus-minus of the D-H parameters depending on the rotation-direction/distance from axis x_{i-1} to axis x_i (based on the right-hand coordinate system)

- <u>D-H Homogeneous Matrix</u>
  - Generation of $\left[H_{(i-1),i}\right]$ where$\{P\}_{i-1} = \left[H_{(i-1),i}\right]\{P\}_i$
    - Exercise in 'physically shifting' Frame *i*-1 into Frame *i*

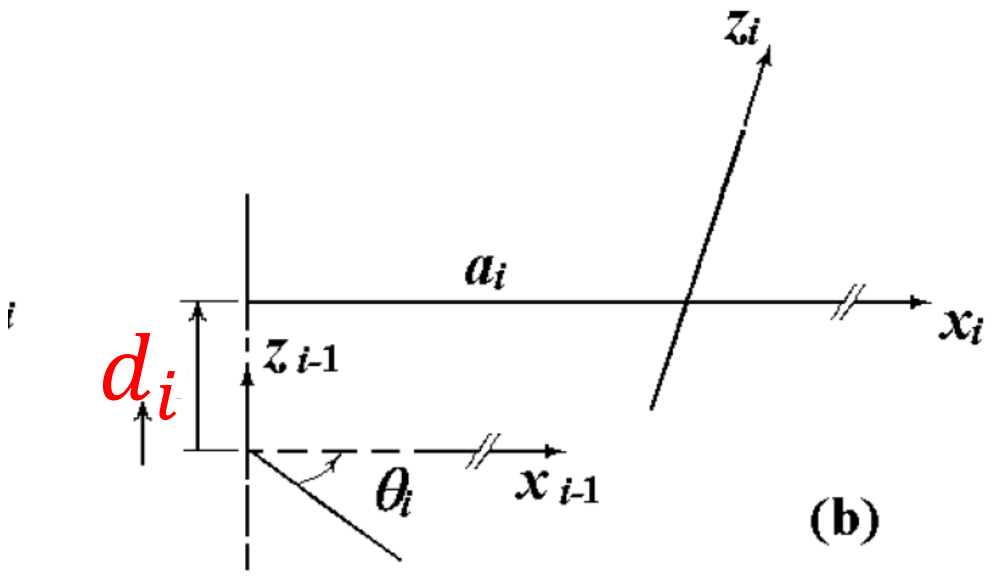


- <u>Step 1</u>
  - Rotate about $z_{i\text{-}1}$ axis θ_i



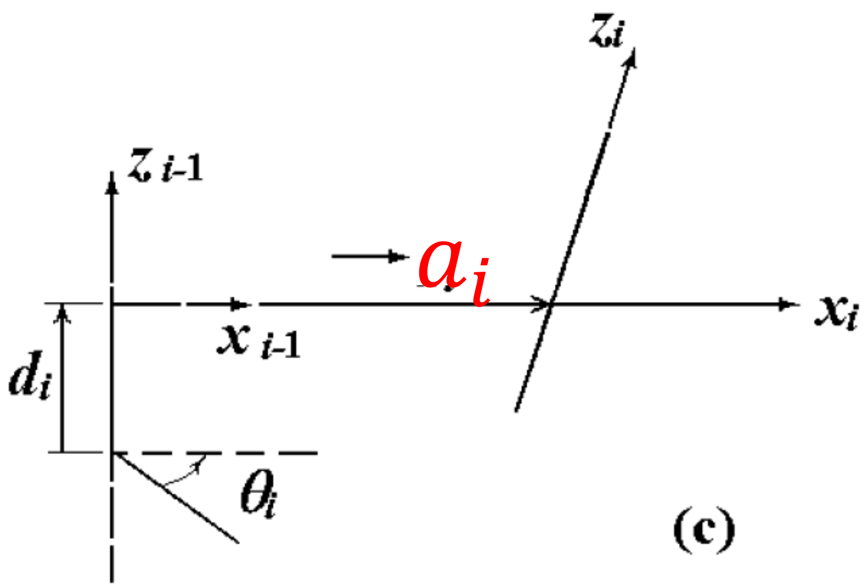
$x_{i\text{-}1}$ axis and x_i axis will become parallel

- <u>Step 2</u>
  - Translate along the $z_{i\text{-}1}$ axis d_i



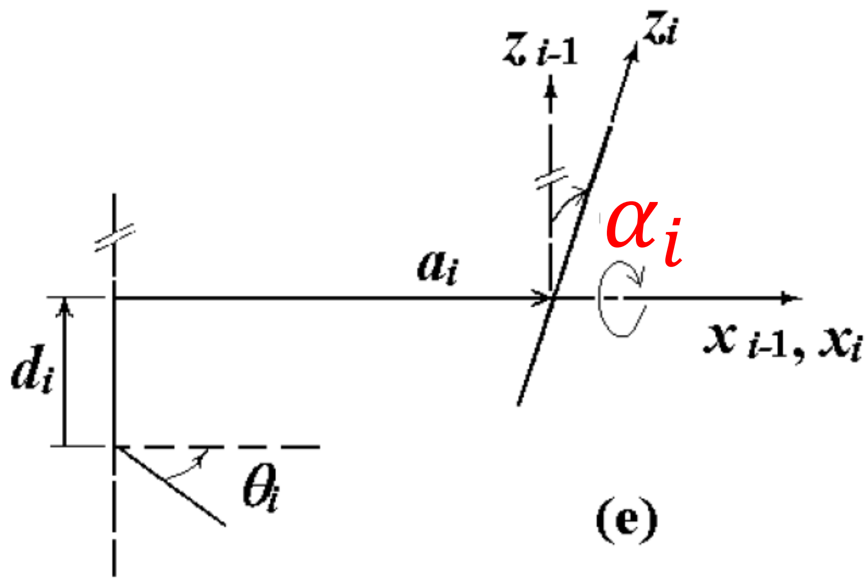
$x_{i\text{-}1}$ axis and x_i axis will become collinear

- <u>Step 3</u>
  - Translate along $x_{i\text{-}1}$ axis a_i



Origins of frame i-1 and i will collocated

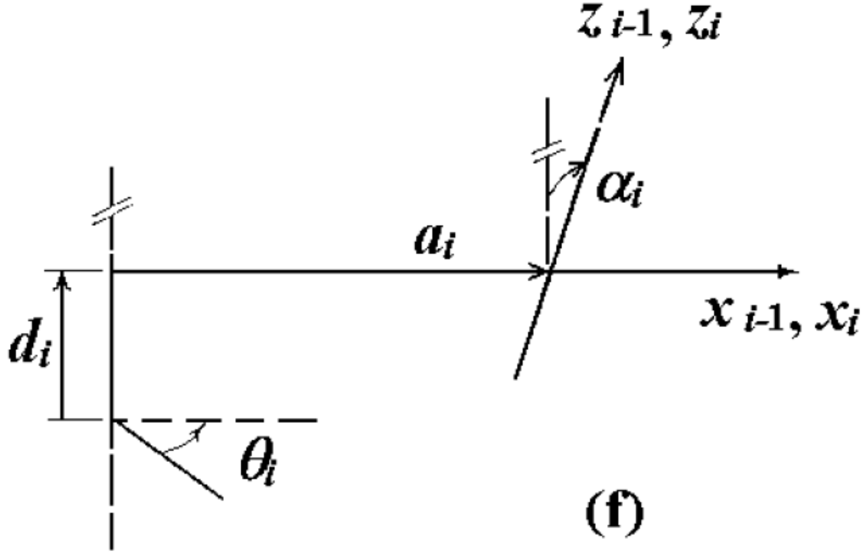
- <u>Step 4</u>
  - Rotate $z_{i\text{-}1}$ axis α_i about x_i axis



(e)

Frame i-1 and Frame i will be exactly the same

- <u>End</u>
  - Transformation of Frame *i*-1 to Frame *i* completed



(f)

- In effect the 4 steps result in the matrix ***T***

$$\boldsymbol{T}_{(i-1),i} = \left[H_{(i-1),i}\right] = [H]_{z_{i-1},\theta_i}[H]_{z_{i-1},d_i}[H]_{x_i,a_i}[H]_{x_i,\alpha_i}$$

$$= \underbrace{\begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Rotation } \theta_i \\ \text{about } z_{i-1}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Translation } d_i \\ \text{along } z_{i-1}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Translation } a_i \\ \text{along } x_{i-1}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Rotation } \alpha_i \\ \text{about } x_i}} \quad (7.23)$$

$$\left[H_{(i-1),i}\right] = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{(7.24)}$$

- For example, the transformation between joints 2 & 3 would be:

$$T_{23} = [H_{23}] = \begin{bmatrix} \cos\theta_3 & -\cos\alpha_3 \sin\theta_3 & \sin\alpha_3 \sin\theta_3 & a_3 \cos\theta_3 \\ \sin\theta_3 & \cos\alpha_3 \cos\theta_3 & -\sin\alpha_3 \cos\theta_3 & a_3 \sin\theta_3 \\ 0 & \sin\alpha_3 & \cos\alpha_3 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.25}$$

- The total transformation between base of the robot and the hand:

$$\boldsymbol{T}_{RH} = \boldsymbol{T}_{01}\boldsymbol{T}_{12}\boldsymbol{T}_{23} \quad \ldots \boldsymbol{T}_{(n-1),n} \tag{7.26}$$

- The inverse of $\left[H_{(i-1),i}\right]$ is

$$\left[H_{i,(i-1)}\right] = \left[H_{(i-1),i}\right]^{-1} = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & -a_i \\ -\cos\alpha_i \sin\theta_i & \cos\alpha_i \cos\theta_i & \sin\alpha_i & -d_i \sin\alpha_i \\ \sin\alpha_i \sin\theta_i & -\sin\alpha_i \cos\theta_i & \cos\alpha_i & -d_i \cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.27)$$

$$\boldsymbol{H}^{-1} = \begin{bmatrix} n_X & n_Y & n_Z & -\boldsymbol{n}^T\boldsymbol{p} \\ s_X & s_Y & s_Z & -\boldsymbol{s}^T\boldsymbol{p} \\ a_X & a_Y & a_Z & -\boldsymbol{a}^T\boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \boldsymbol{n}^T & & -\boldsymbol{n}^T\boldsymbol{p} \\ & \boldsymbol{s}^T & & -\boldsymbol{s}^T\boldsymbol{p} \\ & \boldsymbol{a}^T & & -\boldsymbol{a}^T\boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & -\boldsymbol{n}^T\boldsymbol{p} \\ & \boldsymbol{R}_{3\times3}^T & & -\boldsymbol{s}^T\boldsymbol{p} \\ & & & -\boldsymbol{a}^T\boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- <u>D-H Forward Kinematics</u>
  - The homogeneous matrix $H_{0,i}$
    - The chain product of successive coordinate transformation matrices of $\left[H_{(i-1),i}\right]$

$$
\begin{aligned}
\mathbf{H}_{0i} &= \mathbf{H}_{01}\mathbf{H}_{12}\ldots.\mathbf{H}_{i-1,i} = \prod_{j=1}^{i}\mathbf{H}_{j-1,j} \quad \text{for} \quad i=1,2,\ldots,n \\
&= \begin{bmatrix} \mathbf{x}_i & \mathbf{y}_i & \mathbf{z}_i & \mathbf{p}_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{0i} & {}^{\mathbf{0}}\mathbf{p}_i \\ 0 & 1 \end{bmatrix}
\end{aligned}
\tag{7.28}
$$

where

$[\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i]$ = rotation matrix of the ith coordinate system established at link i with respect to the base coordinate system. It is the upper left 3×3 partitioned matrix of $\mathbf{H}_{0i}$.

$\{\mathbf{p}_i\}$ = position vector that points from the origin of the base coordinate system to the origin of the ith coordinate system. It is the upper right 3×1 partitioned matrix of $\mathbf{H}_{0i}$.

- The forward matrix $H_{0,n}$
  - A manipulator with *n* links and joints
    - The base reference frame will be $O_0x_0y_0z_0$ and the last reference frame at the hand will be $O_nx_ny_nz_n$

$$\mathbf{H}_{0n} = \prod_{i=1}^{n} \mathbf{H}_{(i-1),i}$$

Is there a simpler way?

  - Recall the duality of the homogeneous transformation matrix:
    - A class of matrix operators that can perform simultaneous vector operations resulting in translation &/or rotation
    - Description of the geometric relationship between a body-attached frame *oxyz* and another reference coordinate frame *OXYZ*

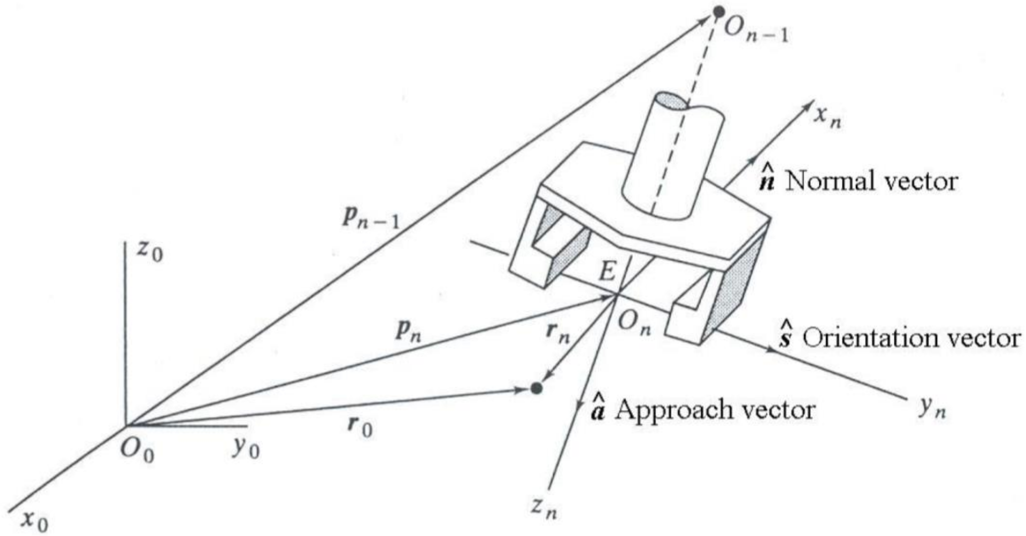


Figure 7A Hand frame of a manipulator with respect to the fixed system 0.

Again, $$\boldsymbol{H} = \begin{bmatrix} n_X & s_X & a_X & p_X \\ n_Y & s_Y & a_Y & p_Y \\ n_Z & s_Z & a_Z & p_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{a} & \boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \boldsymbol{R} & & \boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (7.12)$$

where $\boldsymbol{n}$ = *normal* vector; $\boldsymbol{s}$ = *sliding* (or *orientation*) vector, $\boldsymbol{a}$ = *approach* vector, and $\boldsymbol{p}$ = *position* vector.

• Summary

Step 1: Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

Step 2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-hand frame.

For $i = 1, \ldots, n-1$, perform Steps 3 to 5.

Step 3: Locate the origin O_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate O_i at this interse

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This
of the tool frame expressed in

and z_{i-1} are parallel, locate O_i in any convenient position along z_i.

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through O_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 5: Establish y_i to complete a right-hand frame.

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n-th joint is revolute, set $z_n = \boldsymbol{a}$ along the direction z_{n-1}. Establish the origin O_n conveniently along z_n, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = \boldsymbol{s}$ in the direction of the gripper closure and set $x_n = \boldsymbol{n}$ as $\boldsymbol{s} \times \boldsymbol{a}$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-hand frame.

Step 7: Create a table of link parameters a_i, d_i, α_i, θ_i.

$a_i =$ distance along x_i from O_i to the intersection of the x_i and z_{i-1} axes.

$d_i =$ distance along z_{i-1} from O_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

$\alpha_i =$ the angle between z_{i-1} and z_i measured about x_i (see Figure 3.3).

$\theta_i =$ the angle between x_{i-1} and x_i measured about z_{i-1} (see Figure 3.3). θ_i is variable if joint i is revolute.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into (3.10).

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

Thank you!