Chapter 1

Introduction

To study the orbital motion of a binary black hole system the idea of Newton's two body problem is used. In Newtonian mechanics, the two body problem determines the relative motion of two point masses M_1 and M_2. In the lab frame of reference when the distance between the two masses is measured w.r.t. the origin, then the following coupled differential equations are obtained by Newton's law of gravitation.

$$\begin{aligned} \frac{d^2\overrightarrow{x_1}}{dt^2} &= -Gm_2\frac{\overrightarrow{x_1}-\overrightarrow{x_2}}{r^3} \\ \frac{d^2\overrightarrow{x_2}}{dt^2} &= -Gm_1\frac{\overrightarrow{x_2}-\overrightarrow{x_1}}{r^3} \end{aligned} \tag{1.1}$$

In order to solve the coupled differential, the Centre of Mass (COM) refernce frame is used because the COM ref. frame does not accelerate. The COM of a two body problem is found by

$$\begin{gathered}\vec{R} = \frac{m_1\vec{x_1}+m_2\vec{x_2}}{m_1+m_2} \\ \frac{d^2\vec{R}}{dt^2} = 0\end{gathered} \tag{1.2}$$

Then the relative separation between the two masses and the reduced mass, μ are given by the following equations

$$\begin{gathered}\vec{r} = \vec{x_2} - \vec{x_1} \\ \mu = \frac{m_1 m_2}{M} \\ M = m_1 + m_2\end{gathered} \tag{1.3}$$

The order of the two-body problem is then reduced from second order to first order.

$$\frac{d^2\vec{r}}{dt^2} = -GM\frac{\vec{r}}{r^3} \tag{1.4}$$

However, the Newtonian one body problem does not predict the emission of gravitational waves. According to General Relativity,due to the emission of gravitational waves there is a reduction in the semi-major axis of the one-body problem. Therefore, the Post Newtonian (PN) correction is used to incorporate the general relativistic effects in modelling the orbit of two binary black holes. The PN correction show a shift in the peri-centre of the one body problem. And the emission of gravitational waves is modelled by the PN2.5 correction. After incorporating the PN correction, the acceleration of the COM is given by

$$\begin{aligned}\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = -\frac{GM}{r^2}(A\vec{n} + B\vec{v})\\ \vec{n} = \frac{\vec{r}}{r}\\ A = A_0 + c^{-2}A_1 + c^{-4}A_2 + c^{-5}A_{2.5}\cdots\\ B = B_0 + c^{-2}B_1 + c^{-4}B_2 + c^{-5}B_{2.5}\cdots\cdots\end{aligned} \tag{1.5}$$

where the coefficients are given by

$$\begin{aligned}A_0 = 1\\ A_1 = -\tfrac{3}{2}\dot{r}^2\nu + (1+3\nu)v^2 - 2(2+\nu)\tfrac{GM}{r}\\ A_{2.5} = -\tfrac{8GM}{5r}\nu\dot{r}\left(\tfrac{17GM}{3r} + 3v^2\right)\\ A_2 = \tfrac{15}{8}\dot{r}^4\nu(1-3\nu) + 3\dot{r}^2\nu v^2\left(2\nu - \tfrac{3}{2}\right) + \nu v^4(3-4\nu)\\ +\tfrac{GM}{r}\left(-2\dot{r}^2\left(1+\nu^2\right) - 25\dot{r}^2\nu - \tfrac{13}{2}\nu v^2\right) + \tfrac{G^2M^2}{r^2}\left(9 + \tfrac{87}{4}\nu\right)\\ B_{0.5} = \tfrac{8GM}{5r}\nu\left(\tfrac{3GM}{3r} + v^2\right)\\ B_2 = 3\dot{r}^3\nu\left(\tfrac{3}{2}+\nu\right) + \dot{r}\nu v^2\left(\tfrac{15}{2} + 2\nu\right) + \tfrac{GM\dot{r}}{r}\left(2 + \tfrac{41}{2}\nu + 4\nu^2\right)\end{aligned} \tag{1.6}$$

$$\dot{r} = \vec{n}\cdot\vec{v} = \frac{\vec{r}\cdot\vec{v}}{r} \tag{1.7}$$

Chapter 2

Methodology

The leapfrog algorithm with variable time step is used to solve the first order differential equation of the two body problem in the COM reference frame. The advantage of using the leapfrog algorithm with variable time step is that it reduces the calculation time. When the variable varies slowly, the time step is large as compare to those points where the fluctuation is large . The variable time step used is

$$dt = \eta\sqrt{r_i^3/GM} \tag{2.1}$$

To avoid the calculation of velocity twice, the leapfrog algorithm was modified as follows:

$$
\begin{aligned}
dt &= \eta\sqrt{r_i^3/GM} \\
v'_{i+1} &= v_i + a_i dt/2 \\
r_{i+1} &= r_i + v'_{i+1}dt \\
\vec{n}'_{i+1} &= \tfrac{\vec{r}_{i+1}}{r_{i+1}} \\
\dot{r}_{i+1} &= \tfrac{\vec{r}_{i+1}\cdot\vec{v}'_{i+1}}{r_{i+1}} \\
A' = &\text{ Use Eq.(8) and Eq.(10-13)} \\
B' &= UseEq\cdot\overrightarrow{(9)} \text{ and } Eq\cdot(14-17) \\
a'_{i+1} &= -\tfrac{GM}{r'^2_{i+1}}\left(A'\vec{n}'_{i+1} + B'\vec{v}'_{i+1}\right) \\
v_{i+1} &= v'_{i+1} + a'_{i+1}dt/2 \\
\dot{r}_{i+1} &= \tfrac{\vec{r}_{i+1}\cdot\vec{v}_{i+1}}{r_{i+1}}
\end{aligned}
\tag{2.2}
$$

Then the acceleration at each time step is calculated as

$$
a_{i+1} = -\frac{GM}{r^2_{i+1}}\left(A\vec{n}'_{i+1} + B\vec{v}_{i+1}\right) \tag{2.3}
$$

The suggested units to be used to simulate the black hole binary system are

$$
G = 1, [R] = 1mpc \text{ and } [T] = 1yr \tag{2.4}
$$

Then the unit of mass is given by

$$
[M] = 4.430458954051335^{35} mpc^3/yr^2 \tag{2.5}
$$

and, the unit of velocity is

$$1.0220113556817297 \times 10^{-6}\text{mpc/yr} \tag{2.6}$$

Using the suggested units the speed of light is calculated to be

$$c = 306.603\text{mpc/yr} \tag{2.7}$$

The initial position of the point mass is at the apogee and the initial velocity is in the y-direction. Further, to calculate the eccentricity of the orbit of COM, the following formula is used

$$\begin{aligned} e &= \tfrac{\vec{v}\times\vec{h}}{GM} - \tfrac{\vec{r}}{r} \\ h &= \vec{r} \times \vec{v} \end{aligned} \tag{2.8}$$

Then by capturing how eccentricity evolves with each iteration the advance in the perihelion is calculated by

$$d\phi = \phi_f - \phi_0 \tag{2.9}$$

where,

$$\phi = \tan^{-1}\left(e_y/e_x\right) \tag{2.10}$$

The numerically generated perihelion advance is compared with the theoretical prediction of General Relativity given by,

$$d\phi_{\text{theoretical}} = \frac{6\pi GM}{c^2 a\left(1-e^2\right)} \tag{2.11}$$

Chapter 3

Results

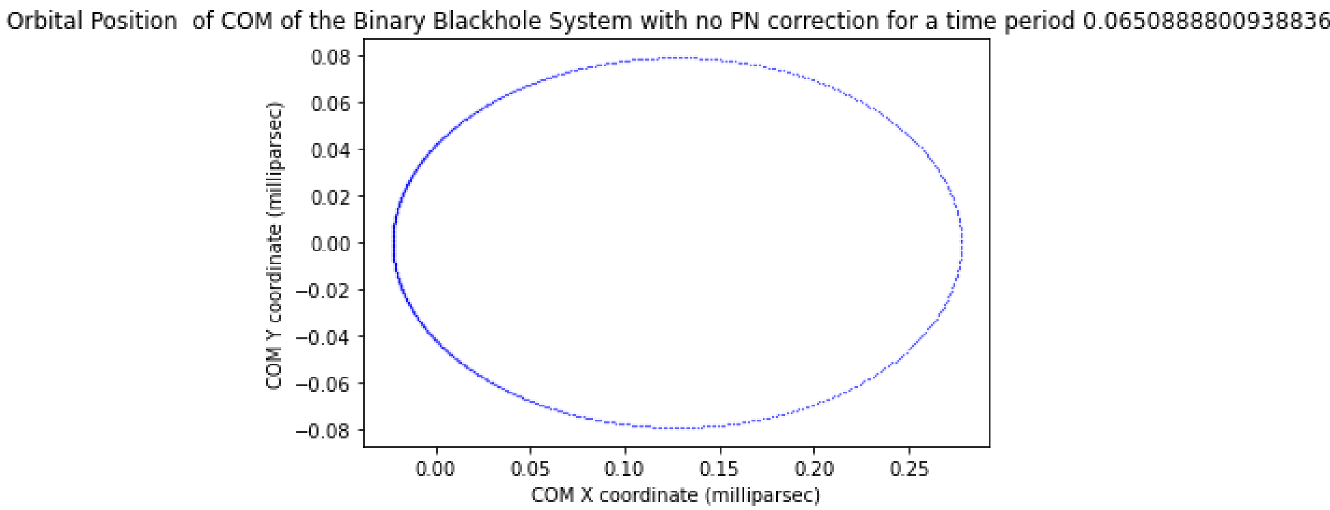

Figure 3.1: Orbit of the COM of the given black hole system plotted using the variable time step Leapfrog integrator with e = 0.15. As the orbit follows Keplerian equations of motion, perihelion advance is not observed with the PN0 correction.

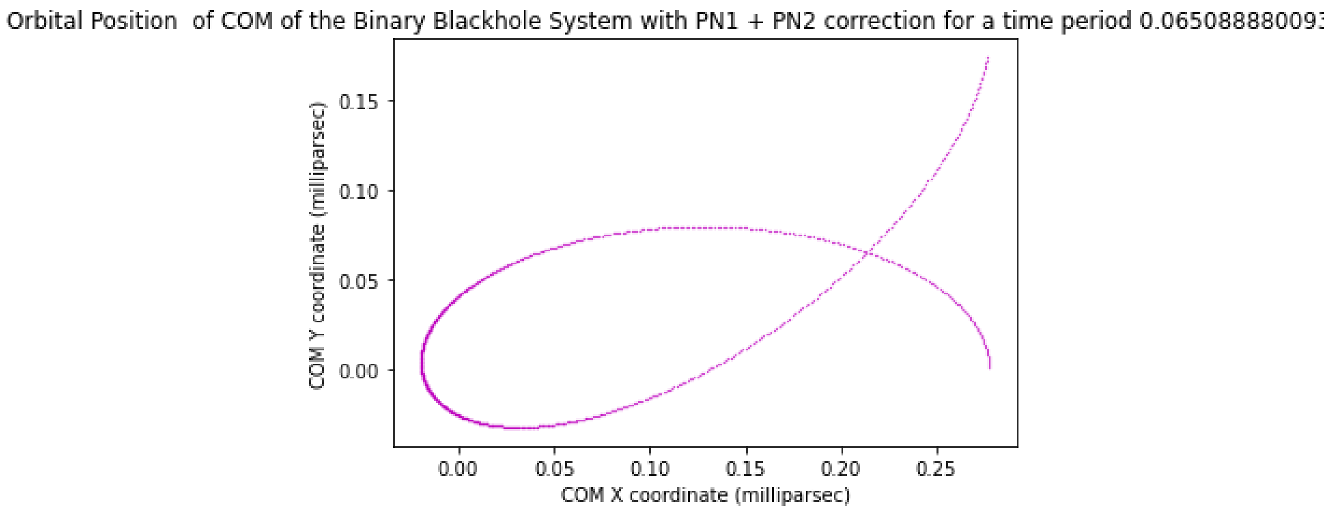

Figure 3.2: Orbit of the COM of the given black hole system plotted using the variable time step Leapfrog integrator with e = 0.15, η = 0.01. The perihelion advance can be observed when both PN1 and PN2 corrections are implemented

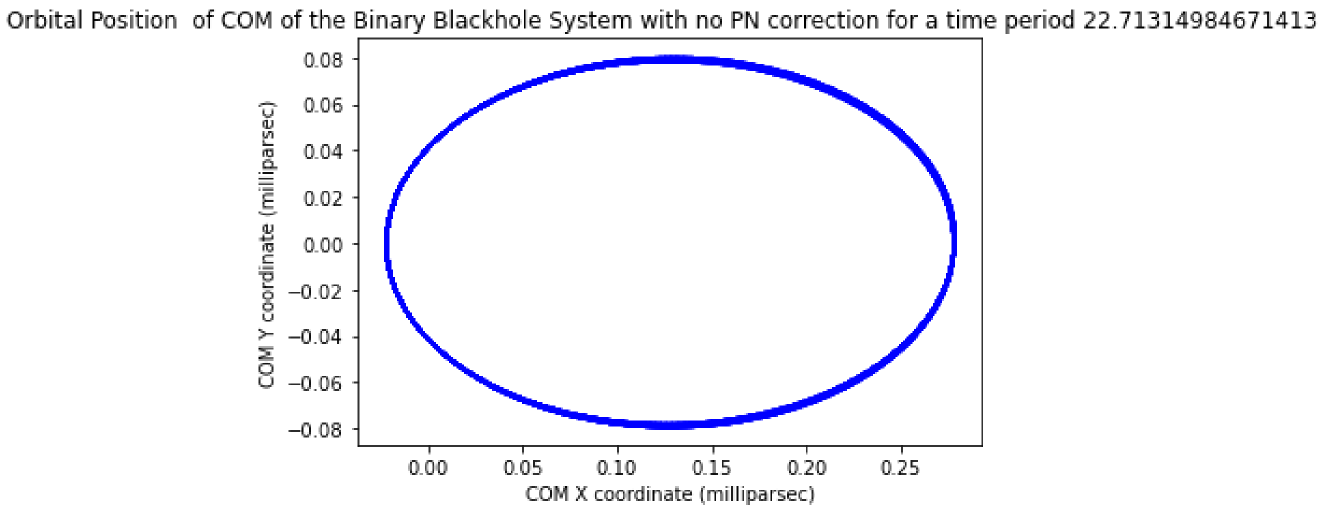

Figure 3.3: Orbit of the COM of the given black hole system plotted using the variable time step Leapfrog integrator with e = 0.15,$\eta = 0.01$.The orbital motion is plotted for a time period equal to t_p, where t_p is the time taken for a 360^o precession. However, a complete precession is not observed with the PN0 correction

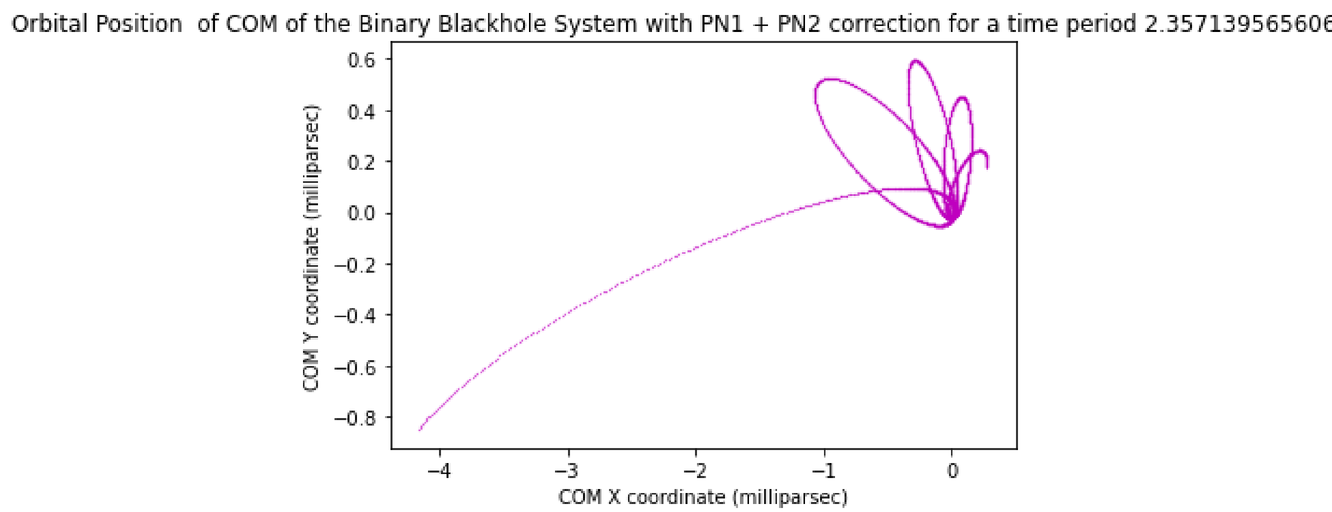

Figure 3.4: Orbit of the COM of the given black hole system plotted using the variable time step Leapfrog integrator with e = 0.15,η = 0.01. A complete 2π radians precession is not observed with the accuracy parameter used

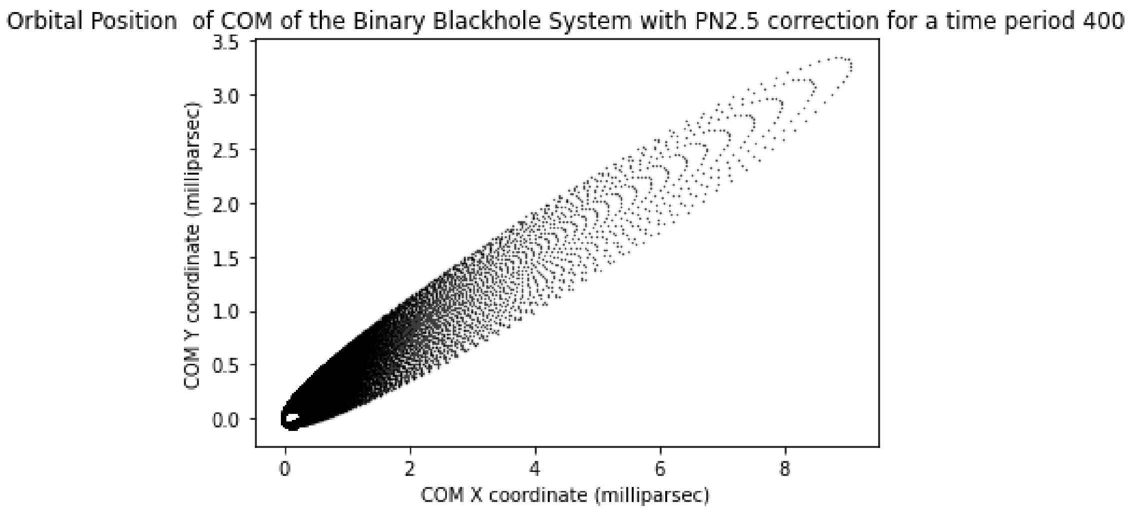

Figure 3.5: Orbit of the COM of the given black hole system plotted using the variable time step Leapfrog integrator with e = 0.15,η = 0.01. It can be observed that the orbital motion spirals inward due to the emission of gravitational waves

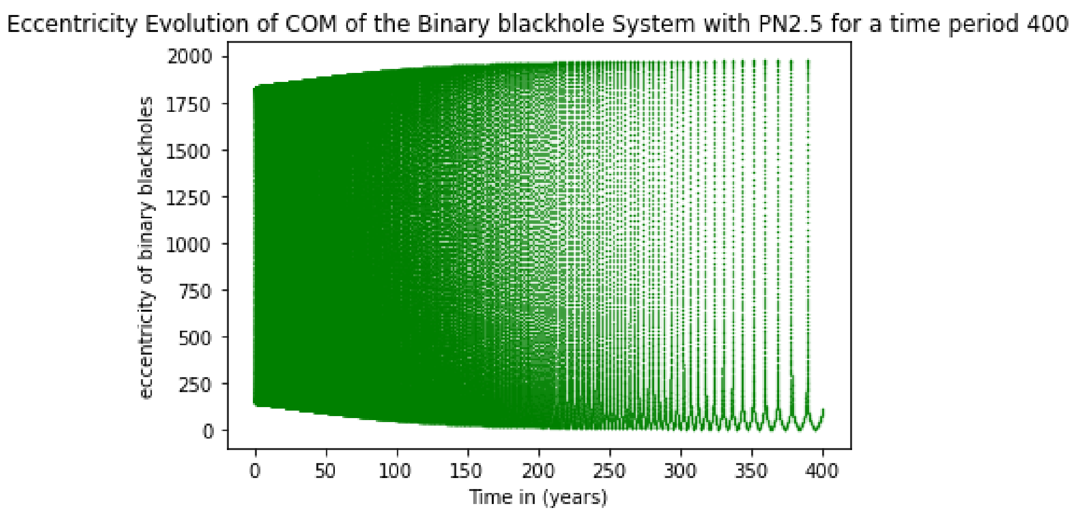

Figure 3.6: Eccentricity vs time of the COM of the given black hole system plotted using the variable time step Leapfrog integrator with e = 0.15,η = 0.01

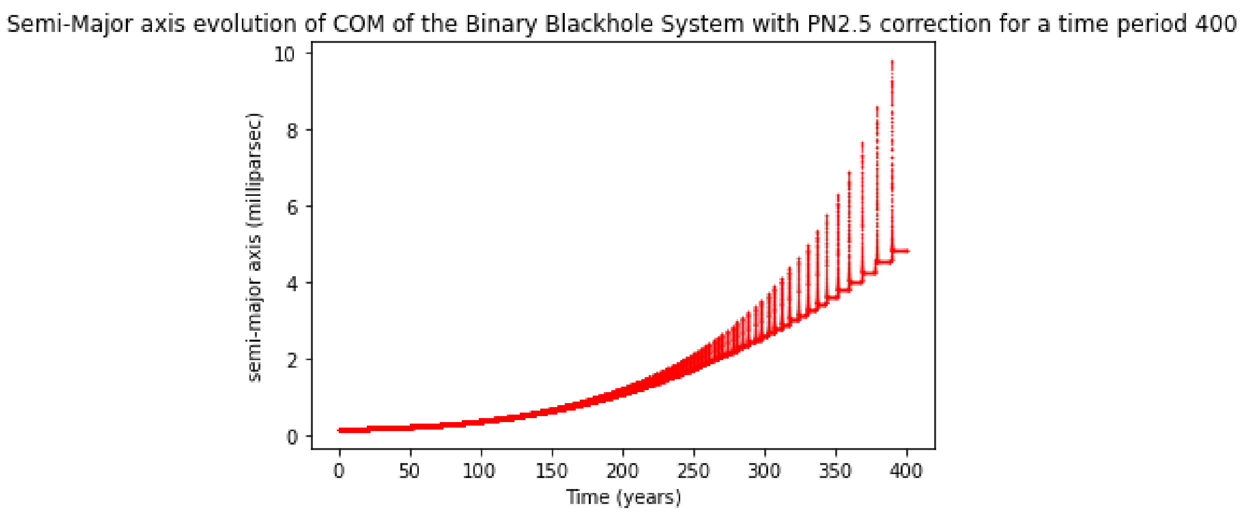

Figure 3.7: Semi-major axis vs time of COM of the given black hole system plotted using the variable time step Leapfrog integrator with e = 0.15,η = 0.01

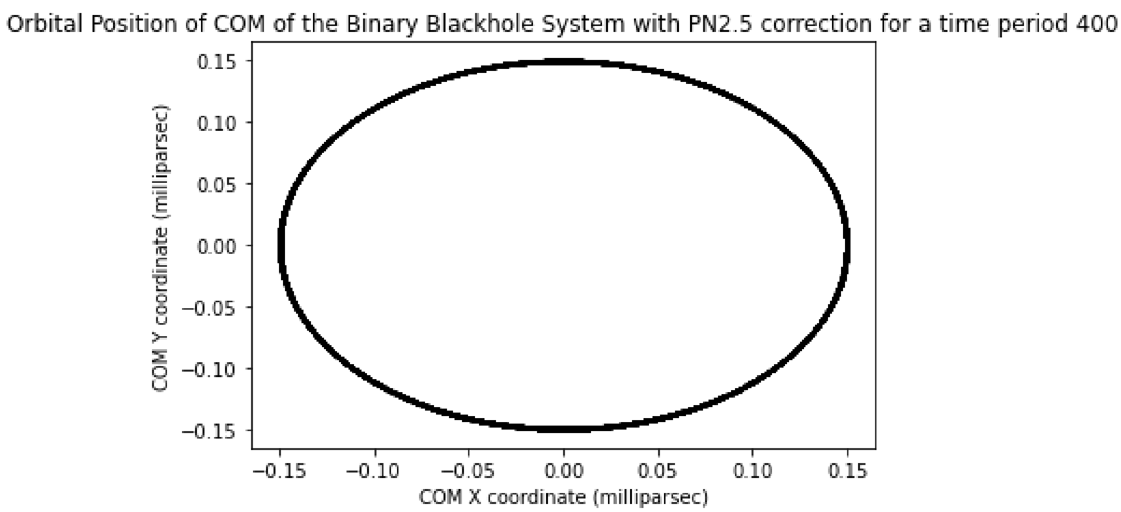

Figure 3.8: Orbit of the COM of the given black hole system plotted using the variable time step Leapfrog integrator with e $= 0$,$\eta = 0.01$

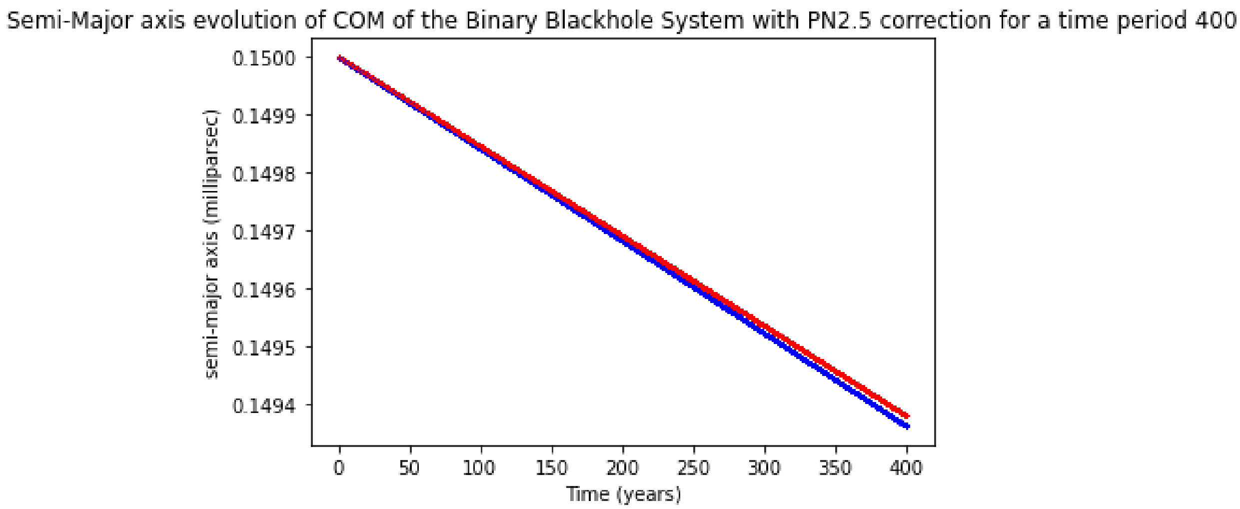

Figure 3.9: Theoretical and numerical Semi-major axis vs time of COM of the given black hole system plotted using the variable time step Leapfrog integrator with e = 0,η = 0.01. This implies that the binary black holes in circular orbits merge with increasing time because the semi-major axis is decreasing linearly with time

Chapter 4

Discussion

It is observed that the accuracy of the plots depends on the accuracy parameter η. When $\eta = 0.0001$ is considered, the PN1 and PN2 corrections do not distinctively show the perihelion advance. Also, the time taken for the orbits to be plotted using the variable leapfrog algorithm is in the magnitude of hours when accuracy parameter as low as $1 * 10^{-6}$ is used. This implies that a better time step for the leapfrog algorithm should be chosen to reduce the calculation time of acceleration. However, the method confirms the emission of gravitational waves because– when the PN2.5 correction is used to calculate the acceleration of the COM of two black holes, the orbit spirals inwards and the semi-major axis decreases linearly with time. The results are in accordance with the theoretical predictions made by General Relativity.

After analysing the evolution of semi-major axis w.r.t. when PN2.5 correction is implemented, the eccentric black hole binaries appear to grow further apart while the circular binary black holes tend to coalesce.However, comparing with literature, the eccentric binaries should also emit gravitational waves and coalesce.

Thus, to improve the calculation time when PN corrections are implemented parallel computing is used to generate accurate simulations. Also, the algorithm should only calculate the necessary order PN corrections based on the requirements. So, the algorithm should contain switches that turn certain calculations off according to the user input.