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① Let $\ell \subseteq \overbrace{\{1,\dots,n\}}^{\text{nodes}}$ s.t. $|\ell| = \ell$.

Define $X_{\ell+1} := \mathbb{1}\left[\ell+1 \text{ forms a path of length } \ell\right]$

Let $N = \sum_{(\ell+1)} X_{(\ell+1)}$

$n^{\ell+1}p^{\ell} = 1$

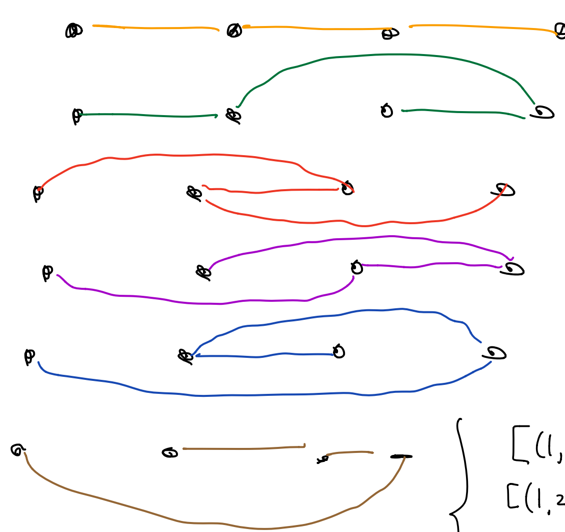
$p = n^{-\frac{\ell+1}{\ell}}$

Then by first moment method we have

$$P\{N \geq 1\} \leq E\{N\} = \underbrace{\binom{n}{\ell+1}}_{\text{choose nodes}} \times \underbrace{\frac{(\ell+1)!}{2}}_{\text{constant depicting number of unique paths on } (\ell+1) \text{ nodes}} p^{\ell} = \Theta\left(n^{\ell+1}p^{\ell}\right)$$

$$= \Theta\left(n\delta^{\ell}\right)$$

$$\boxed{P\{N \geq 1\} \to 0 \quad \text{if} \quad p = o\left(n^{-\frac{\ell+1}{\ell}}\right)}$$



[(1,4),(4,3),(3,2)]

[(1,2)(2,3)(3,4)]
[(1,2)(2,4)(4,3)]
[(1,3),(3,2),(2,4)]
[(1,3)(3,4)(4,2)]
[(1,4)(4,2)(2,3)]

$\boxed{\text{2nd moment method}}$:

Note that $\boxed{E\{N\} = \binom{n}{l+1}\frac{l!}{2}p^l \sim \frac{n^{l+1}}{l+1}p^l \xrightarrow{n\to\infty} \infty}$

$$P\{N=0\} = P\{N - E(N) = -E(N)\}$$

$$\leq \frac{Var(N)}{(E(N))^2} \qquad Var(N) = E\{N^2\} - (E\{N\})^2$$

Define $s, t \subseteq V$ and $X_s = \mathbb{1}[s \text{ forms a path}]$

Suffices to show:

$$\sum_{s\sim t} E\{X_s X_t\} = o\left(E(N)\right)^2$$

We want to show X_s and X_t becomes asymptotically independent.

Since $n^{-\frac{l+1}{l}} < \frac{1}{n}$

WLOG assume $p = o\left(\frac{1}{n}\right)$, the setting where no cycles form asymptotically.

Define $I = s \cap t$. I must be connected, otherwise there is a cycle which contradicts the setting we're in.

Hence I must be a tree.

Each leaf must be connected to the sets that exclude I in order for a path to form. Note that then I

must be a path of length $|I|-1$ and $s \setminus I$ and $t \setminus I$

connects to it at leaves.

$$P\{\text{forming a paths in } s \text{ and } t\} = \left(p^{l-|I|}(l-|I|)!\right)^2$$

Hence

$$E\{X_s X_t\} = \frac{1}{2}|I\ |!\, p^{|I|-1} p^{2l-2|I|} \left[(l-|I|)\right]^2$$

$$\sim p^{2l-|I|-1}\left[(l-|I|)!\right]^2 |I|!$$

$$\sim p^{2l-|I|-1}$$

Fix s. The number of sets t w/ r elements in common for

$$0 < r < (l+1) \text{ is precisely}$$

$$\binom{l+1}{r}\binom{n-(l+1)}{l+1-r} \leq \frac{(l+1)^r\,(n-(l+1))^{l+1-r}}{r!\,(l+1-r)!}$$

Hence:

$$\sum_{s \sim t} E\{X_s X_t\} \leq \sum_{r=1}^{(l+1)-1} \binom{n}{l+1} \frac{(l+1)^r\,(n-(l+1))^{l+1-r}}{r!\,(l+1-r)!}$$

$$\times\, E\{X_s X_t \mid |s \cap t| = r\}$$

$$\sim \sum_{r=1}^{(l+1)-1} n^{l+1}\, n^{l+1-r}\, p^{2l-r-1}$$

$$\leq l\, n^{l+1}\, n^{l+1-l}\, p^{2l-l-1}$$

$$\sim n^{l+2} p^{l-1} \qquad = o\left(E\{X\}^2\right)$$

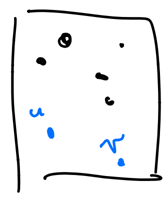
∎

Exercise 5.53 A RANDOM GRAPH

Consider a graph on n nodes in which each node independently picks one other node to connect to. Double edges are collapsed into one edge. Obtain first order asymptotics for the expected number of connected components.

<u>Proof</u>: Let's determine how p behaves asymptotically.

Let u, v be two nodes in $G_{n,p}$.



The absence of the edge (u,v) depends u not picking v <u>and</u> v not picking u:

$$P\{(u,v) \text{ does not exist}\} = \left(\frac{n-2}{n-1}\right)\left(\frac{n-2}{n-1}\right)$$

$$\Rightarrow \quad P\{(u,v) \text{ exists}\} = 1 - \left(\frac{n-2}{n-1}\right)\left(\frac{n-2}{n-1}\right) \qquad \delta = c/n$$

$$= 1 - \frac{n^2-4n+2}{n^2-2n+1}$$

$$= \frac{n^2-2n+1-n^2+4n-2}{n^2-2n+1}$$

$$= \frac{2n-1}{n^2-2n+1} = \frac{2-\frac{1}{n}}{n-2+\frac{1}{n}} \sim \frac{2}{n}$$

Hence $p \sim \frac{c}{n}$ where $c = 2$.

Anyways, notice that every connected component is a cycle. Hence to find the # of components we count the number of cycles:

Define N_k := number of cycles of length k.

$$E\{N\} = \sum_{k=2}^{n} E\{N_k\}$$

$$= \sum_{k=2}^{n} \binom{n}{k} \frac{k!}{2k} \left(\frac{1}{(n-1)} \right)^k$$

as seen in class: $\sim \sum_{k=2}^{n} \frac{\cancel{n^k}}{\cancel{k!}} \frac{\cancel{k!}}{2k} \frac{1}{\cancel{n^k}}$

$$= \frac{1}{2} H_n - \frac{1}{2}$$

$$\sim \boxed{\frac{1}{2} \log(n)}.$$

Exercise 5.34 Coloring Random Graphs.

A vertex coloring of a graph using K colors is an assignment of colors to vertices such that no two adjacent vertices have the same color. Consider a graph $G_{n,p}$ with $p = \frac{c}{n}$, c constant.

(i) Show that for all c large enough (and say for which!), the probability that $G_{n,p}$ can be 3-colored tends to 0.

(Hint: Use the first moment method on the number of partitions of nodes into three sets that have no edges within)

Proof: $\underline{\underline{X}} = \mathbb{1}\left[\begin{array}{l} A, B, C \text{ partition } \{1, 2, \ldots, n\} \\ \text{and no edges lie with } A, B \text{ and } C \end{array} \right]$

$E\{X\}$

$$\leq \sum_{K=0}^{n} \sum_{j=0}^{n-K} \binom{n}{K} \binom{n-K}{j} (1-p)^{\frac{K(K-1)}{2} + \frac{(j)(j-1)}{2} + \frac{(n-K-j)(n-K-j-1)}{2}}$$

$$\leq \sum_{K=0}^{n} \sum_{j=0}^{n-k} \underbrace{\frac{n^K}{K!} \frac{(n-k)^j}{j!}}\, e^{-\frac{c}{2n}\left(\underbrace{K(K-1)} + j(j-1) + \underbrace{(n-k-j)(n-K-j-1)}\right)}$$

$\cancel{K^2} - K + \cancel{j^2} - j$

$n^2 - nk - nj - n$

$- nk + \cancel{k^2} + kj + k$

$- nj + jk + j^2 + j$

$$= \sum_{K=0}^{n} \sum_{j=0}^{n-k} \frac{n^k}{k!} \frac{(n-k)^j}{j!} \exp\left\{ -\frac{ck^2}{2n} - \frac{cj^2}{2n} - \frac{cn}{2} + cj + ck + \frac{c}{2} - \cancel{\frac{cjk}{n}} \right\}$$

$$\leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\left(n e^{-\frac{ck}{2n}+c}\right)^k}{k!} \frac{\left([n-k] e^{-\frac{cj}{2n}+c}\right)^j}{j!} \exp\left\{-\frac{cn}{2}+\frac{c}{2}\right\}$$

$$\leq \left[\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\left(n e^{-c\left(\frac{k}{2n}-1\right)}\right)^k}{k!} \frac{\left([n-k] e^{-c\left(\frac{j}{2n}-1\right)}\right)^j}{j!}\right] \exp\left\{-\frac{cn}{2}+\frac{c}{2}\right\}$$

$$\leq \left[\sum_{k>4n}^{\infty} \frac{\left(n e^{-c\left(\frac{1}{1}-1\right)}\right)^k}{k!} + \sum_{k=0}^{4n} \frac{\left(n e^{-c\left(\frac{k}{2n}-1\right)}\right)^k}{k!}\right]\left[\sum_{j>4n}^{\infty} \frac{\left([n-k] e^{-c\left(\frac{1}{1}-1\right)}\right)^j}{j!} + \sum_{j=0}^{4n} \frac{\left([n-k] e^{-c\left(\frac{j}{2n}-1\right)}\right)^j}{j!}\right] \exp\left\{-\frac{cn}{2}+\frac{c}{2}\right\}$$

$$\leq \left[\sum_{k>2n}^{\infty} \frac{\left(n e^{-c\left(\frac{1}{1}-1\right)}\right)^k}{k!} + \sum_{k=0}^{2n} \frac{\left(n e^{-\frac{c}{2n}}\right)^k}{k!}\right]\left[\sum_{j>2n}^{\infty} \frac{\left([n-k] e^{-c\left(\frac{1}{1}-1\right)}\right)^j}{j!} + \sum_{j=0}^{2n} \frac{\left([n-k] e^{-\frac{c}{2n}}\right)^j}{j!}\right] \exp\left\{-\frac{cn}{2}+\frac{c}{2}\right\}$$

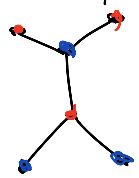
$$\leq \left[\sum_{k>2n}^{\infty} \frac{\left(n e^{-c\left(\frac{1}{1}-1\right)}\right)^k}{k!} + \sum_{k=0}^{\infty} \frac{\left(n e^{-\frac{c}{2n}}\right)^k}{k!}\right]\left[\sum_{j>2n}^{\infty} \frac{\left([n] e^{-c\left(\frac{1}{1}-1\right)}\right)^j}{j!} + \sum_{j=0}^{\infty} \frac{\left([n] e^{-\frac{c}{2n}}\right)^j}{j!}\right] \exp\left\{-\frac{cn}{2}+\frac{c}{2}\right\}$$

$$= \left(e^n + e^{n e^{-\frac{c}{2n}}}\right)\left(e^n + e^{n e^{-\frac{c}{2n}}}\right)\left(e^{-\frac{cn}{2}+c/2}\right)$$

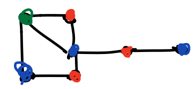
$$= \left(e^{2n} + 2e^{n + ne^{-c/2n}} + e^{2n e^{-\frac{c}{2n}}} \right) \left(e^{-\frac{cn}{2} + c/2} \right) \qquad (*)$$

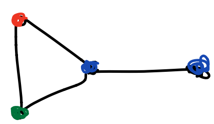
So choose $c \geq 4$ to make $(*) \longrightarrow 0$.

5.34 (ii) Show for c small enough, one can indeed 3-color a $G_{n,p}$ with prob. going to one. (Hint: argue in terms of trees and cycles. How many colors are needed for a tree, for a component w/ one cycle, and for a component w/ more than one cycle.)

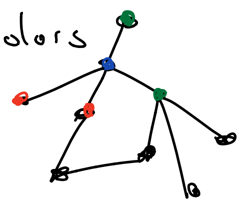


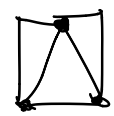
requires 2 colors for trees

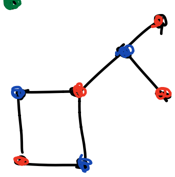




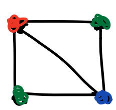
triangles require 3-colors

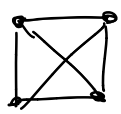






requires 2 colors





$\Rightarrow$ not 3-colorable (has more than one cycle)

* See notes for this result *

If we let $c \leq 1$: the number of nodes that do not live in isolated trees is $o(n)$. In addition, w/ high probability no component in the graph has more than one cycle.

Proof: Trees only require 2 colours. A component containing one cycle requires at most 3 colours.

Thus if $c \leq 1$ $\quad P\{G_{n,\frac{c}{n}} \text{ is 3-colourable}\} \xrightarrow{n\to\infty} 1$.

∎

Exercise 5.43 · Figs. Connectivity

Assume that a given degree sequence $d_1, d_2, \dots, d_n$ is given, and that its sum (denoted by $2e$) is even. Then one can define a random graph in which node i has degree d_i in the following manner:

Attach to node i d_i half edges. Match the half-edges up uniformly at random to make full edges. Call such a random graph a fixed-degree graph, or fig, for short. Observe that loops and multiple edges can occur.

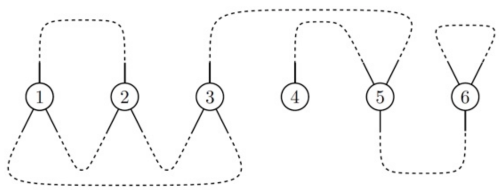


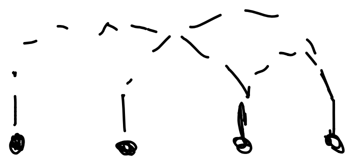
Figure 5.7. Half-edges are shown in solid lines and the edges matching them up in dashed lines. In this particular example, the graph is connected. There is a loop at vertex 6, and there is a double edge between 1 and 2.

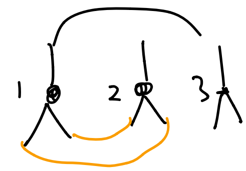
An equivalent definition is this: $\forall i$, throw d_i balls into an urn with the number i written on them. Remove the balls from the urn one by one (uniformly at random) and line them up to form a list of length $2e$. Break up the sequence up in consecutive pairs to define the edges. In the figure above, a possible sequence could have been:

$$(23)(54)(66)(35)(56)(12)(21)(31)$$

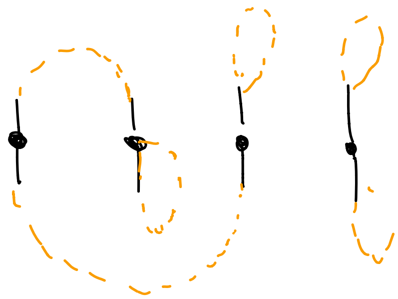
A K-regular fig is one in which all degrees are equal to K.

When $k=0$, there are no edges. When $k=1$ and n is even, we obtain a uniform random matching of the n vertices.





When $k=2$:



$(1\ 2)(1\ 2)(1\ 3)$

One obtains a collection of disjoint cycles.

The first interesting case is $k=3$. Show that the probability that a 3-regular fig is connected tends to one.

$\boxed{\text{Soln \#1}}$

Proof: Note that a graph is not connected

$\iff \exists$ partitions $A \sqcup B = \{1, \dots, n\}$ s.t. A and B are disconnected, i.e. $A \not\Leftrightarrow B$

Define $X :=$ # ways G can be partitioned into A, B s.t. $A \not\Leftrightarrow B$

Hence it suffices to show

$$E\{X\} \xrightarrow{n\to\infty} 0.$$

Let $X_A := \mathbb{1}[A \not\Leftrightarrow G \setminus A]$ i.e. $A \subseteq \{1, \dots, n\}$

$$E\{X\} = \sum_{\substack{A \subseteq V \\ |A| \text{ is even}}} E\{X_A\}$$

$|A|$ must be even or else cannot be disconnected.

The # of ways of connecting a 3-regular graph is

$$\frac{(3n)!}{(6\sqrt{8})^n}$$

$\leftarrow$ draw $3n$ balls from urn

$\leftarrow$ accounts for overcounting

$$\Rightarrow \log \frac{(3n)!}{(6\sqrt{8})^n} \sim 3n\log(3n) - n\left(3 + \log(6\sqrt{8})\right)$$

Let $D = 3 + \log(6\sqrt{8})$

$$\log\left[P\{A \not\Leftrightarrow G \mid A\}\right]$$

$$= 3|A|\log(3|A|) - |A|D + 3(n-|A|)\log(3(n-|A|)) - (n-|A|)D - 3n\log(n) + nD$$

$$= 3|A|\log(3|A|) - 3|A|\log(3(n-|A|)) + 3n\left(\log(3(n-|A|)) - \log(3n)\right)$$

$$= 3|A|\log\left(\frac{|A|}{n-|A|}\right) + 3n\log\left(\frac{n-|A|}{n}\right)$$

$$= 3\log\left(\frac{|A|^{|A|}(n-|A|)^{n-|A|}}{n^n}\right)$$

$$\sim \frac{|A|^{3|A|}(n-|A|)^{3(n-|A|)}}{n^{3n}}$$

Thus

$$E\{X\} \sim \sum_{|A|=1}^{n-1} \binom{n}{|A|} \frac{|A|^{3|A|}(n-|A|)^{3(n-|A|)}}{n^{3n}}$$

$$\leq \sum_{|A|=1}^{n-1} \frac{n^n}{|A|^{|A|}(n-|A|)^{n-|A|}} \frac{|A|^{3|A|}(n-|A|)^{3(n-|A|)}}{n^{3n}}$$

$$\leq \sum_{|A|=1}^{n-1} \boxed{\frac{|A|^{2|A|}(n-|A|)^{2(n-|A|)}}{n^{2n}}}$$ → maximized when $|A|=2$

$$\leq \sum_{|A|=1}^{n-1} \frac{16(n-2)^{n-2}}{n^n}$$

$$\leq \sum_{|A|=1}^{n-1} \frac{(n)^{n-2}}{n^n}$$

$$\leq \sum_{|A|=1}^{n-1} \frac{1}{n^2}$$

$$= O\left(\frac{1}{n}\right)$$

<u>ie</u> $\lim_{n\to\infty} E\{X\} = 0$

$\boxed{\text{Solution 2}}$: Let $V = \{1, \dots, 2n\}$

Define $v(2k) = (2k-1)(2k-3)\cdots(3)(1)$

$$= \frac{(2k)!}{2^k \cdot k}$$

$\alpha\left(\frac{2}{e}\right)^k k^k \leq v(2k) \lesssim A\left(\frac{2}{e}\right)^k k^k$ by Stirling $\forall k \in \mathbb{N}$

Then define $A := \left[\begin{array}{l} A \sqcup B = \{1, \dots, n\} \\ |A| = 2i, \ |B| = 2j, \ 2i + 2j = 2n \end{array}\right]$

$$P\{A\} = \binom{2n}{2i} \frac{v(6i)\, v(6j)}{v(6n)}$$

$$\lesssim \frac{i^i j^j}{n^n} \quad \text{(absorbing constants)}$$

$\frac{i^i j^j}{n^n}$ is maximized when $i = 1$ and $j = n-1$

or $j = 1$ and $i = 1$

Since

$$\log(i^i j^j) = \underbrace{\underbrace{i \log i}_{\text{convex}} + \underbrace{j \log j}_{\text{convex}}}_{\text{sum of convex is convex}}, \quad i + j = n$$

(max takes value at boundary)

$$P\{G_{2n} \text{ is not connected}\} \lesssim A \sum_{i=1}^{n-1} \frac{i^i j^j}{n^n}$$

$$= \cancel{A}\frac{(n-1)^{n-1}}{n^n} + \frac{\cancel{A}}{2}\underbrace{\sum_{i=2}^{n-2}}_{(n-3)\text{ terms}} \frac{i^i j^j}{n^n}$$

Now $\max\left\{ \frac{i^i \cdot j^j}{n^n} \right\}$ occurs when $i = n-2 \wedge j = 2$

<u>or</u> $j = n-2 \wedge i = 2$

Hence, we have:

$$P\{G_{2n} \text{ not connected}\} \le \left(\frac{n^{n-1}}{n^n} + \underline{(n-3)} \frac{(n-2)^{n-2}}{n^n} \right)$$

$$= O\left(\frac{1}{n}\right)$$

$$\longrightarrow 0$$

■