## §2.14. Assignment 1, Fall 2020

**Exercise 2.59.** THE BERNOULLI DISTRIBUTION. Let $X$ be a random variable such that for all sttrictly positive integers $k$, $\mathbb{E}\{X^k\} = p$, where $0 \le p \le 1$. Using only material from this chapter, show that $X$ must be Bernoulli $(p)$.

**Exercise 2.60.** THE UNIT CUBE. Let $Q = [0,1]^d$ be the unit cube of $\mathbb{R}^d$, and let $X = (X_1, \ldots, X_d)$ be uniformly distributed inside $Q$.

(i) Show that $\mathbb{P}\{X_1 + \cdots + X_d \le 1\} = 1/d!$.

(ii) Derive the distribution function of $d \min_i X_i$. What distribution do you get when $d \to \infty$?

(iii) For symmetry, assume that $X$ is uniform in $[-1,1]^d$. Even though this is cubic in shape, show that $X$ lives with high probability near the surface of a ball: if $||X||$ is the Euclidean norm of $X$, and $\epsilon > 0$, then $\mathbb{P}\{(1-\epsilon)\sqrt{d/3} \le ||X|| \le (1+\epsilon)\sqrt{d/3}\} \to 1$ as $d \to \infty$.

**Exercise 2.61.** THE HYPERCUBE. Let $X = (X_1, \ldots, X_d)$ be uniform on the hypercube $\{0,1\}^d$.

(i) What is the distribution of $\sum_{i=1}^d X_i$?

(ii) Let $X, Y$ be independent and distributed as $X$, and let $H(X,Y)$ be the Hamming distance, i.e., the number of coordinates in which $X$ and $Y$ differ. Show that $\mathbb{E}\{|H(X,Y) - d/2|\} = O(\sqrt{d})$ (so, almost all pairs of points are at Hamming distance close to $d/2$ from each other...).

**Exercise 2.62.** BRAIN TEASER: OPTIONAL. Let $X$ be binomial $(n,p)$. There is a relatively simple formula for the probability that $X$ is a multiple of three (zero included):

$$\frac{1}{3} + \frac{2}{3}\theta^n \phi.$$

Here $\theta \in [0,1]$ is a simple function of $p$ and $\phi$ is a function of $n$ and $p$ taking values in $[-1,1]$. You are asked to determine $\theta$ and $\phi$.