Module 1.
Basic Statistics

Methodology Training
EXL Decision Analytics

Year 2013

EXL

GO NEXT. NOW.

Objectives and Scope

Course Goals

- Introduction to basic statistical terms
- Provide a structured overview of statistical concepts used during application of EXL DA methodology
- Explain interpretation and basis of statistical distributions and hypothesis testing
- Provide helpful "tricks of the trade"

Beyond the Scope of this Training

- Comprehensive coaching on Statistics
- Derivation of statistical formulas or terms (unless required as part of methodology explanation)

Self Study Goals

- In-depth research on advanced Statistical concepts
- Innovations and new techniques related to methodology
- Discussion on advanced concepts can be taken up offline

Table of Contents

Chapter 1: Dataset Basics

1.1 Data Set Dimensions

1.1.1. Data Set

A data set contains information in a set of rows and columns

- Row → Observation
- Column → Variable

← Variables →
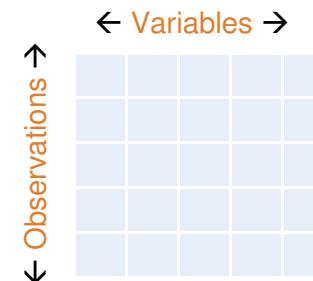
↑ Observations ↓

1.1.2. Observation

Observations are the objects described by a set of data. They may be anything.

Few examples:

- Individuals (like customers of a bank, employees of a company, students of a class)
- Transactions
- Accounts
- Zip Codes

1.1.3. Variable

A variable is any characteristic of an observation and can take different values for different observations.

Few examples:

- A customer's tenure or an employee's salary or a student's marks
- A transaction's date, time or amount
- An account's balance or spending limit
- A zip code's population

1.2 Variable Type

1.2.1. Categorical Variable

A categorical variable places each observation into one of the several groups or categories, like the gender variable with male or female categories

Such variables are best represented by a **Pie Chart** or a **Bar Graph**

Gender Distribution

Male 40%
Female 60%

Gender Distribution
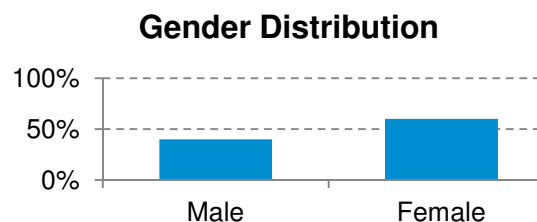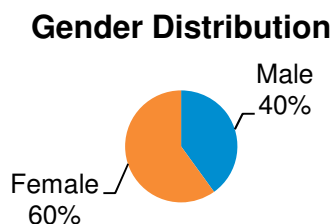
1.2.2. Quantitative Variable

A quantitative variable has numerical values that measure some characteristic of each observation, like height in centimeters or salary in dollars per year

Such variables are best represented by a **Histogram**

Income Distribution
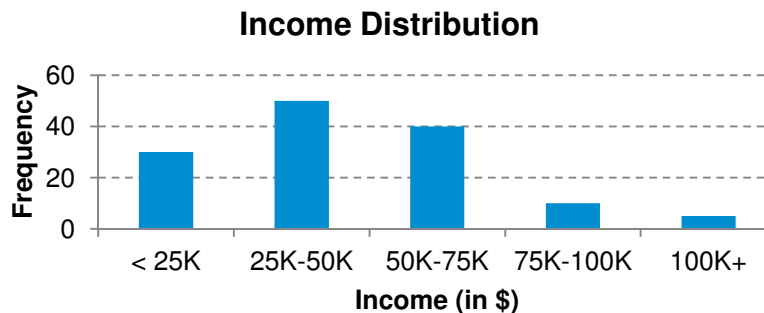
Things to Remember

Before drawing the histogram, the quantitative variable is divided into equal classes

Exercise

Exercise 1. In the 'Bar Graph' of Gender Distribution (in Section 1.2.1), there are two categories. In the 'Histogram' of Income Distribution (in Section 1.2.2), there are five categories. Otherwise, how are they different from each other?

[Hint: Does the sequence of categories matter?]

1.3 Scales of Measurement

1.3.1. Properties of Measurement Scales

	Four Key Properties Relating to Scales of Measurement			
	Each number has a particular meaning	Numbers have an inherent order from smaller to larger	Differences between numbers (units) anywhere on the scale are the same	Zero point represents the absence of the characteristic being measured
Scales	**Identity**	**Magnitude**	**Equal Intervals**	**True Zero**
Nominal Scale	?	?	?	?
Ordinal Scale	?	?	?	?
Interval Scale	?	?	?	?
Ratio Scale	?	?	?	?

1.3.2. Nominal Scale

Example: 'Marital Status'

- Consider a variable 'Marital Status' with three categories: Single, Married and Divorced
- Arbitrarily, a number is assigned to the three categories

Scenario A: Single denoted by '0', Married denoted by '1' and Divorced denoted by '9'

Marital Status

| 100% |
| 50% |
| 0% |

0 (Single)   1 (Married)   9 (Divorced)
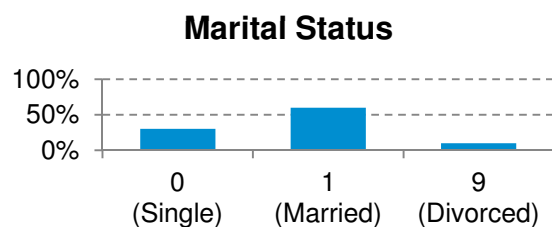
Scenario B: Single denoted by '1', Married denoted by '0' and Divorced denoted by '2'

Marital Status

| 100% |
| 50% |
| 0% |

0 (Married)   1 (Single)   2 (Divorced)
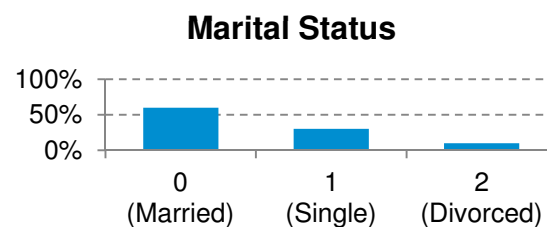
As Observed

1. Scale points '0' , '1' and '9' have specific meanings with respect to a scenario

2. Scale points are mere names of the categories. The categories on nominal scale can not be rank-ordered

3. Difference between two consecutive scale points is not 'equal'. In fact, calculation of such difference is not even meaningful.

4. The scale point '0' is not the 'true' zero point (Here '0' does not mean 'absence' of marital status)

1.3.3. Ordinal Scale

Example: 'Preference Rank' for Car's Color

- Suppose a car buyer is asked to provide preference ranking for colors: white, black, red and blue

Color	Buyer's State of Mind	Scenario A	Scenario B
		Preference Rank (on a scale of 1 to 4)	Preference Rank (on a scale of 0 to 3)
Black	I simply love black	1	0
White	I strongly like white too	2	1
Red	Red looks very sporty	3	2
Blue	I hate blue	4	3

As Observed

1. Every scale point (preference rank) has a particular meaning with respect to a scenario

2. Each number on the scale is different from the other and all of them can be rank-ordered

3. Rank difference between the last two preferences can not be assumed to be equivalent to rank difference between the first two preferences. Clearly, the customer's preference for black over white is not as strong as his preference for red over blue.

4. The scale point '0' is not the 'true' zero point (In Scenario B, black color has '0' rank, which does not mean the absence of preference. Rather it is the most preferred color.)

1.3.4. Interval Scale

Example: Temperature Measurement on Celsius Scale

■ The thermometer identifies how many units of mercury correspond to the temperature measured
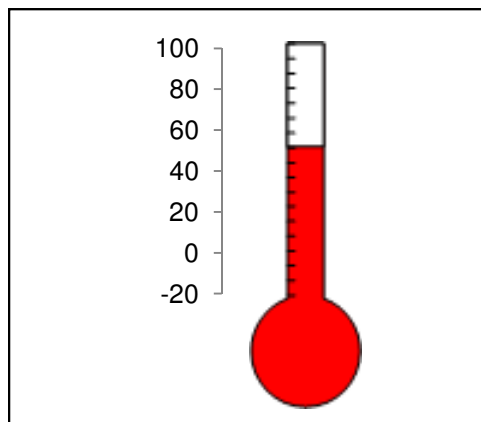
As Observed

1. Every scale point has a particular meaning

2. Each number on scale is different from the other and all of them can be rank-ordered (80℃ is hotter than 60℃)

3. There is the same 10-degree difference in temperature between 20° and 30° as between 50° and 60°

4. 0℃ is not a 'true' zero point. It does not indicate absence of temperature; it is an arbitrary point on the scale

1.3.5. Ratio Scale

Example: Purchasing Power

■ The purchasing power is measured in terms of money

Purchasing Power

$0 $25 $50 $75 $100 $125 $150
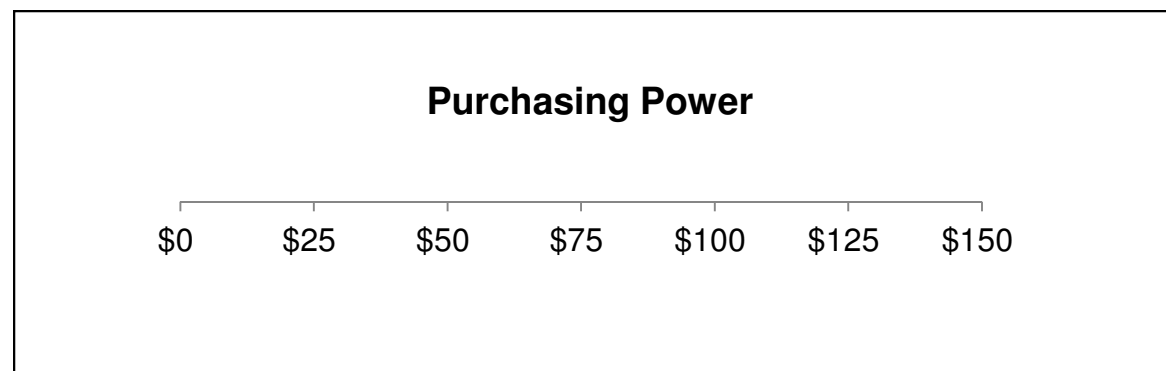
As Observed

1. Every scale point has a particular meaning

2. Numbers on scale can be rank-ordered ($150 has more purchasing power than $125)

3. There is the same $25 difference in purchasing power between $100 and $125 as between $50 and $75

4. $0 is a 'true' zero point. $0 means no money and absolutely no ability to purchase anything

1.3.6. Scales of Measurement: Summary

Four Key Properties Relating to Scales of Measurement

	Each number has a particular meaning	Numbers have an inherent order from smaller to larger	Differences between numbers (units) anywhere on the scale are the same	Zero point represents the absence of the characteristic being measured
Scales	**Identity**	**Magnitude**	**Equal Intervals**	**True Zero**
Nominal Scale	✔	✘	✘	✘
Ordinal Scale	✔	✔	✘	✘
Interval Scale	✔	✔	✔	✘
Ratio Scale	✔	✔	✔	✔

Exercise

Exercise 2. List down at least 3 examples each for

a. Nominal Scale Variable

b. Ordinal Scale Variable

c. Interval Scale Variable

d. Ratio Scale Variable

Chapter 2: Univariate Analysis

2.1 Univariate Analysis

2.1.1. Plotting and Visualizing a Distribution

Recall the Histogram of Dollar Income plotted in Section 1.2.2.

Income Distribution

After Adding a Polynomial Trend Line

Income Distribution
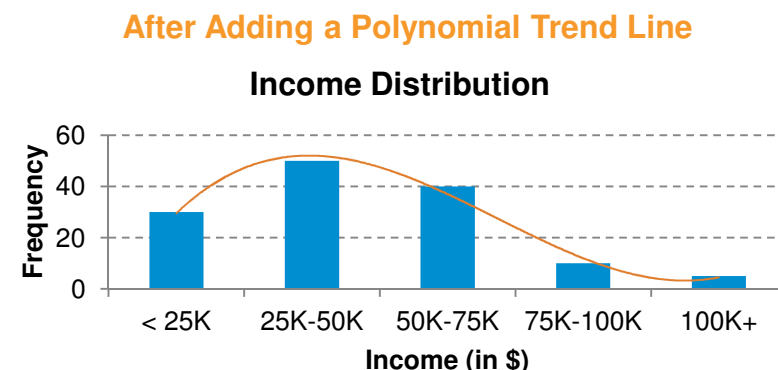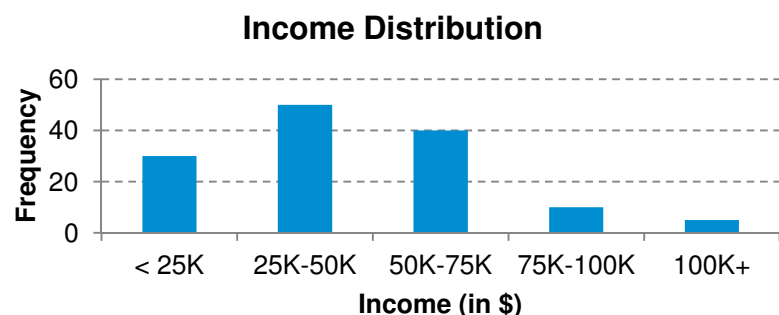
2.1.2. Examining a Distribution

Look for the **overall pattern** and for **striking deviations** from that pattern

Examining a Distribution	
Overall Pattern	**Striking Deviations** (Outliers)

Center (Measures of Central Tendency)	**Spread** (Measures of Dispersion)	**Shape** (Skewness / Kurtosis)
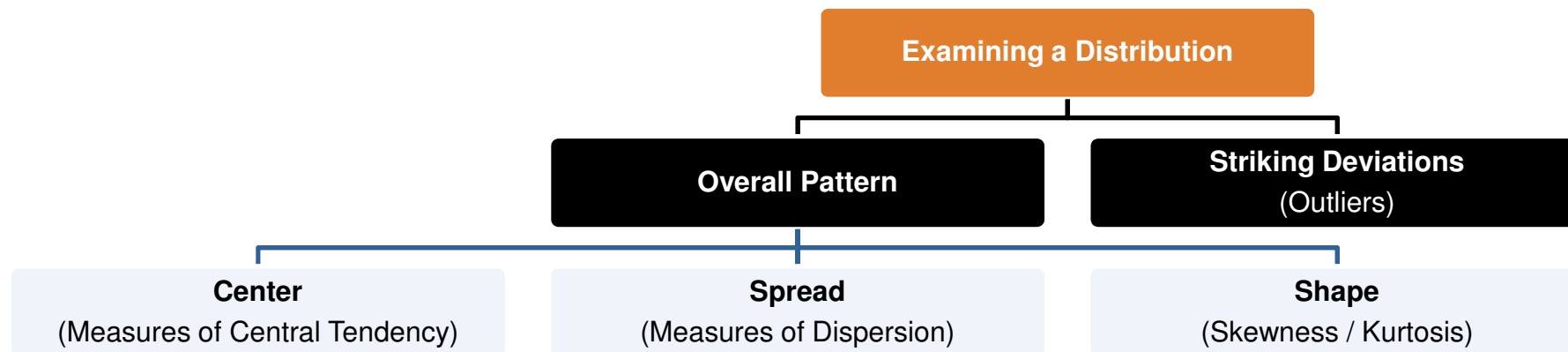
2.1.3. Percentiles

A percentile is the value of a variable below which a certain percent of observations fall.

The pth percentile is a value such that
- At most (p)% of the observations are less than this value; and
- At most (1 – p)% are greater

when the data is sorted

Examples:
- P1 (1st percentile) cuts off the lowest 1% of sorted data
- P98 (98th percentile cuts off the lowest 98% of sorted data

Term Used	Number of Splits
Quartiles	4
Deciles	10
Pentiles	20
Percentile	100

SAS Tip

PROC UNIVARIATE

Percentile

- P100 or Maximum
- P75 or Q_3
- P50 or Median or Q_2
- P25 or Q_1
- P0 or Minimum
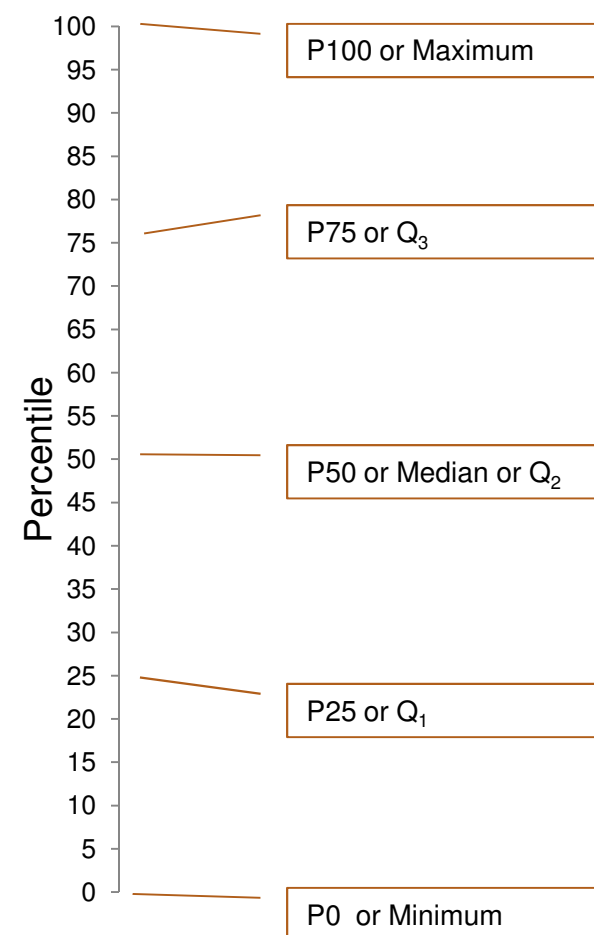
2.2 Center (Measures of Central Tendency)

2.2.1. Mean

Mean is arithmetic average of the observations.

$$\overline{X} = \frac{\sum\limits_{i=1}^{n} X_i}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

n : *Number of observations*

X_i : *i^{th} observation*

Think about it..

If frequency of any observation is more than 1 (e.g. if two students scored the same marks), how would the formula for mean calculation look like?

- The Most Common Measure of Central Tendency
- Affected by Extreme Values (Outliers)
- 'Center of Gravity' of a Distribution - the point on which the distribution would balance

Example: Find average marks scored by a group of 4 students: A (1 mark), B (3 marks), C (6 marks), D (10 marks)

0 1 2 3 4 5 6 7 8 9 10

Mean = 5
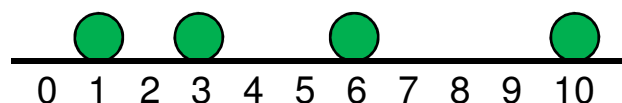
$$\overline{X} = \frac{1+3+6+10}{4} = \frac{20}{4} = 5$$

2.2.2. Median

Median is the midpoint of a distribution.

$$M = \left(\frac{n+1}{2}\right)^{th} Observation$$

- A Robust Measure of Central Tendency
- Not Affected by Extreme Values (Outliers)

🔔 Things to Remember
Before calculating median, always arrange all observations in order of size, from smallest to largest. However, if you are using PROC MEANS in SAS, there is no need to sort manually. SAS does it automatically.

If number of observations (n) is <u>odd</u>, the median (M) is the center observation in the ordered list

0 1 2 3 4 5 6 7 8 9 10
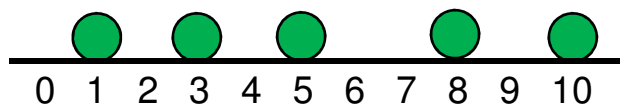
Median = 5

If number of observations (n) is <u>even</u>, the median (M) is the mean of the two center observations in the ordered list

0 1 2 3 4 5 6 7 8 9 10

Median = (5+7)/2 = 6
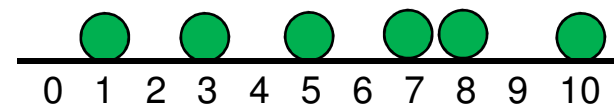
2.2.3. Mode

Mode is the most frequently occurring observation.

There may be multiple modes

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

Mode = 6

Mode = 12
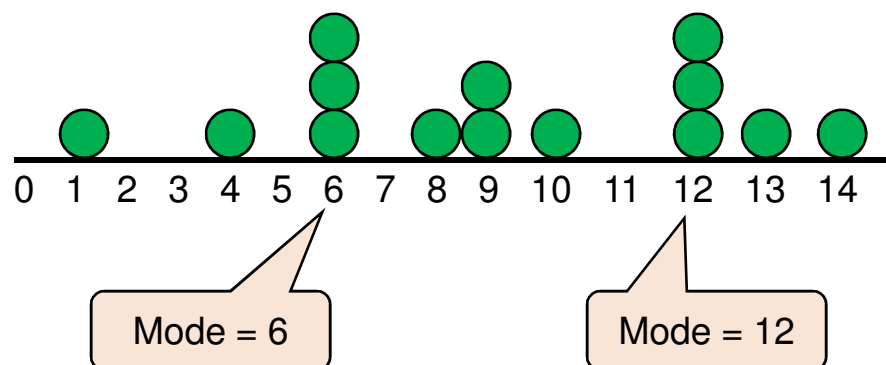
There may not be a mode

0 1 2 3 4 5 6

No Mode
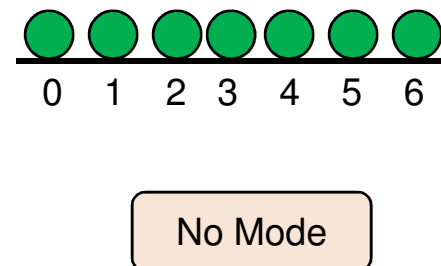
- Not Affected by Extreme Values (Outliers)
- Can be used for Numerical as well as Categorical Data

2.3 Spread (Measures of Dispersion)

2.3.1. Range

Range is the difference between the maximum and the minimum values

$$Range = MaxValue - MinValue$$

- By definition, Range is affected by Extreme Values
- Independent of data distribution

0 1 2 3 4 5 6 7 8 9 10

Range = 10 - 1 = 9
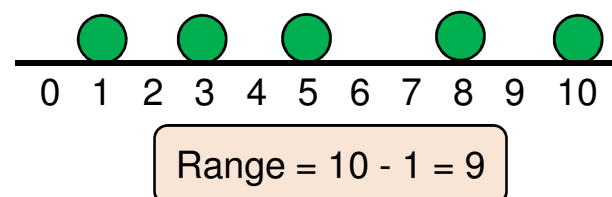
2.3.2. Inter-Quartile Range

Inter-Quartile Range is the difference between the first and the third quartile

- Mid-spread (Spread in the middle 50%)
- Independent of Extreme Values

$$IQR = Q_3 - Q_1$$

$$Q_i = \left(\frac{i \times (n+1)}{4} \right)^{th} Observation$$

Inter-Quartile Range

P0   P25   P50   P75   P100

25%	25%	25%	25%

(Q_1) (Q_2) (Q_3)

Median of 1st Half   Overall Median   Median of 2nd Half
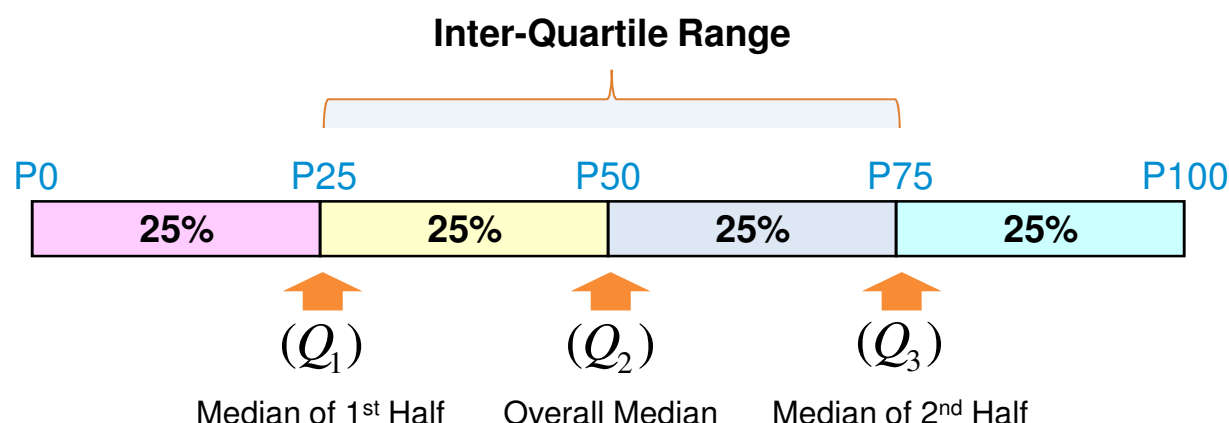
2.3.3. Variance

Variance is an average of the squares of the deviations of a set of observations from their mean.

$$\sigma^2 = \frac{(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 + \ldots + (X_n - \overline{X})^2}{n-1} = \frac{\sum\limits_{i=1}^{n}(X_i - \overline{X})^2}{n-1}$$

- Affected by Extreme Values (outliers)
- Variance = 0 if there is no spread (when all observations have the same value); Otherwise Variance > 0

2.3.4. Standard Deviation

Standard deviation is the square root of the variance.

$$\sigma = \sqrt{\frac{\sum\limits_{i=1}^{n}(X_i - \overline{X})^2}{n-1}}$$

SAS Tip
PROC MEANS

- Affected by Extreme Values (outliers)
- Std. Deviation = 0 if there is no spread (when all observations have the same value); Otherwise Std. Dev. > 0
- Std. Deviation has the same units of measurement as the original observations

2.3.5. Coefficient of Variation

Coefficient of variation (CV) is a normalized measure of dispersion of a probability distribution.

$$CV = \frac{\sigma}{\overline{X}} \times 100 \%$$

- CV is unit-less and hence can be used for comparing variation in data sets with different units
- When the mean value is close to zero, CV will approach infinity and is hence sensitive to small changes in the mean
- CV is sensitive to outliers

2.4 Shape (Skewness / Kurtosis)

2.4.1. Skewness

Skewness is the degree of departure from symmetry of a distribution.

■	Positively Skewed Distribution	: Tail pulled in the positive direction
■	Negatively Skewed Distribution	: Tail pulled in the negative direction
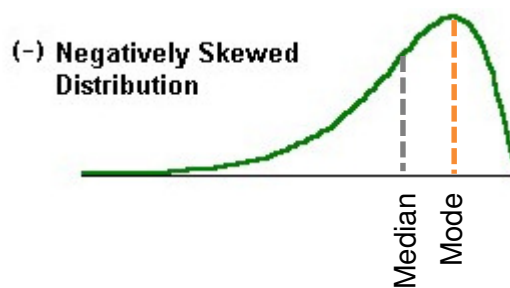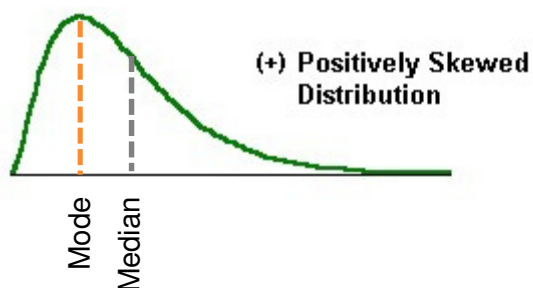
2.4.2. Kurtosis

Kurtosis is the degree of peakedness of a distribution.

■	**Mesokurtic Distribution**	: Normal distribution
■	**Leptokurtic Distribution**	: Higher peak and heavier tails
■	**Platykurtic Distribution**	: Lower peak and lighter tails

Mesokurtic Curve	: Kurtosis = 3
Leptokurtic Curve	: Kurtosis > 3
Platykurtic Curve	: Kurtosis < 3
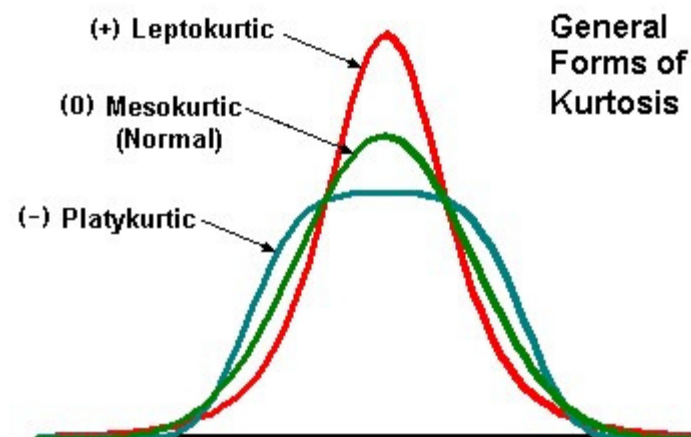
Example

Example: For 16 students, marks in Mathematics are given below:

40, 31, 19, 22, 47, 25, 24, 32, 43, 12, 20, 5, 6, 21, 43, 67

Let us calculate measures of central tendency and dispersion, without using SAS.

Arrange marks in ascending order:

5, 6, 12, 19, 20, 21, 22, 24, 25, 31, 32, 40, 43, 43, 47, 67

Inter-Quartile Range = $Q_3 - Q_1$ = 41.5 – 19.5 = 22

Q_1 = 19.5

Q_3 = 41.5

5 6 12 19 20 21 22 24 25 31 32 40 43 43 47 67

Min = 5

Median = 24.5

Mode = 43

Max = 67

Range = Max - Min = 67 – 5 = 62
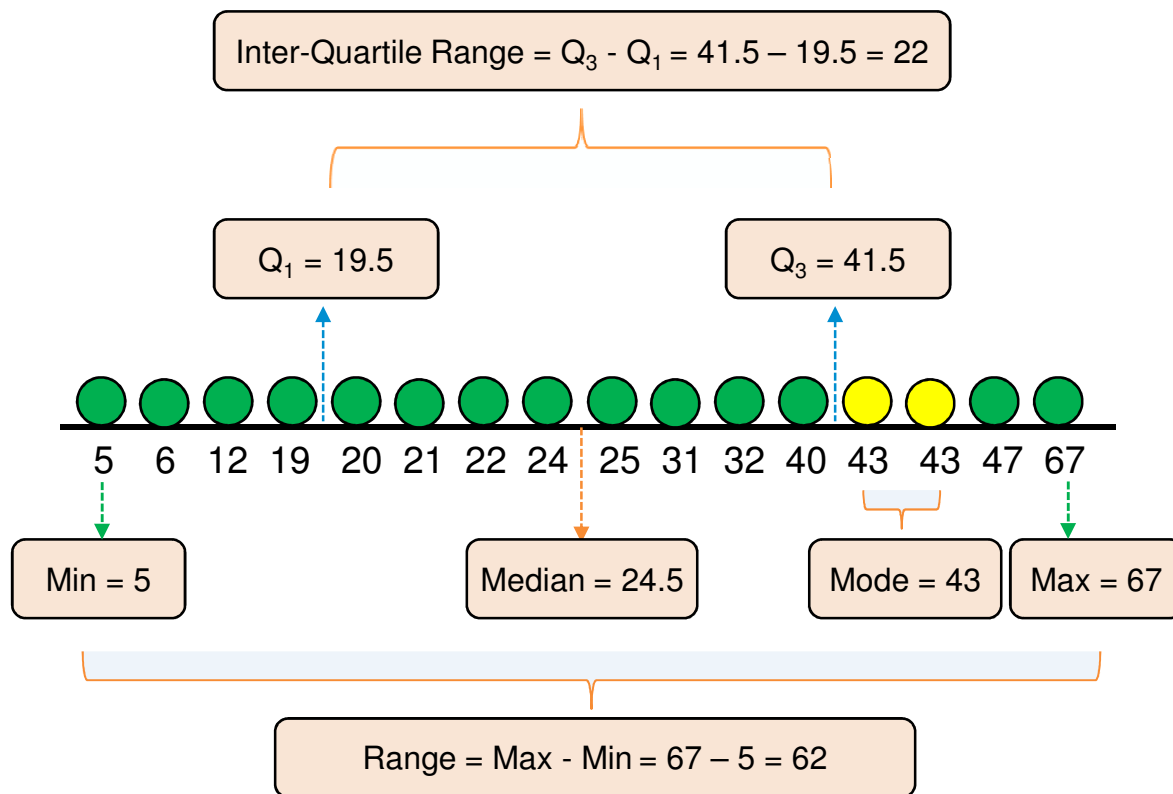
	A	B	C
1	**Marks (X)**	**X - Mean**	**(X-Mean)²**
2	5	-23.6	555.2
3	6	-22.6	509.1
4	12	-16.6	274.3
5	19	-9.6	91.4
6	20	-8.6	73.3
7	21	-7.6	57.2
8	22	-6.6	43.1
9	24	-4.6	20.8
10	25	-3.6	12.7
11	31	2.4	5.9
12	32	3.4	11.8
13	40	11.4	130.8
14	43	14.4	208.4
15	43	14.4	208.4
16	47	18.4	339.9
17	67	38.4	1477.4
18	**ΣX = 457**	**Σ(X - Mean) = 0.0**	**Σ(X-Mean)² = 4019.9**

$$\overline{X} = \frac{457}{16} = 28.56 \qquad \sigma = \sqrt{\frac{4019.9}{(16-1)}} = 16.37 \qquad CV = \frac{16.37}{28.56} = 57\%$$

Example

For the same example, let us compute measures of center, spread and shape using SAS.

Convert Raw Data to SAS Format

```
data outlib.marks_data;
infile datalines;
input marks;
datalines;
40
31
19
22
47
25
24
32
43
12
20
5
6
21
43
67
;
```

Run PROC UNIVARIATE on SAS Data Set

```
proc univariate data = outlib.marks_data modes;
    var marks;
    output out   = outlib.marks_univariate
    n            = n
    min          = min
    mean         = mean
    median       = median
    mode         = mode
    max          = max
    range        = range
    q1           = q1
    q3           = q3
    qrange       = qrange
    std          = std_dev
    skewness     = skewness
    kurtosis     = kurtosis
    ;
run;
```

Output Data Set

Output

	n	mean	std_dev	skewness	kurtosis	max	q3	median	q1	min	range	qrange	mode
1	16	28.5625	16.37058	0.68004	0.542626	67	41.5	24.5	19.5	5	62	22	43

marks_univariate.sas7bdat

2.5 Box Plot

A **box-plot** is a graph of the summary of a variable's distribution

- A central box spans the quartiles Q_1 and Q_3
- A line in the box marks the median M
- A symbol (+) marks the mean
- Lines extend from the box out to the min. and max. values

Let us create the box plot graph for the previous example.

univariate.sas

```sas
data outlib.marks_data;
set outlib.marks_data;
  format Boxplot $1.;
  Boxplot = "";
run;


ods select ssplots ;
proc univariate data = outlib.marks_data;
var marks;
by Boxplot;
run;
```

ℹ Some More Information

Above snippet works well in Base SAS.

For using PROC BOXPLOT, you will need SAS/GRAPH

univariate.lst - Notepad

```
The UNIVARIATE Procedure
    Variable:  marks

  Schematic Plots
```

Maximum = 67

$Q_3 = 41.5$

Mean = 28.6

Median (Q_2) = 24.5

$Q_1 = 19.5$

Minimum = 5
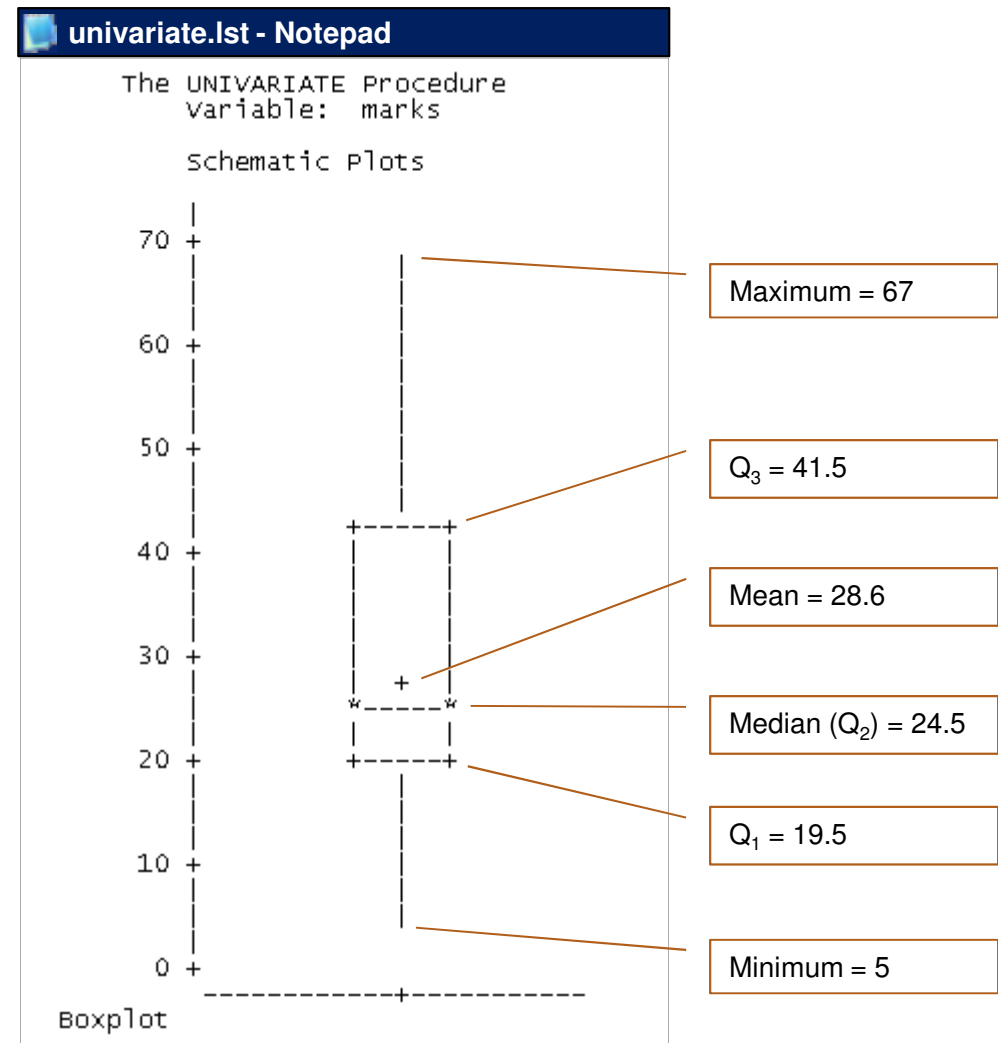
Exercise

Exercise 3. For 15 randomly chosen employees working in Delhi NCR, per month salary (in thousand rupees) is given below:

4, 25, 30, 30, 30, 31, 32, 35, 50, 50, 50, 55, 60, 74, 110

a. In MS Excel spreadsheet (without using inbuilt functions), calculate

   i. Mean

   ii. Median

   iii. Mode

   iv. Range

   v. Inter-Quartile Range

   vi. Variance

   vii. Standard Deviation

   viii. Coefficient of Variation

b. In SAS (using SAS procedures), compute and report all of the above metrics along with Skewness and Kurtosis

c. Using SAS, create box-plot graph

Exercise 4. In a class of 50 students, all scored 70 marks in Statistics. Calculate Mean, Median, Mode, Range, Inter-Quartile Range, Variance, Standard Deviation and Coefficient of Variation.

[Hint: Do you really need a calculator or any software to answer this?]

Chapter 3: Sampling Distributions

3.1 Sampling

3.1.1. Sample Selection

> **True Population:** The entire collection of data, which we wish to analyze, describe or draw conclusions about

> **Sample Population:** The part of true population that we actually examine in order to gather information and draw valid conclusions about the larger group
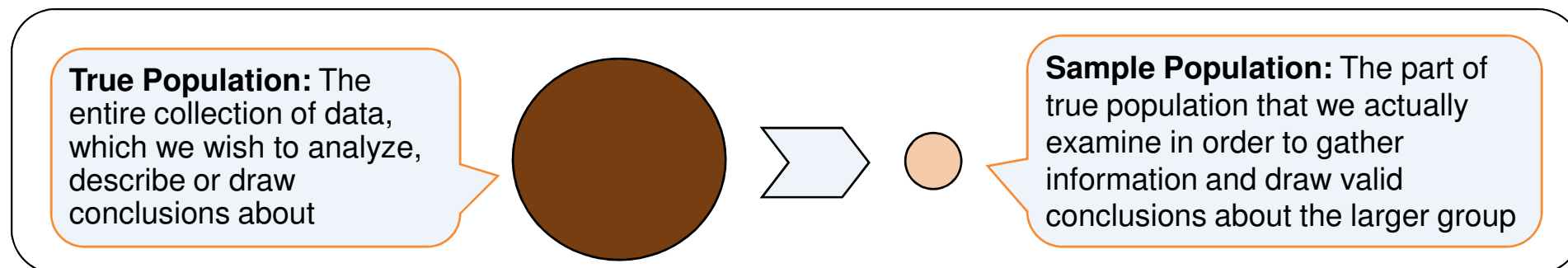
Two Most Frequently Used Methods of Sample Selection

SAS Tip

PROC SURVEYSELECT

Simple Random Sampling:

- This method selects units with equal probability and without replacement
- Each possible sample of n different units out of N has the same probability of being selected
- Selection probability for each unit = n / N

Stratified Random Sampling:

- This method selects random samples independently within strata
- Selection probability for a unit in stratum h = n_h / N_h

Other Methods include Sequential Random Sampling, Systematic Random Sampling etc.

3.1.2. Parameter and Statistic

A parameter is a value, usually unknown (and which therefore has to be estimated), that has to be estimated with minimum error using statistic from one or more samples.

Parameter	Statistic
■ Describes a characteristic of a population	■ Describes a characteristic of a sample
■ Fixed Value within the population	■ Value varies from sample to sample within the same population
■ Generally represented as Greek alphabets (e.g. Σ)	■ Generally represented as English alphabets (e.g. S)

Example:

A properly chosen sample of 1600 people across the country was asked if they regularly watch a certain television program, and 24% said yes.

Parameter : The true proportion of all people in the country who watch the program

Statistic   : 24% (as obtained from the sample of 1600 people)

3.1.3. Sampling Distribution of a Statistic

The sampling distribution of a statistic is the distribution of values taken by the statistic in all possible samples of the same size (n) from the same population.

Sampling distribution is nothing but the **probability distribution of the statistic** - It tells what values a statistic takes and how often it takes those values in repeated sampling.

Example: A box consists of 10 counters. The color of the counter can be

■	Yellow	(1)	(Score Value = 1)
■	Green	(2)	(Score Value = 2)
■	Red	(5)	(Score Value = 5)
■	Blue	(4)	(Score Value = 4)

Suppose there are 2 yellow, 2 green, 3 red and 3 blue counters in the box. A person is asked to draw 2 counters from this box.

All possibilities of drawing 2 counters

Frequency Distribution

				1+4+6+6 = 17
(1)	(1)(1) **1**	(1)(2) **4**	(1)(5) **6**	(1)(4) **6**
(2)	(2)(2) **1**	(2)(5) **6**	(2)(4) **6**	1+6+6=13
(5)	(5)(5) **3**	(5)(4) **9**		3+9=12
(4)	(4)(4) **3**			3

Determine the sampling distribution of the mean score of this person.

17 + 13 + 12 + 3 = 45

Frequency Distribution and Mean of Scores Obtained in Each Sample

					1+4+6+6 = 17
					1+6+6=13
					3+9=12
					3

- ▪ Frequency
- ▪ Mean Scores

17 + 13 + 12 + 3 = 45

If a random sample of size 'n' is drawn from a population with mean μ and standard deviation σ, then the sampling distribution of the means has mean μ and standard deviation σ/ √n

Sampling Distribution of Mean Score

Mean	Frequency	Probability
1	1	1/45
1.5	4	4/45
2	1	1/45
2.5	6	6/45
3	12	12/45
3.5	6	6/45
4	3	3/45
4.5	9	9/45
5	3	3/45
Sum	**45**	**1**

3.1.4. Standard Error

Standard error is the standard deviation of the sampling distribution of the statistic. If statistic is the sample mean, and the samples are uncorrelated, the standard error is given by:

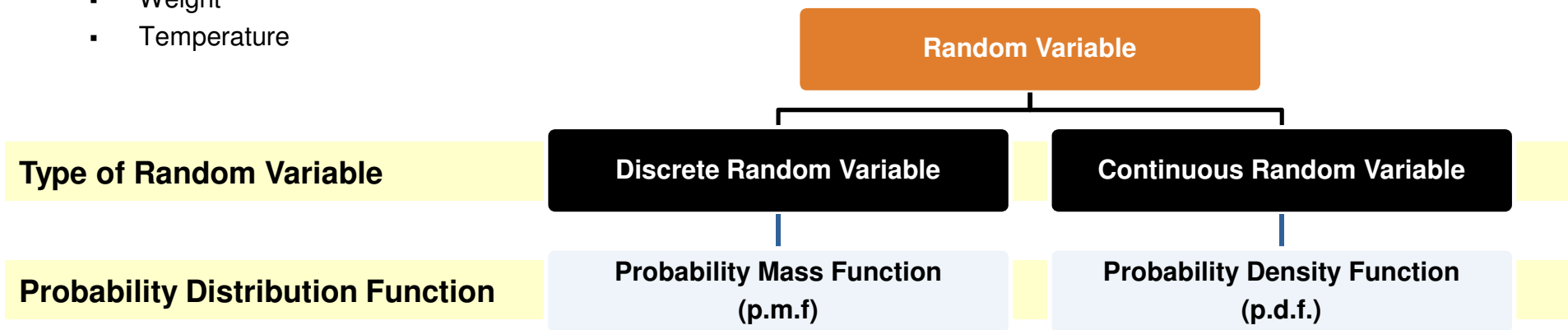$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

3.2 Random Variable & Probability Distributions

3.2.1. Random Variable

A random variable is a variable whose value is a numerical outcome of a random phenomenon. It may be discrete or continuous.

1. **Discrete Random Variables:** Random variables that have a finite (countable) list of possible outcomes, with probabilities assigned to each of these outcomes. Examples:
   - Number of Cars Owned (0, 1, 2, 3, …)
   - Attendance (in terms of numbers of days) in a month (1, 2, 3, … , 31)

2. **Continuous Random Variables:** Random variables that can take on any value in an interval, with probabilities given as areas under a density curve. Example:
   - Weight
   - Temperature

	Random Variable	
Type of Random Variable	**Discrete Random Variable**	**Continuous Random Variable**
Probability Distribution Function	**Probability Mass Function (p.m.f)**	**Probability Density Function (p.d.f.)**

The **probability distribution of a random variable** is the mathematical function describing the possible values of a random variable and their associated probabilities

3.2.2. Probability Mass Function (p.m.f.)

A probability mass function (p.m.f.) is a function that gives the probability that a **discrete random variable** X is exactly equal to some value x.

Probability Mass Function : $f(x) = P(X = x)$ such that the following two conditions satisfy :

1. $0 \le f(x) \le 1$ for every $x \in S$

2. $\sum_{x \in S} f(x) = 1$

where S is the Sample Space

Coin Toss Example: Toss three unbiased coins and observe 'Number of Heads'.

Random Variable: Number of Heads Observed (which may take four values: 0, 1, 2 or 3)

Sample Space = {TTT, HTT, THT, TTH, HHT, HTH, THH, HHH}

Probability Mass Function

x	f(x)
0	f(0) = P(X = 0) = P(TTT) = 1/8
1	f(1) = P(X = 1) = P(HTT, THT, TTH) = 3/8
2	f(2) = P(X = 2) = P(HHT, HTH, THH) = 3/8
3	f(3) = P(X = 3) = P(HHH) = 1/8
Sum	**Σf(X) = 1/8 + 3/8 + 3/8 + 1/8 = 1**

Probability Mass Function
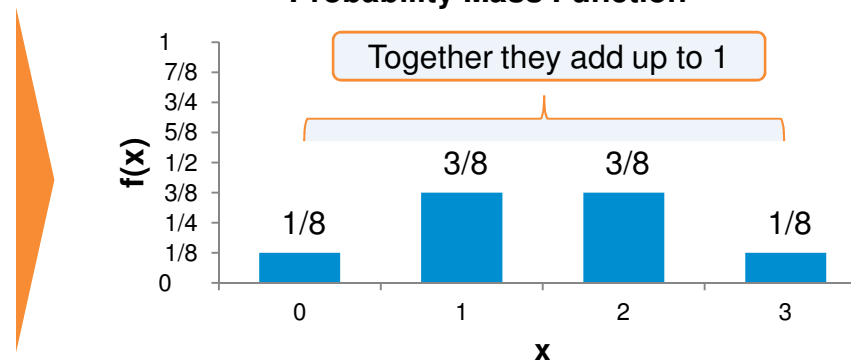
3.2.3. Probability Density Function (p.d.f.)

A probability density function is a function that describes the relative likelihood for a **continuous random variable** X to take values within a particular interval [a, b].

$$\text{Probability Density Function}: P(a \leq X \leq b) = \int_{a}^{b} f(x)dx \text{ such that the following two conditions satisfy}:$$

1. $f(x)$ is a non - negative integrable function

2. $\int_{-\infty}^{\infty} f(x)dx = 1$

In case of a continuous random variable

- There are infinite number of outcomes and hence a probability can not be assigned to each individual
- Probabilities are assigned to intervals of outcomes by using areas under density curves
- A density curve has area exactly 1 underneath it, corresponding to total probability 1

Probability Density Function

Area under the entire curve = 1

Area under the shaded area = P(a ≤ X ≤ b)
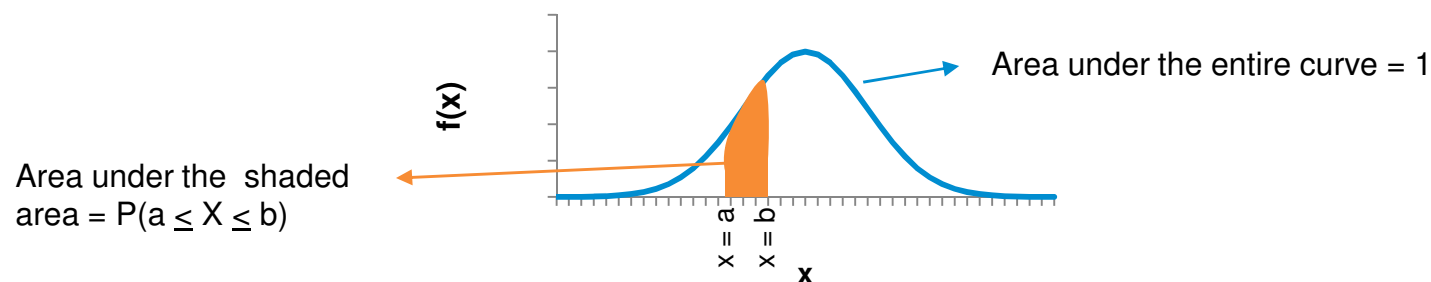
3.2.4. Cumulative Distribution Function (c.d.f.)

A cumulative distribution function describes the probability that a random variable X with a given probability distribution will be found at a value less than or equal to x. Intuitively, it is the "area so far" function of the probability distribution.

Cumulative Distribution Function: $F(x) = P(X \leq x)$

For Discrete Random Variable: $F(x_i) = P(X \leq x_i) = \sum_{k=1}^{i} P(X = x_k)$ [Note: $P(X = x_k) = F(x_k) - F(x_{k-1})$]

For Continuous Random Variable: $F(x_i) = P(X \leq x_i) = \int_{-\infty}^{x_i} f(t)\, dt$ [Note: f is probability density function]

Properties of CDF: $0 \leq F(x) \leq 1$, $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$

c.d.f. for Discrete Random Variable

c.d.f. for Continuous Random Variable
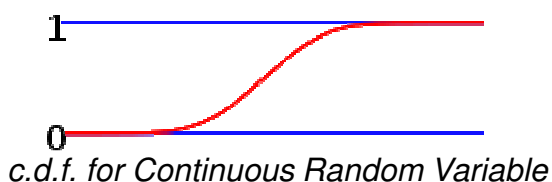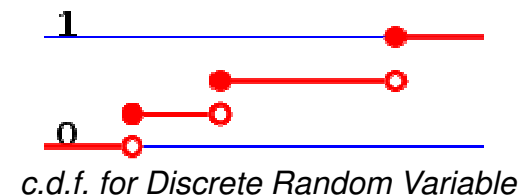
Coin Toss Example Continued…

x	f(x)	F(x)
0	f(0) = 1/8	F(1) = 1/8
1	f(1) = 3/8	F(2) = 1/8+3/8 = 4/8
2	f(2) = 3/8	F(3) = 4/8+3/8 = 7/8
3	f(3) = 1/8	F(4) = 7/8+1/8 = 1
	Σf(X) = 1	

C.D.F. for a Discrete Variable
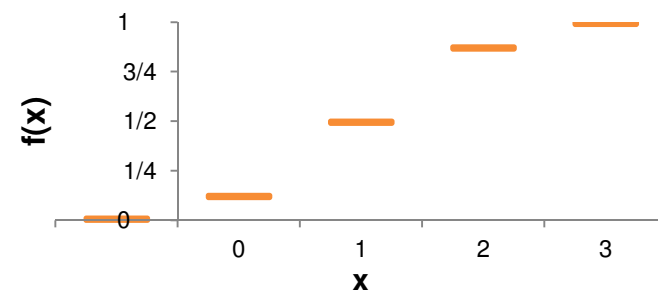
3.3 List of Distributions

3.3.1. Examples of Discrete Distributions

1. Discrete Uniform Distribution   [Details]
2. Binomial Distribution   [Details]
3. Hypergeometric Distribution   [Details]
4. Poisson Distribution   [Details]
5. Geometric Distribution   [Details]
6. Negative Binomial Distribution   [Details]

Note: 'Details' link leads to Appendix A.2

3.3.2. Examples of Continuous Distributions

1. Continuous Uniform Distribution   [Details]
2. Normal Distribution   [Details]
3. Chi-Square Distribution   [Details]
4. F Distribution   [Details]
5. Student's t Distribution   [Details]
6. Exponential Distribution   [Details]
7. Gamma Distribution   [Details]
8. Beta Distribution   [Details]

Note: 'Details' link leads to Appendix A.3

Exercise

Exercise 5. In an excel spreadsheet, plot
1. Probability Mass Function of Binomial Distribution
2. Probability Mass Function of Hypergeometric Distribution
3. Probability Mass Function of Poisson Distribution
4. Probability Mass Function of Negative Binomial Distribution
5. Probability Density Function of Normal Distribution
6. Probability Density Function of Exponential Distribution
7. Probability Density Function of Gamma Distribution

[Hint A: Explore Excel functions: BINOMDIST, HYPGEOMDIST, POISSON, NEGBINOMDIST, NORMDIST, EXPONDIST and GAMMADIST]

[Hint B: Refer Appendix A.2 and A.3]

Chapter 4: Hypothesis Testing

4.1 Key Concepts

4.1.1. Statistical Hypothesis

A statistical hypothesis is some assumption or statement about a population parameter.

- This assumption may or may not be true
- It is tested on the basis of the evidence from a random sample

Example: Average sales amount (i.e. sales amount per transaction) of a grocery store is $150.

Statistical hypotheses are of two types:

A. Null Hypothesis

Hypothesis which is being tested. It is denoted by H_0.

B. Alternative Hypothesis

Counter-proposition to the null hypothesis. It is denoted by H_1 or H_A.

Example: Let the average sales amount of a grocery store be denoted by μ. There could be three scenarios.

1. Left-Tailed (One Tailed) : $H_0 : \mu \geq 150$ and $H_1 : \mu < 150$
2. Right-Tailed (One Tailed) : $H_0 : \mu \leq 150$ and $H_1 : \mu > 150$
3. Two-Tailed : $H_0 : \mu = 150$ and $H_1 : \mu \neq 150$

Can We Accept the Null Hypothesis?
Based on a sample, we can not conclude that H_0 is true. So, instead of concluding "we accept H_0", we say "we fail to reject H_0".
Note: "Acceptance of H_0" applies H_0 is true, while "Failure to reject H_0" implies evidence from sample is not sufficient for rejecting H_0.

4.1.2. Tests of Significance

Test of significance is, in general, a procedure to assess the significance of difference between two or more values.

List of Some Commonly Used Tests: See Appendix A.1

Example: The sales amount of a grocery store is normally distributed with standard deviation of $30.2. A sample of 40 sales receipts has an average sales amount of $137. Based on sample information, one may want to test whether the mean of sales at the grocery store is different from $150.

In this case,

- Assumed population mean (μ_0) = 150 and Sample mean ($\bar{x}$) = 137
- Objective is to test the significance of difference between μ_0 and $\bar{x}$
- Population is normally distributed and its variance is known

Therefore, Z-Test should be applied

✓ If the difference is not found to be significant, we do not reject H_0 and it is attributed to pure chance due to fluctuations in sampling
✓ If the difference is found to be significant, we reject H_0 and it is attributed to some non-random cause

4.1.3. Test Statistic

A test statistic is a value calculated from a sample to test the validity of null hypothesis. Since a test statistic is a random variable, it has a probability distribution.

Example:

Null Hypothesis   : Average sales amount of grocery store = $150   [i.e. $H_0 : \mu = 150$]

Alternative Hypothesis : Average sales amount of grocery store ≠ $150   [i.e. $H_1 : \mu \neq 150$]

Additional Information :

1. Sales amount is normally distributed with standard deviation of $30.2   [i.e. $X \sim N(\mu, 30.2^2)$]

2. From a sample of 40 transactions, average sales amount = $137   [i.e. $\bar{x} = 137$]

Test Statistic   : $Z = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \dfrac{137 - 150}{30.2/\sqrt{40}} = -2.722$

Properties

All good test statistics should have two properties:

(a) They should tend to behave differently when H_0 is true from when H_1 is true; and

(b) Their probability distribution should be calculable under the assumption that H_0 is true. It is also desirable that tables of this probability distribution exist

4.1.4. Types of Error, Level of Significance and Power of Test

There can be two types of errors in Hypothesis Testing

■ Type I Error   : Error of rejecting H_0 when it is, in fact, true
■ Type II Error   : Error of not rejecting H_0 when it is, in fact, false

Size of the Test (or Level of Significance) = α
- Probability of committing a Type I Error
- Maximize size of Type I Error at risk

Power of the Test = 1-β
- Probability of not committing a Type II Error
- Ability of the Test to Reject a False Null Hypothesis

		Situation	
		H_0 is true	H_0 is false
Decision	H_0 is not rejected	Right Decision	Type II Error P(Type I Error) = β
	H_0 is rejected	Type I Error P(Type I Error) = α	Right Decision

■ For any sample size, it is not possible to minimize both types of errors simultaneously
■ In practice, a Type I error is likely to be more serious than a Type II error
■ Classical Approach:
  1. Keep the probability of committing a type I error at a fairly low level (0.01, 0.05 or 0.10)
  2. Then try to minimize the probability of having a type II error (i.e. maximize the power of the test)

Set α to 1%, 5% or 10% and then try to minimize β

4.1.5. Confidence Interval

Confidence interval is the interval in which we can be 100(1- α)% sure that values of a particular statistic will lie.

Note: α is the level of significance.

Significance Level	Confidence Level
0.01	99%
0.05	95%
0.10	90%

95% Confidence Interval for Mean (μ) of a Large Sample:

μ : Population Mean

σ : Population Standard Deviation

n : Sample Size

x̄ : Sample Mean

$$X_i \sim N(\mu, \sigma^2) \Rightarrow \overline{X} \sim N(\mu, \sigma^2/n) \Rightarrow Z_i = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

From Normal Distribution Table,

$$P(\,-1.96 \leq Z_i \leq 1.96\,) = 0.95$$

$$\Rightarrow P(\,-1.96 \leq \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \leq 1.96\,) = 0.95$$

$$\Rightarrow P(\,\overline{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}\,) = 0.95$$

Interpretation of 95% Confidence Interval:

A 95% Confidence Interval is interpreted as the range of values within which the population parameter will fall 95% of the time, if the procedure of calculating confidence intervals is repeated on multiple samples

4.1.6. Critical Regions

Confidence interval is called the acceptance region and the areas outside it are called the critical regions or regions of rejection of the null hypothesis. The lower and upper limits of the acceptance region are called the critical values.

Example: Continuing with Grocery store example:

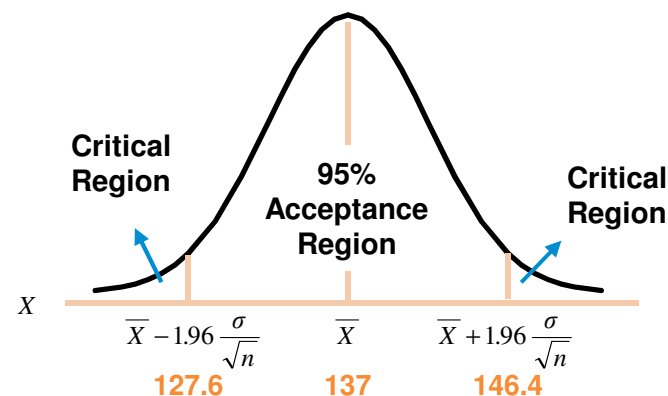$H_0 : \mu = 150$, $H_1 : \mu \neq 150$, $X \sim N(\mu, 30.2^2)$, $\bar{x} = 137$, n

$$P(\,\overline{X} - 1.96\,\frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + 1.96\,\frac{\sigma}{\sqrt{n}}\,) = 0.95$$

$$\Rightarrow P(\,137 - 1.96 \times \frac{30.2}{\sqrt{40}} \leq \mu \leq 137 + 1.96 \times \frac{30.2}{\sqrt{40}}\,) = 0.95$$

$$\Rightarrow P(\,137 - 9.4 \leq \mu \leq 137 + 9.4\,) = 0.95$$

$$\Rightarrow P(\,127.6 \leq \mu \leq 146.4\,) = 0.95$$

95% Confidence Interval for μ : [127.6, 146.4]

Critical Region   95% Acceptance Region   Critical Region

X

$\overline{X} - 1.96\,\dfrac{\sigma}{\sqrt{n}}$ $\overline{X}$ $\overline{X} + 1.96\,\dfrac{\sigma}{\sqrt{n}}$

127.6   137   146.4

Observation : $\mu = 150$ lies in the critical region
Conclusion : H_0 is rejected at 5% significance level

4.1.7. The p-Value or Exact Level of Significance

Instead of preselecting α (the level of significance) at arbitrary levels, such as 1%, 5% or 10%, one can obtain the p (probability) value, or exact level of significance of a test statistic. The p value is defined as the lowest significance level at which a null hypothesis can be rejected.

Example:

Null Hypothesis   : Average sales amount of grocery store = $150   [i.e. $H_0 : \mu = 150$]

Alternative Hypothesis : Average sales amount of grocery store ≠ $150   [i.e. $H_1 : \mu \neq 150$]

Additional Information :

1.   Sales amount is normally distributed with standard deviation of $30.2   [i.e. $X \sim N(\mu, 30.2^2)$]

2.   From a sample of 40 transactions, average sales amount = $137   [i.e. $\bar{x} = 137$]
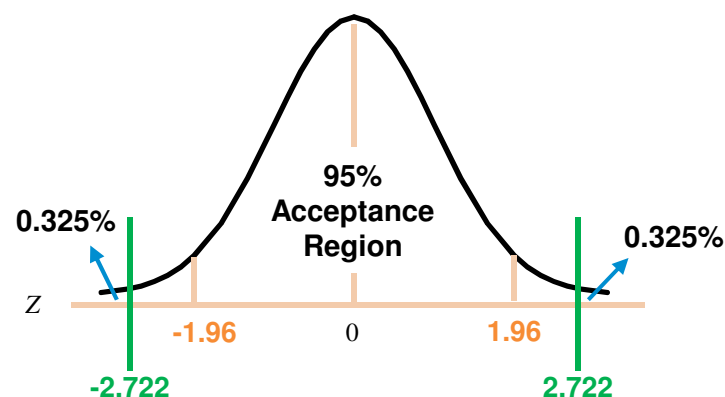
Test Statistic   : $Z = \dfrac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \dfrac{137 - 150}{30.2 / \sqrt{40}} = -2.722$

It is a two-tailed test. From table, p-value = 0.0065

Interpretation:

p-value = 0.65% is the exact level of significance.

- Since 0.65% < 1%, we reject at H_0 not only at 5% level, but even at 1% level of significance
- That is, we reject H_0 not with just 95% confidence, but with 99% confidence.

0.325%   95% Acceptance Region   0.325%

z -1.96   0   1.96

-2.722   2.722

$P(Z \leq -1.96) = 2.5\%$, $P(Z \geq 1.96) = 2.5\%$: Adding up to 5%

$P(Z \leq -2.722) = 0.325\%$, $P(Z \geq 2.722) = 0.325\%$: Adding up to 0.65%

4.2 Step-by-Step Process

Steps in Hypothesis Testing

Process of hypothesis testing can be summarized in following five steps:

Step 1: Set up Null Hypothesis (H_0) and Alternative Hypothesis (H_1), keeping in mind if it is going to be a single (left or right) tailed or a two-tailed test

Step 2: Calculate the test statistic and determine its probability distribution

Step 3: Choose the appropriate (1%, 5% or 10%) level of significance (α)

Step 4: Evaluate the test statistic by either confidence interval approach or by p value approach

Step 5: Conclude by either rejecting or not rejecting the null hypothesis at the given level of significance

Exercise

Exercise 6. An insurance company is reviewing its current policy rates. When originally setting the rates they believed that the average claim amount was $1,800. They are concerned that the true mean is actually higher than this, because they could potentially lose a lot of money. They randomly select 40 claims, and calculate a sample mean of $1,950. Assuming that the standard deviation of claims is $500, set α = 0.05 and test to see if the insurance company should be concerned.

[Hint: At 5% significance level, the critical value for a one tailed test from the table of z-scores = 1.645]

Chapter 5: Scatter Plots and Correlations

5.1 Scatter Plot

5.1.1. Explanatory and Response Variables

A response variable measures an outcome of a study. An explanatory variable explains or influences changes in a response variable.

- A **response variable** is also called **dependent variable** and is generally denoted by **Y**
- An **explanatory variable** is also called **independent variable** and is generally denoted by **X**

Analysis of relationship between two variables is referred to as Bivariate Analysis. To run this analysis, both variables are measured on the same individuals.

Relationships between two quantitative variables are best displayed graphically through a scatterplot

5.1.2. Plotting and Visualizing a Scatter Plot

A scatterplot shows the relationship between two quantitative variables measured on the same individuals.
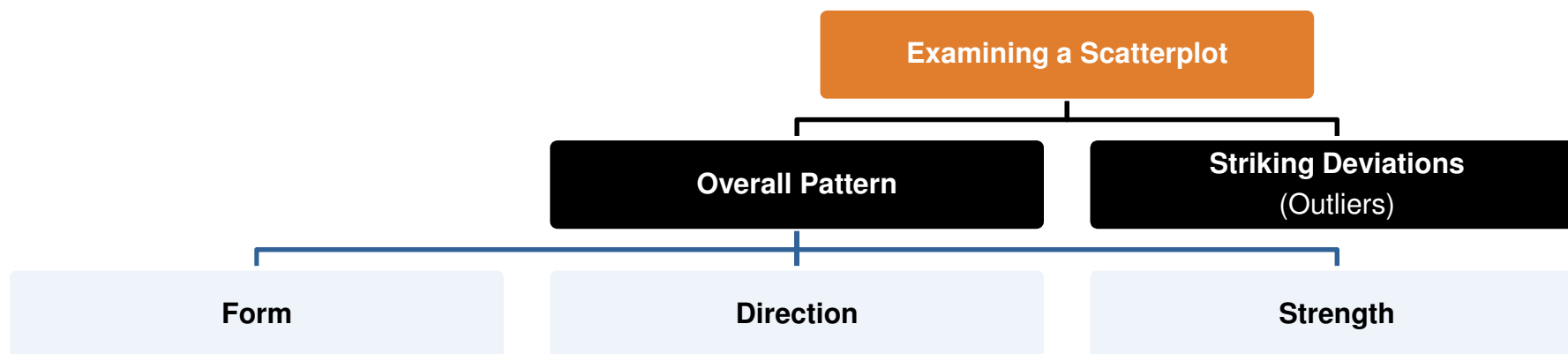
- The values of one variable appear on the horizontal axis, and the values of the other variable appear on the vertical axis
- Each individual in the data appears as the point in the plot fixed by the values of both variables for that individual
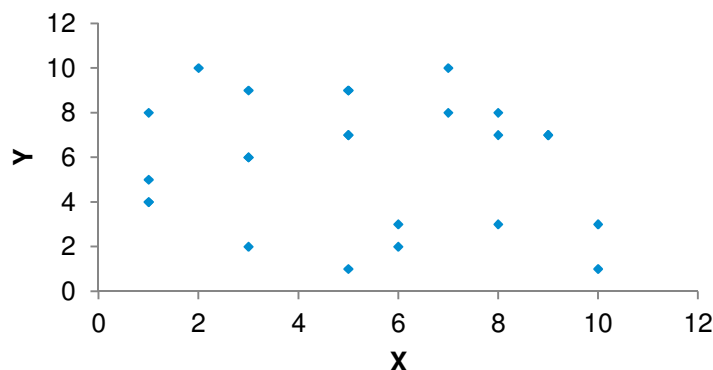
- ✓ Always plot the explanatory variable on the horizontal axis (the x axis) of a scatterplot
- ✓ If there is no explanatory-response distinction, either variable can go on the horizontal axis

5.1.3. Examining a Scatterplot

Look for the **overall pattern** and for **striking deviations** from that pattern

```
                    ┌─────────────────────────────┐
                    │   Examining a Scatterplot   │
                    └─────────────────────────────┘
                      ┌────────────┴────────────┐
            ┌──────────────────┐      ┌──────────────────────┐
            │  Overall Pattern │      │  Striking Deviations │
            │                  │      │      (Outliers)      │
            └──────────────────┘      └──────────────────────┘
         ┌──────────┴──────────┐              │
  ┌────────────┐      ┌────────────┐   ┌────────────┐
  │    Form    │      │  Direction │   │  Strength  │
  └────────────┘      └────────────┘   └────────────┘
```

- **Form:** Linear relationships, where the points show a straight-line pattern, are an important form of relationship between two variables. Curved relationships and clusters are other forms to watch for.

- **Direction:** If the relationship has a clear direction, there is either positive association (high values of the two variables tend to occur together) or negative association (high values of one variable tend to occur with low values of the other variable)

- **Strength:** The strength of a relationship is determined by how close the points in the scatterplot lie to a simple form such as a line

5.2 Correlation

5.2.1. Pearson's Correlation Coefficient

Pearson's Correlation Coefficient (r) measures the strength and direction of the linear association between two quantitative variables X and Y. Although correlation can be calculated for any scatterplot, Pearson's correlation coefficient measures only straight-line relationships.

$$r = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \overline{X})^2 \sum_{i=1}^{n}(Y_i - \overline{Y})^2}} \in [-1, +1]$$

SAS Tip
PROC CORR **(Use OUTP Option)**

- Correlation indicates the direction of a linear relationship by its sign
  - r > 0 for a positive association
  - r < 0 for a negative association
- Correlation indicates the strength of a linear relationship by magnitude of its absolute value
  - Correlation ranges from -1 to +1
  - The Closer to –1, the Stronger the Negative Linear Relationship
  - The Closer to +1, the Stronger the Positive Linear Relationship
  - The Closer to 0, the Weaker Any Linear Relationship
  - r = ±1 means perfect correlation, which occurs only when the points on a scatterplot lie exactly on a straight line
- Correlation ignores the distinction between explanatory and response variables
- The value of r is not affected by changes in the unit of measurement of either variable
- Correlation is not resistant, so outliers can greatly change the value of r

Graphical Illustration

$r = -1$

$r = -.6$

$r = 0$

$r = .6$

$r = 1$
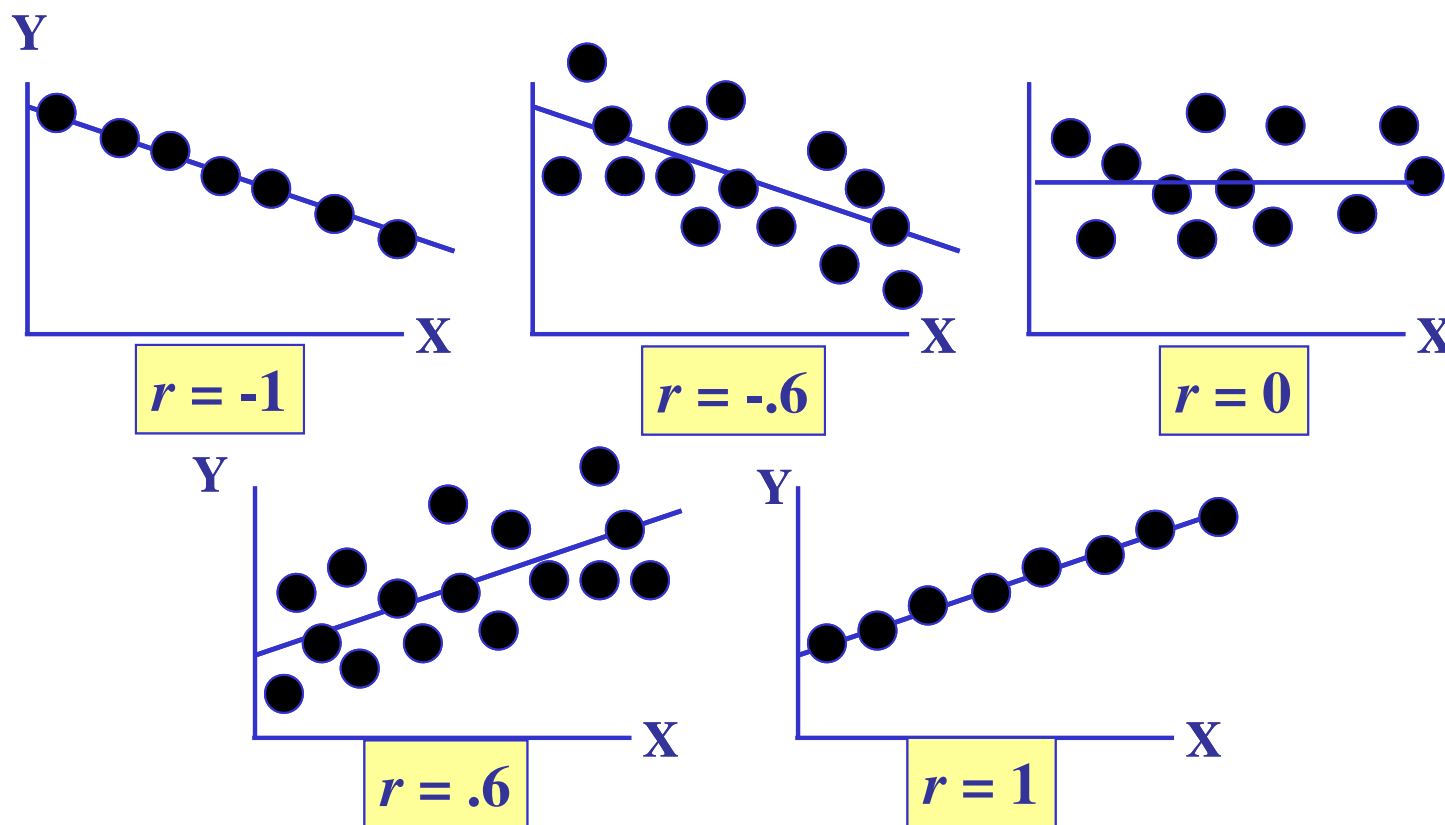
Example

	A	B	C	D	E	F	G
1	**X**	**Y**	**$X_i - X_{mean}$**	**$Y_i - Y_{mean}$**	**$(X_i - X_{mean})*(Y_i - Y_{mean})$**	**$(X_i - X_{mean})2$**	**$(Y_i - Y_{mean})2$**
2	101.0	99.2	24.7	(35.3)	(871.6)	609.5	1246.5
3	100.1	99.0	23.8	(35.5)	(844.6)	565.9	1260.7
4	100.0	100.0	23.7	(34.5)	(817.4)	561.1	1190.7
5	90.6	111.6	14.3	(22.9)	(327.3)	204.2	524.7
6	86.5	122.2	10.2	(12.3)	(125.4)	103.8	151.4
7	89.7	117.6	13.4	(16.9)	(226.3)	179.2	285.8
8	90.6	121.1	14.3	(13.4)	(191.5)	204.2	179.7
9	82.8	136.0	6.5	1.5	9.7	42.1	2.2
10	70.1	154.2	(6.2)	19.7	(122.3)	38.6	387.9
11	65.4	153.6	(10.9)	19.1	(208.4)	119.1	364.6
12	61.3	158.5	(15.0)	24.0	(360.2)	225.4	575.7
13	62.5	140.6	(13.8)	6.1	(84.2)	190.8	37.1
14	63.6	136.2	(12.7)	1.7	(21.5)	161.6	2.9
15	52.6	168.0	(23.7)	33.5	(794.2)	562.2	1121.9
16	59.7	154.3	(16.6)	19.8	(328.8)	276.0	391.8
17	59.5	149.0	(16.8)	14.5	(243.7)	282.6	210.1
18	61.3	165.5	(15.0)	31.0	(465.3)	225.4	960.6
19					(6023.1)	4551.5	8894.2

$r = (-6023.1) / (4551.5 * 8894.2)^{1/2}$

$r = -95\%$

5.2.2. Spearman's Rank Correlation Coefficient

Spearman's rank correlation coefficient (rho), is a non-parametric measure of statistical dependence between two variables. It assesses how well the relationship between two variables can be described using a monotonic function.

$$\rho = 1 - \frac{6\sum_{i=1}^{n} d_i}{n(n^2 - 1)}, \text{ where } d_i = x_i - y_i$$

> **SAS Tip**
> **PROC CORR**
> **(Use OUTS Option)**

Example:

	A	B	C	D	E
1	Candidate	Judge A's Ranking	Judge B's Ranking	d	d²
2	A	1	2	-1	1
3	B	2	1	1	1
4	C	3	4	-1	1
5	D	4	3	1	1
6	E	5	6	-1	1
7	F	6	7	-1	1
8	G	7	5	2	4
9	Sum				10

ρ = 1 − [6(10) / 7(49-1)]

ρ = 82.14%

Appendix

A.1 List of Some Commonly Used Tests

Statistical Tests

Investigation of significance of

- Difference between an assumed population mean μ_0 and a sample mean $\bar{x}$,
  - When the population variance is known   [Z-test]
  - When the population variance is unknown   [t-test]
- Difference between two sample means, one from each population,
  - When the population variances are known and equal   [Z-test]
  - When the population variances are known and unequal   [Z-test]
  - When the population variances are unknown but equal   [t-test]
  - When the population variances are unknown and unequal   [t-test]
  - When observations for the two sample are obtained in pairs   [t-test]
- Difference between an assumed population proportion and an observed sample proportion   [Z-test]
- Difference between two sample proportions, one from each population   [Z-test]
- Difference between two counts   [Z-test]
- Difference between an assumed population variance and a sample variance   [χ^2-test]
- Difference between two sample variances, one from each population   [F-test]
- A variable in Regression Model (i.e. difference between a regression coefficient and zero)   [t-test]
- Overall regression model   [F-test]

A.2 Examples of Discrete Distributions

A.2.1. Discrete Uniform Distribution
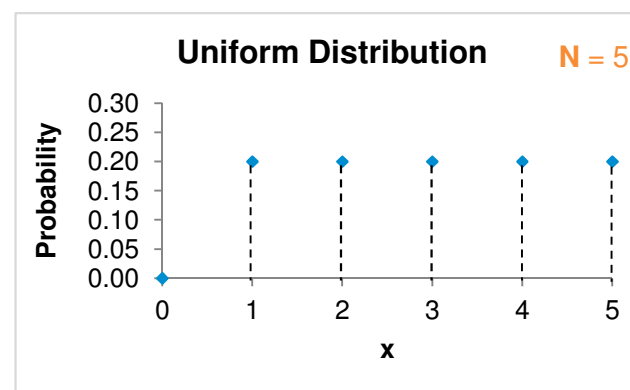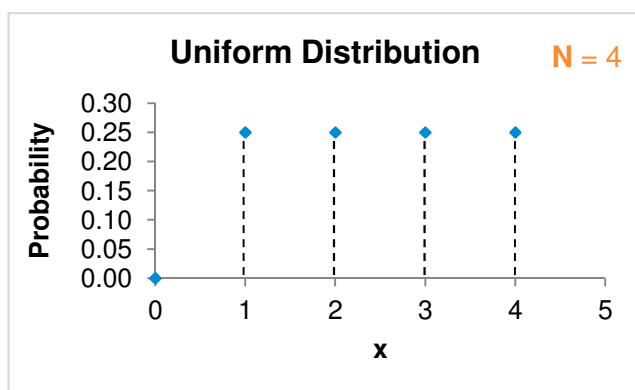
Probability Mass Function

$$P(X = k) = \frac{1}{N}, \quad k = 1, 2, \dots, N$$
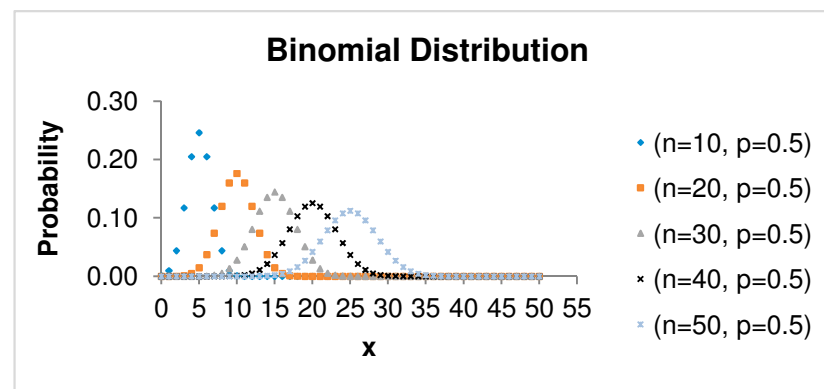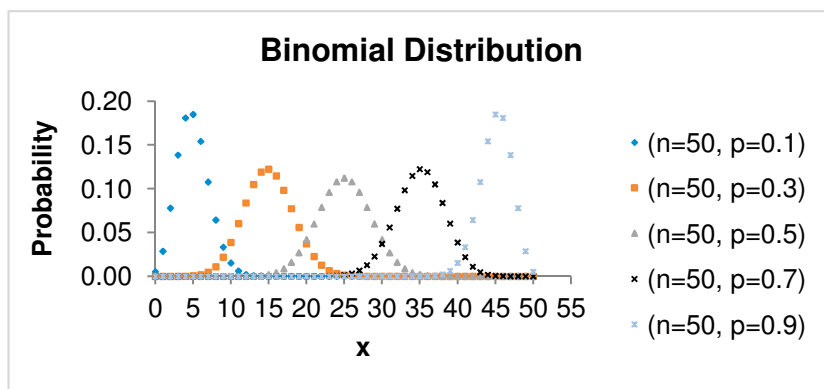
Main Properties

- This distribution is used to model experimental outcomes which are "equally likely"

Applications

- Tossing of a fair and unbiased die. For the given sample space {1, 2, 3, 4, 5, 6}, each number occurs with a probability of 1/6

A.2.2. Binomial Distribution

Binomial Distribution

- ◆ (n=50, p=0.1)
- ■ (n=50, p=0.3)
- ▲ (n=50, p=0.5)
- × (n=50, p=0.7)
- ✳ (n=50, p=0.9)

Binomial Distribution

- ◆ (n=10, p=0.5)
- ■ (n=20, p=0.5)
- ▲ (n=30, p=0.5)
- × (n=40, p=0.5)
- ✳ (n=50, p=0.5)

Probability Mass Function

$$P(X = k) = {}^nC_k\, p^k (1-p)^{n-k}, \quad k = 0, 1, ..., n$$

n : Number of trials
p : Probability of success on a single trial

Main Properties

- Binomial distribution is used when there are exactly two mutually exclusive outcomes of a trial, labeled as 'success' and 'failure'
- Mean = n p and variance = n p (1-p)
- The binomial distribution is probably the most commonly used discrete distribution

Applications

- Customer Retention (Identification of Attrition Cases)
- Fraud Detection (Identification of Defaulters)

A.2.3. Hypergeometric Distribution

Hypergeometric Distribution

- ♦ (n=100, M=50, N=1000)
- ■ (n=100, M=100, N=1000)
- ▲ (n=100, M=150, N=1000)
- × (n=100, M=200, N=1000)
- × (n=100, M=250, N=1000)

Hypergeometric Distribution

- ♦ (n=50, M=200, N=1000)
- ■ (n=75, M=200, N=1000)
- ▲ (n=100, M=200, N=1000)
- × (n=125, M=200, N=1000)
- × (n=150, M=200, N=1000)
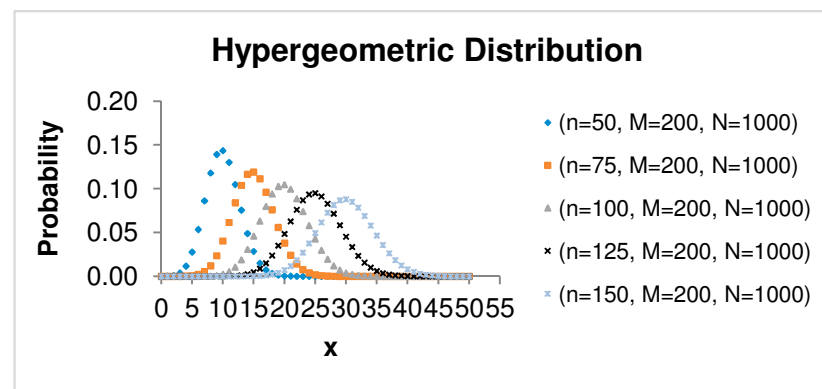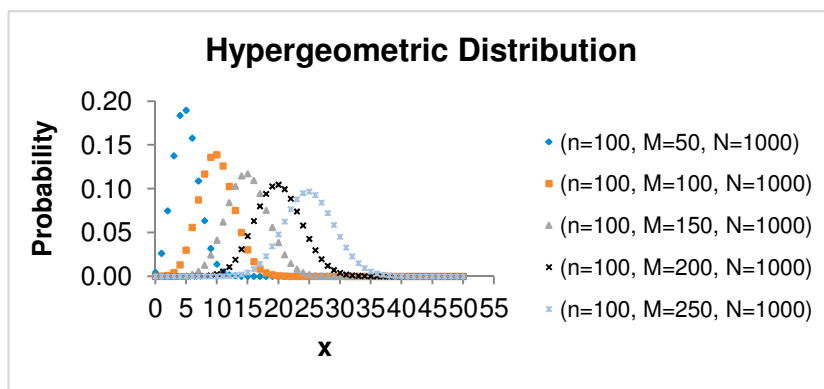
Probability Mass Function

$$P(X=k) = \frac{{}^{M}C_k \; {}^{N-M}C_{n-k}}{{}^{N}C_n}, \quad L \leq k \leq U, \text{ where } L = \max\{0, M\text{-}N+n\} \text{ and } U = \min\{n, M\}$$

N : Number of items in population
M : Number of defective items in population
n : Number of items in a sample
k : Number of defective items in the sample

Main Properties

- The Hypergeometric (n, M, N) distribution models the number of items of a particular type that are there in a sample of size n where that sample is drawn from a population of size N of which M are also of that particular type

- Mean = n (M / N) and Variance = n (M / N) [1 – (M / N)] [(N-n) / (N-1)]

Applications

- Sampling without replacement

- From a lot of N items with M defective pieces, what is the probability of getting k defective items by the customer if he purchases n units?

A.2.4. Poisson Distribution

Poisson Distribution
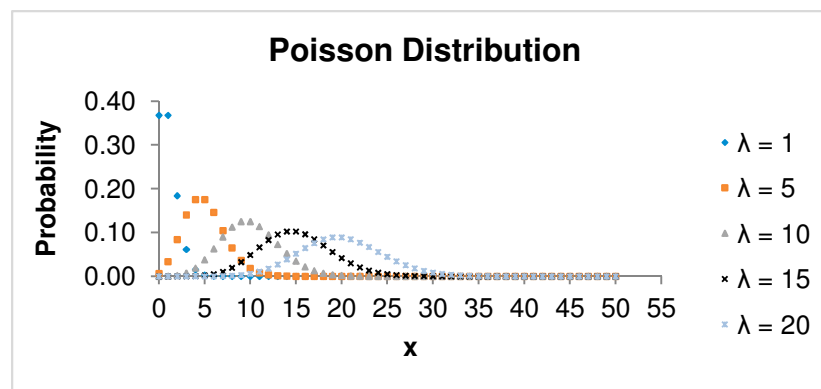
Probability Mass Function

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, ...$$

λ : The shape parameter which indicates the average number of events in the given time interval
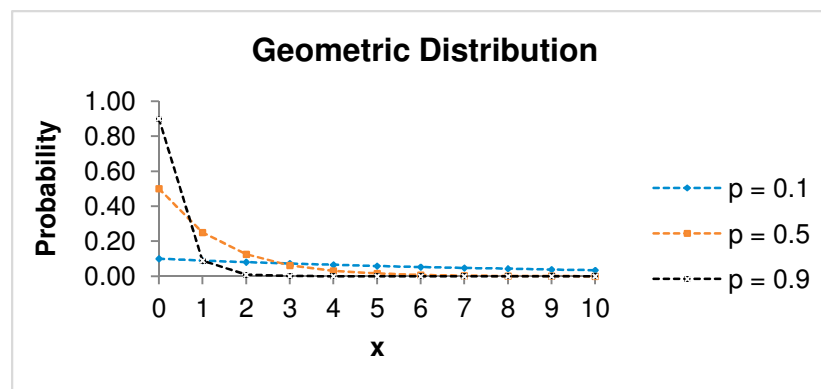
Main Properties

- Poisson distribution is used to model the number of events occurring within a given time interval
- Mean = Variance = λ
- Poisson distribution is very commonly used to model count data

Applications

- Count of phone calls arriving at a call centre per minute
- Count of insurance claims made by customers in a given period of time

A.2.5. Geometric Distribution

Probability Mass Function

$$P(X=k)=(1-p)^k\, p,\quad k=0,1,2,...$$

p : Probability of success on each trial

Main Properties

- Geometric distribution is used for modeling number of failures until the first success. It expresses the probability of exactly k failures until the first success to occur or equivalently, the probability that exactly (k + 1) trials are required to get the first success
- Mean = (1-p) / p and Variance = (1-p) / p^2

Applications

- Number of phone calls required before making a sale
- Number of dry wells an oil company will drill in a particular area before getting an oil-producing well
- Number of proposals to get a 'Yes'

A.2.6. Negative Binomial Distribution

Negative Binomial Distribution

- (r=3, p=0.5)
- (r=5, p=0.5)
- (r=10, p=0.5)
- (r=15, p=0.5)
- (r=20, p=0.5)

Negative Binomial Distribution

- (r=10, p=0.3)
- (r=10, p=0.4)
- (r=10, p=0.5)
- (r=10, p=0.6)
- (r=10, p=0.7)
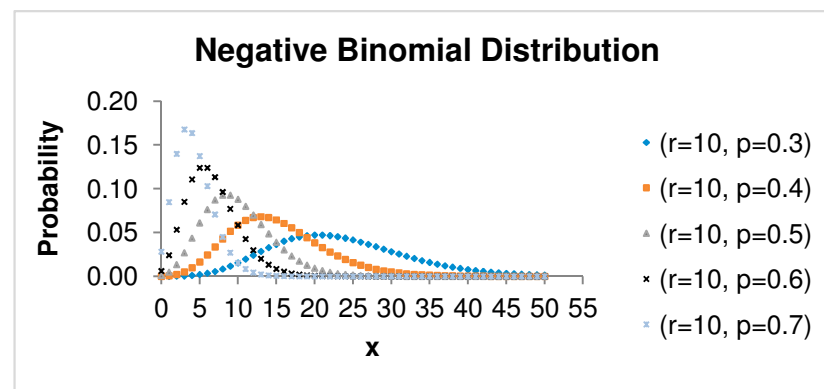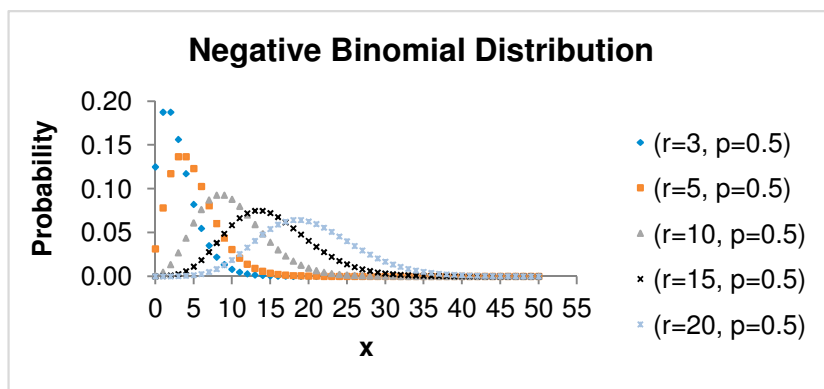
Probability Mass Function

$$P(X=k)={}^{r+k-1}C_k\, p^r (1-p)^k, \quad k=0,1,2,...; 0<p<1$$

p : Probability of success on each trial
r : Threshold number of successes
k : Number of failures until the rth success

Main Properties

- Negative Binomial distribution gives the probability of observing k failures before the rth success or equivalently, probability that k+r trials are required until the rth success to occur
- Mean = r (1-p) / p and Variance = r (1-p) / p^2

Applications

- Games (what's the probability that a basket ball player makes his 3rd free throw on his 5th shot?)
- Marketing (what's the probability that a door-to-door salesman sells the last candy bar at the 10th house?)

A.3 Examples of Continuous Distributions

A.3.1. Continuous Uniform Distribution

Uniform Distribution

a = 0 and b = 2

Standard Uniform Distribution

a = 0 and b = 1
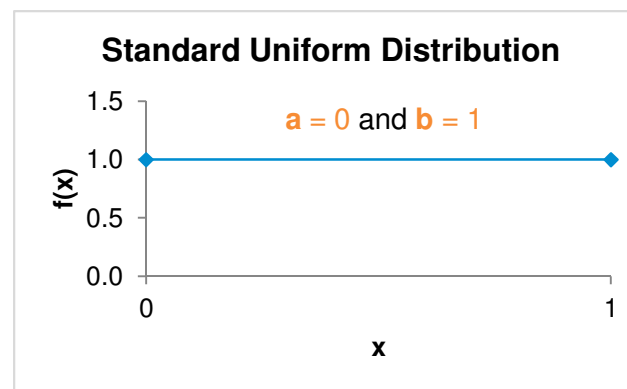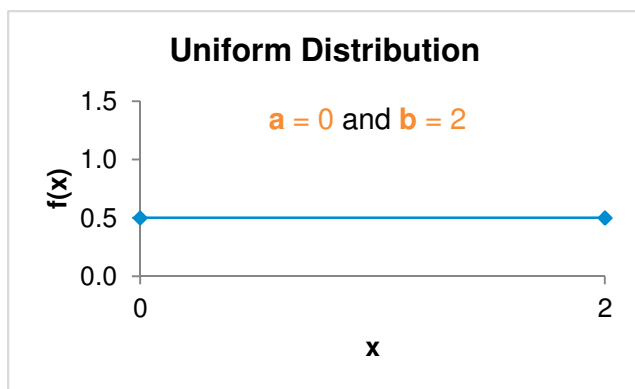
Probability Density Function

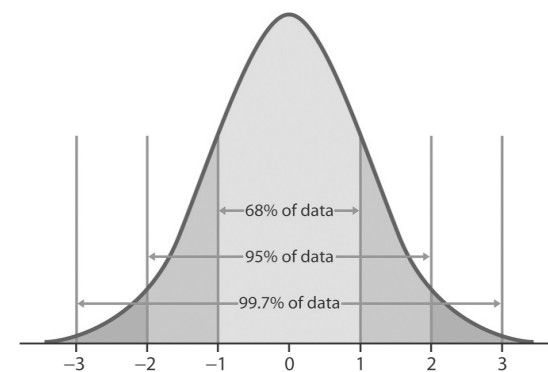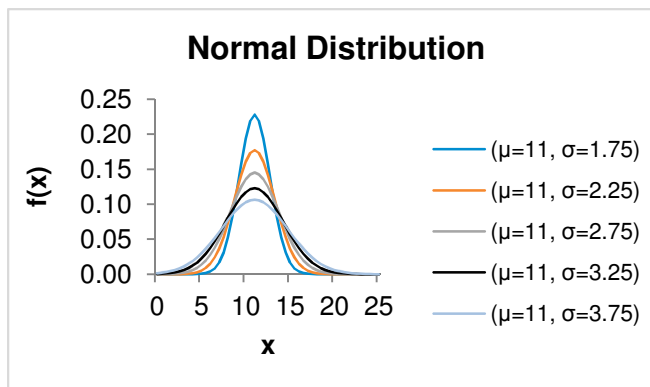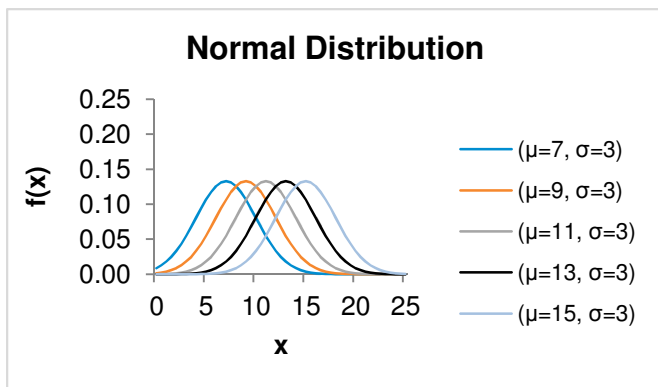$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

Main Properties

- The uniform distribution defines equal probability over a given range for a continuous distribution
- 'a' is the location parameter and 'b-a' is the scale parameter
- In case of a standard uniform distribution, a = 0 and b = 1

Applications

- Generation of random numbers: Almost all random number generators generate random numbers on the (0,1) interval

A.3.2. Normal Distribution

Normal Distribution

- (μ=7, σ=3)
- (μ=9, σ=3)
- (μ=11, σ=3)
- (μ=13, σ=3)
- (μ=15, σ=3)

Normal Distribution

- (μ=11, σ=1.75)
- (μ=11, σ=2.25)
- (μ=11, σ=2.75)
- (μ=11, σ=3.25)
- (μ=11, σ=3.75)

68% of data
95% of data
99.7% of data

Standard Normal Distribution

μ = 0 and σ² = 1

$$Z = \frac{\overline{X} - \mu}{\sigma}$$

Probability Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$$

μ : Mean
σ : Standard Deviation

Main Properties

- It is symmetrical around its mean value and is also called the 'bell-shaped' curve
- 68%, 95% and 99.7% of area under the normal curve lies within $\mu \pm \sigma$, $\mu \pm 2\sigma$, and $\mu \pm 3\sigma$ respectively
- Normal distribution with mean = 0 and variance = 1 is called standard normal distribution
- **Central Limit Theorem**: Let $X_1, X_2, \ldots, X_n$ denote n independent random variables, all of which have same PDF with mean = μ and variance = σ^2. Let X-bar = $\Sigma X_i / n$. Then as n increases indefinitely (i.e. as n → ∞), X-bar approaches the normal distribution with mean = μ and variance = σ^2 / n
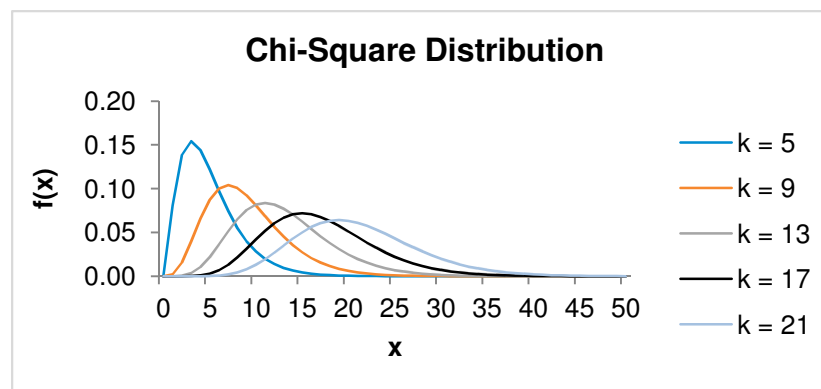
Applications

- Any process yielding values that tend to be symmetric around a central value (the mean) generally follow a Normal Distribution like height of individuals and marks of candidates in an entrance exam
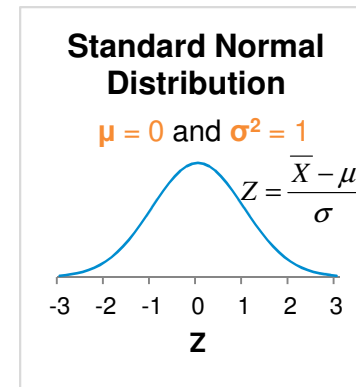
A.3.3. Chi-Square (χ^2) Distribution

Chi-Square Distribution

Probability Density Function

$$f(x) = \frac{1}{2^{k/2}\,\Gamma(k/2)}\, e^{-\frac{x}{2}} x^{\frac{k}{2}-1}, \quad x > 0, k > 0, \Gamma \text{ is the Gamma Function}: \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

k : Degrees of freedom

Main Properties

- If $Z_1, Z_2, \ldots, Z_k$ are independent standardized normal variables, then $Z = \Sigma Z_i^2$ is said to possess the χ^2 distribution with df = k
- The χ^2 distribution is a skewed distribution, the degree of skewness depending on the df. For comparatively few df, the distribution is highly skewed to the right; but as the df increases, the distribution becomes increasingly symmetrical
- Mean = k and Variance = 2k

Applications

- Hypothesis Testing (e.g. to test if a sample variance is equal to an assumed population variance)

A.3.4. F Distribution
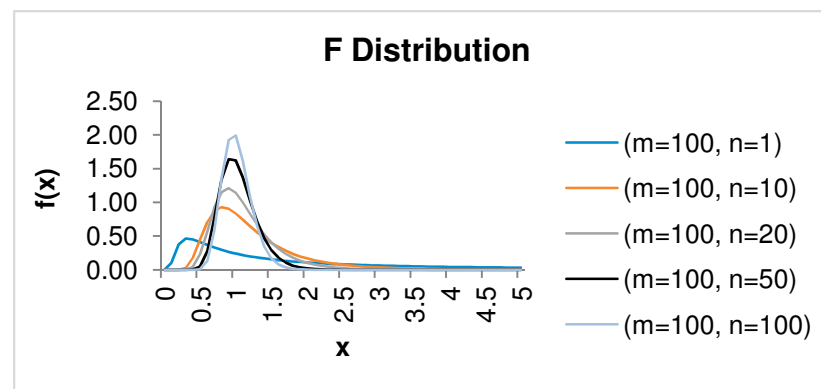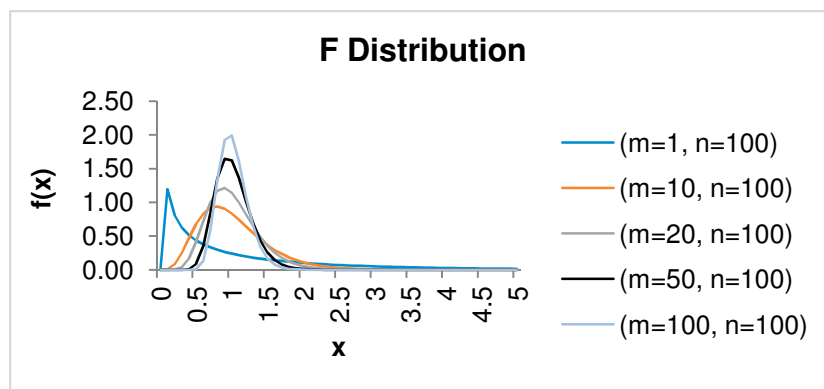
Probability Density Function

$$f(x) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \times \frac{\left(\frac{m}{n}\right)^{m/2} x^{\frac{m}{2}-1}}{\left(1+\frac{mx}{n}\right)^{\frac{m+n}{2}}}, \quad m > 0, n > 0, x > 0, \Gamma \text{ is the Gamma Function}: \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

m : DF of χ^2 random variable in the numerator
n : DF of χ^2 random variable in the denominator

Main Properties

- If X_1 and X_2 are independently distributed chi-square variables with m and n df respectively, then $F = [(X_1 / m) / (X_2 / n)]$ follows (Fisher's) F distribution with m and n df
- Like χ^2 distribution, F distribution is skewed to the right. But as m & n become large, F distribution approaches normal distribution

Applications

- Hypothesis Testing (e.g. Test of equality of variance of two populations, ANOVA test etc)

A.3.5. Student's t Distribution
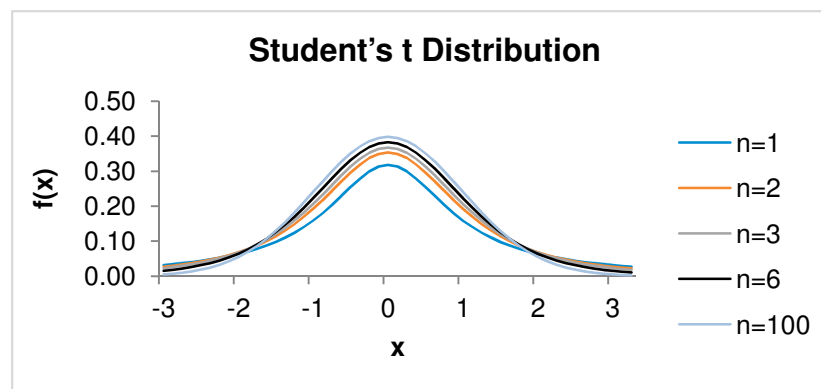
Probability Density Function

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}} \times \left(1 + \frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}, \ -\infty < x < \infty, \ n \geq 1, \ \Gamma \text{ is the Gamma Function}: \Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$$

n : DF of χ^2 random variable in the denominator

Main Properties

- Let Z and S be independent random variables such that Z ~ N(0,1) and n S^2 ~ χ^2_n. The distribution of t = Z / S is called Student's t distribution with df = n. As df increases, t distribution approximates the normal distribution.

- Mean = 0 for n > 1, Variance = n / (n-2) for n > 2, Median = 0, Skewness = 0

Applications

- Hypothesis Testing (e.g. Estimation of population parameters when sample size is small or when population variance is unknown)

A.3.6. Exponential Distribution
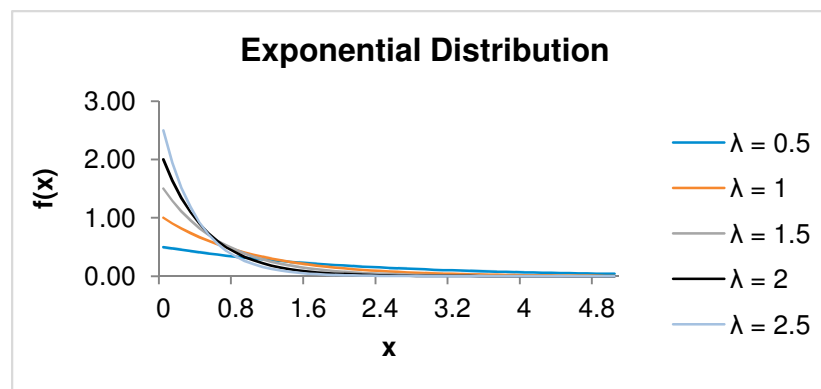
Probability Density Function

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \lambda > 0$$
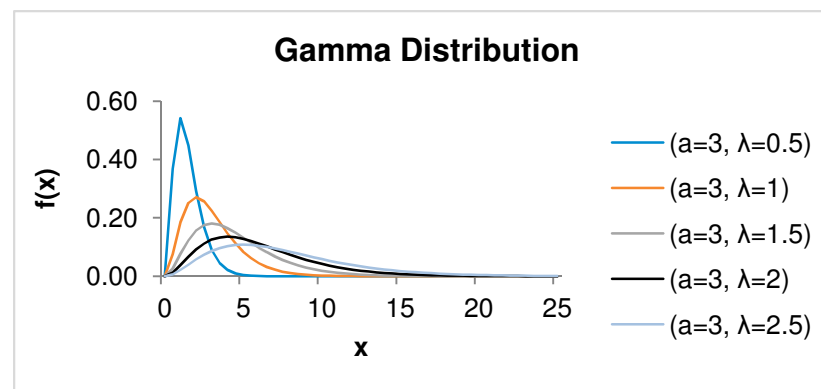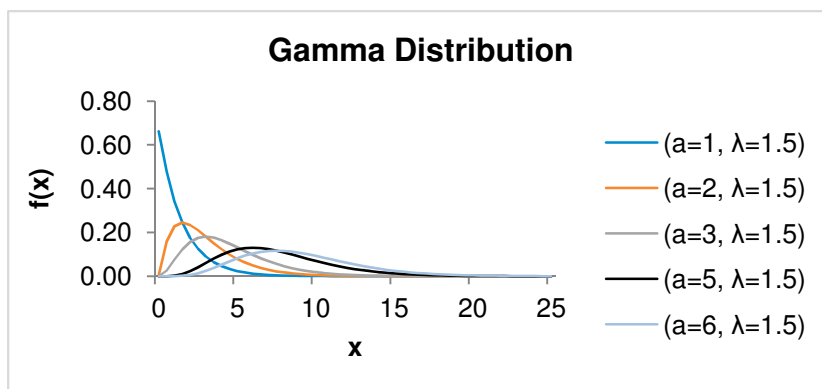
Main Properties

- The exponential distribution is used to model a random variable with mean $1 / \lambda$, that represents the waiting time until the first even to occur, where events are generated by a Poisson process with mean λ (i.e. events occur continuously and independently at a constant average rate λ)
- Mean = $1 / \lambda$ and Variance = $1 / \lambda^2$
- In case of a standard exponential distribution, $\lambda = 1$

Applications

- The exponential distribution is primarily used in reliability applications

Gamma Distribution

(a=1, λ=1.5)
(a=2, λ=1.5)
(a=3, λ=1.5)
(a=5, λ=1.5)
(a=6, λ=1.5)

Gamma Distribution

(a=3, λ=0.5)
(a=3, λ=1)
(a=3, λ=1.5)
(a=3, λ=2)
(a=3, λ=2.5)

Probability Density Function

$$f(x) = \frac{\lambda^a e^{-\lambda x} x^{a\text{-}1}}{\Gamma(a)}, \quad x > 0, \lambda > 0, a > 0$$
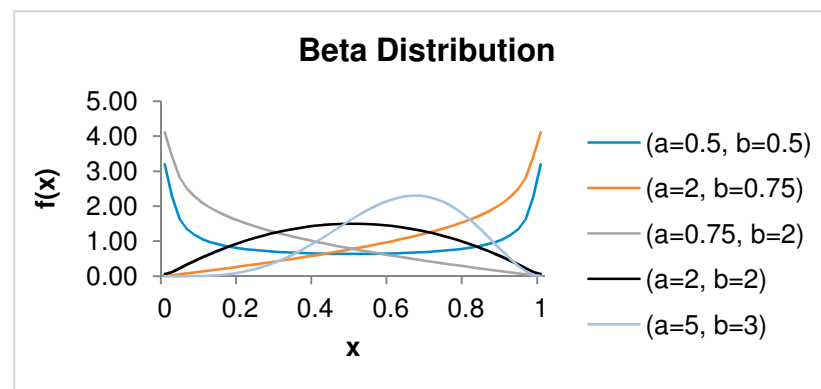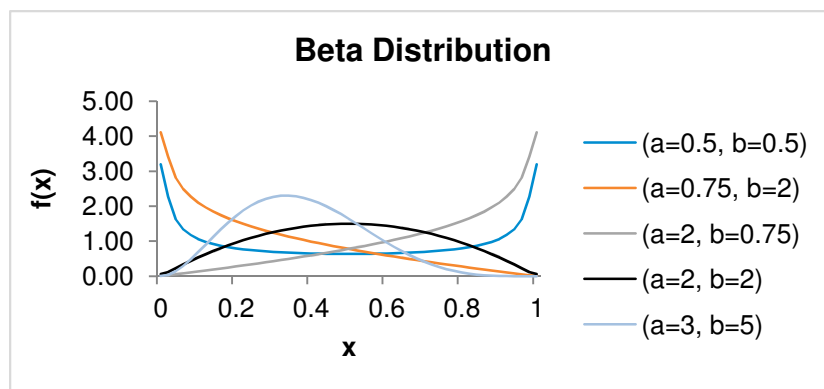
Main Properties

- The gamma distribution can be viewed as a generalization of the exponential distribution. The gamma random variable represents the waiting time until the ath event to occur.
- Mean = a / λ and Variance = a / λ²

Applications

- For modeling size of insurance claims
- For modeling size of rainfalls

A.3.8. Beta Distribution

Beta Distribution

Beta Distribution

Probability Density Function

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1, a > 0, b > 0$$

Main Properties

- a and b are shape parameters. The density curve is
  - U-shaped when a < 1 and b < 1
  - Symmetric about 0.5 when a = b > 1
  - J-shaped when (a-1)(b-1) < 0
  - Unimodal for all other values of a and b
- Mean = a / (a + b) and Variance = a b / [(a + b)^2(a + b + 1)]

Applications

- For modeling events which are constrained to take place within an interval defined by a minimum and maximum value

Thanks

For queries, contact Varun Aggarwal at Varun.Aggarwal@exlservice.com