Lecture 18

Section 4.2 (Part I)

Definition: Let A be an $m \times n$ matrix. The null space of A, namely Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = 0$, i.e.

$$\text{Nul } A = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = 0\}.$$

Example: Let

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{u} = \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}.$$

Is $\mathbf{u} \in$ Nul A?

Solution: We test if $A\mathbf{u} = \mathbf{0}$. Notice that

$$A\mathbf{u} = \begin{pmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 6 + (-2) \cdot 1 + 4 \cdot (-1) \\ 3 \cdot 6 + 7 \cdot 1 + 0 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 25 \end{pmatrix} \neq \mathbf{0}.$$

So $\mathbf{u} \notin$ Nul A.

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. In other words, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Remark: Notice that the linear equations in the above theorem must be homogeneous (i.e. they have no constant term). Otherwise, $\mathbf{0}$ will not contained in the set of solutions, violating the definition of a subspace.

Proof: Since A has n columns, then Nul $A \subseteq \mathbb{R}^n$. Notice that $\mathbf{0} \in$ Nul A, as $A\mathbf{0} = \mathbf{0}$. Now, suppose $\mathbf{u}, \mathbf{v} \in$ Nul A. Then, $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. We see that

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

So, $\mathbf{u} + \mathbf{v} \in$ Nul A, i.e. Nul A is closed under vector addition. Additionally, suppose c is a scalar. Then,

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c(\mathbf{0}) = \mathbf{0}.$$

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Example: Find a spanning set for Nul A where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{pmatrix}.$$

Solution: We consider the augmented matrix $[A\,\mathbf{0}]$. We first put this matrix into reduced row echelon form. We see that

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 7 & 0 \end{pmatrix} \xrightarrow{r_2 \to r_2 - 2 \cdot r_1} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \to r_1 - 2 \cdot r_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

So, we have that $x_1 + x_3 = 0$ and $x_2 + x_3 = 0$. We identify the free variables by the columns which do not have pivots. It follows that x_3 is a free variable. So,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

So, $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a spanning set for Nul A.

Example: Find a spanning set for Nul A where

$$A = \begin{pmatrix} 2 & 4 & 3 & 9 \\ 4 & 8 & 7 & 13 \end{pmatrix}.$$

Solution: As before, we consider the augmented matrix $[A\,\mathbf{0}]$. We first put this matrix into reduced row echelon form. We see that

$$\begin{pmatrix} 2 & 4 & 3 & 9 & 0 \\ 4 & 8 & 7 & 13 & 0 \end{pmatrix} \xrightarrow{r_2 \to r_2 - 2 \cdot r_1} \begin{pmatrix} 2 & 4 & 3 & 9 & 0 \\ 0 & 0 & 1 & -5 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \to r_1 - 3 \cdot r_2} \begin{pmatrix} 2 & 4 & 0 & 24 & 0 \\ 0 & 0 & 1 & -5 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \to \frac{r_1}{2}} \begin{pmatrix} 1 & 2 & 0 & 12 & 0 \\ 0 & 0 & 1 & -5 & 0 \end{pmatrix}.$$

So, we have that $x_1 + 2x_2 + 12x_4 = 0$ and $x_3 - 5x_4 = 0$. We see that columns 1 and 3 contain pivots, and so x_2 and x_4 are free variables. Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 - 12x_4 \\ x_2 \\ 5x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -12 \\ 0 \\ 5 \\ 1 \end{pmatrix}.$$

So, $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -12 \\ 0 \\ 5 \\ 1 \end{pmatrix} \right\}$ is a spanning set for Nul A.

Example: Find a spanning set for Nul A where

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 & 5 \\ 2 & 6 & 1 & 7 & 13 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution: We consider the augmented matrix $[A\,\mathbf{0}]$. We first put this matrix into reduced row echelon form. We see that

$$\begin{pmatrix} 1 & 3 & 0 & 2 & 5 & 0 \\ 2 & 6 & 1 & 7 & 13 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 \to r_2 - 2\cdot r_1} \begin{pmatrix} 1 & 3 & 0 & 2 & 5 & 0 \\ 0 & 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have that $x_1 + 3x_2 + 2x_4 + 5x_5 = 0$ and $x_3 + 3x_4 + 3x_5 = 0$. Columns 1 and 3 have pivots, implying that x_2, x_4, and x_5 are free variables. Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 - 5x_5 \\ x_2 \\ -3x_4 - 3x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -5 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

So, $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a spanning set for $\operatorname{Nul} A$.

Remark: The size of the spanning set of $\operatorname{Nul} A$ is equal to the number of free variables of the equation $A\mathbf{x} = \mathbf{0}$.

Example: Suppose

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \, x_1 + 3 = x_2 - 7x_3 \right\}.$$

Is W a subspace of $\mathbb{R}^3$?

Solution: Clearly, $W \subseteq \mathbb{R}^3$. However, notice that $\mathbf{0} \notin W$, as $0 + 3 \neq 0 - 7 \cdot 0$. We can see that W does not satisfy the aforementioned theorem, as $x_1 + 3 - x_2 + 7x_3 = 0$ is not a homogeneous equation (we have 3 as the constant term). Note that, if instead, the condition to belong to the set W was $x_1 = x_2 - 7x_3$, then W would be a subspace of $\mathbb{R}^3$ (as $x_1 - x_2 + 7x_3 = 0$ has no constant term).