Lecture 19

Section 4.2 (Part II)

Definition: The column space of an $m \times n$ matrix A, denoted by $\operatorname{Col} A$, is the set of all linear combinations of the columns of A. In other words, if $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$, where each $\mathbf{a}_i$ is a column of A, then $\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.

Theorem: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Remark: Let A be an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$. Note that

$$A\mathbf{x} = \begin{pmatrix} \sum_{i=1}^n a_{1i}x_i \\ \sum_{i=1}^n a_{2i}x_i \\ \vdots \\ \sum_{i=1}^n a_{mi}x_i \end{pmatrix} = \sum_{i=1}^n x_i \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}.$$

It follows that

$$\operatorname{Col} A = \{\mathbf{y} \mid \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

Example: Find A such that the given set is $\operatorname{Col} A$.

$$\left\{ \begin{pmatrix} 8a + 22b \\ -3b \\ a + 4b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

Solution: Notice that

$$\begin{pmatrix} 8a + 22b \\ -3b \\ a + 4b \end{pmatrix} = a \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 22 \\ -3 \\ 4 \end{pmatrix}.$$

So,

$$\left\{ \begin{pmatrix} 8a + 22b \\ -3b \\ a + 4b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\} = \left\{ a \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 22 \\ -3 \\ 4 \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}. \tag{1}$$

Since $\operatorname{Col} A$ is defined to be all linear combinations of the columns of A, we can reconstruct A using (1). It follows that

$$A = \begin{pmatrix} 8 & 22 \\ 0 & -3 \\ 1 & 4 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 22 & 8 \\ -3 & 0 \\ 4 & 1 \end{pmatrix}.$$

Note: The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^m$.

Definition: If A is an $m \times n$ matrix, each row has n entries, and so each row can be associated to a vector in $\mathbb{R}^n$, namely a **row vector**.

Definition: The row space of an $m \times n$ matrix A is the set of all linear combinations of the row vectors of A, denoted by Row A.

Remark: The row space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

Example: Suppose

$$A = \begin{pmatrix} 3 & 5 & 0 & 9 \\ 2 & 86 & 1 & 1 \end{pmatrix}.$$

(a) Find k such that Nul A is a subspace of $\mathbb{R}^k$.
(b) Find k such that Col A is a subspace of $\mathbb{R}^k$.

Solution: (a) Because there are 4 columns in A, then $k = 4$.
(b) Because there are 2 rows in A, then $k = 2$.

Example: Suppose

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 2 \end{pmatrix}.$$

(a) Is $\mathbf{v} = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ in Col A? Can $\mathbf{v}$ be in Nul A?
(b) Is $\mathbf{u} = \left(\begin{smallmatrix} 1 \\ -2 \\ 1 \end{smallmatrix}\right)$ in Nul A? Can $\mathbf{u}$ be in Col A?

Solution: (a) We consider $[A\,\mathbf{v}]$. Let's put this matrix into reduced row echelon form. So,

$$\begin{aligned}
\begin{pmatrix} 2 & 4 & 6 & 2 \\ 1 & 3 & 2 & 1 \end{pmatrix} &\xrightarrow{r_1 \to r_1 - r_2} \begin{pmatrix} 1 & 1 & 4 & 1 \\ 1 & 3 & 2 & 1 \end{pmatrix} \\
&\xrightarrow{r_2 \to r_2 - r_1} \begin{pmatrix} 1 & 1 & 4 & 1 \\ 0 & 2 & -2 & 0 \end{pmatrix} \\
&\xrightarrow{r_2 \to \frac{r_2}{2}} \begin{pmatrix} 1 & 1 & 4 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \\
&\xrightarrow{r_1 \to r_1 - r_2} \begin{pmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.
\end{aligned}$$

Our system of equations is consistent, and so $\mathbf{v} \in$ Col A. Note that Nul A is a subspace of $\mathbb{R}^3$, and so $\mathbf{v}$ cannot be in the null space of A, as it only has 2 entries.
(b) We see that

$$A\mathbf{u} = \begin{pmatrix} 2 - 8 + 6 \\ 1 - 6 + 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \neq \mathbf{0}.$$

So, $\mathbf{u} \notin \text{Nul}\, A$. Note that Col A is a subspace of $\mathbb{R}^2$, and so $\mathbf{u}$ cannot be in the column space, as it has 3 entries.

Definition: A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x} \in V$ a unique vector $T(\mathbf{x}) \in W$ where

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$, and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u} \in V$ and scalars c.

Definition: The kernel of a linear transformation T is defined to be

$$\ker T = \{\mathbf{u} \in V \mid T(\mathbf{u}) = \mathbf{0}\}.$$

Definition: The range of a linear transformation T is defined to be

$$\operatorname{im} T = \{T(\mathbf{u}) \mid \mathbf{u} \in V\}.$$

Note: If T is associated to a matrix, then the kernel and the range of T correspond to the null space and the column space of T, respectively.

Note: The kernel of T is a subspace of V, and the range of T is a subspace of W.

Example: Let V be the vector space of all real-valued functions f defined on an interval $[a, b]$ such that each f is differentiable and each f' is continuous on $[a, b]$. Let W be the vector space of all continuous functions on $[a, b]$. Suppose $T : V \to W$ such that $T(f) = f'$ for all $f \in V$. Then, we have that $T(f + g) = T(f) + T(g)$ for all $f, g \in V$, as the derivative is distributive over addition. Additionally, $T(cf) = cT(f)$ for all $f \in V$ and scalars c, due to the constant multiple rule of differentiation. One can show that $\ker T$ is the set of constant functions on $[a, b]$ and $\operatorname{im} T = W$.