Solving linear systems II

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Last week

- Solving linear systems
  - Gaussian elimination
  - LU factorization

Review: Gaussian elimination

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
a_{21}	a_{22}	a_{23}	a_{24}	b_2
a_{31}	a_{32}	a_{33}	a_{34}	b_3
a_{41}	a_{42}	a_{43}	a_{44}	b_4

- A linear system with 4 unknowns and 4 equations
- Steps
  1. Subtract multiples $l_{i1} = a_{i1} / a_{11}$ of row 1 from row i, $i = 2, \ldots, 4$.
  2. Set $a'_{ik} = a_{ik} - l_{i1}\, a_{1k}$, $i, k = 2, \ldots, 4$.
  3. Set $b'_i = b_i - l_{i1}\, b_1$, $i = 2, \ldots, 4$.

Review: Gaussian elimination

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
0	a'_{22}	a'_{23}	a'_{24}	b'_2
0	a'_{32}	a'_{33}	a'_{34}	b'_3
0	a'_{42}	a'_{43}	a'_{44}	b'_4

- A linear system with 4 unknowns and 4 equations
- Steps
  1. Subtract multiples $l'_{i2} = a'_{i2} / a'_{22}$ of row 2 from row i, $i = 3, \ldots, 4$.
  2. Set $a''_{ik} = a'_{ik} - l'_{i2} a'_{2k}$, $i, k = 3, \ldots, 4$.
  3. Set $b''_i = b'_i - l'_{i2} b'_2$, $i = 3, \ldots, 4$.

Review: Gaussian elimination

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
0	a'_{22}	a'_{23}	a'_{24}	b'_2
0	0	a''_{33}	a''_{34}	b''_3
0	0	a''_{43}	a''_{44}	b''_4

- A linear system with 4 unknowns and 4 equations
- Steps
  1. Subtract multiples $l''_{i3} = a''_{i3} / a''_{33}$ of row 3 from row i, $i = 4$.
  2. Set $a'''_{ik} = a''_{ik} - l''_{i3}\, a''_{3k}$, $i, k = 4$.
  3. Set $b'''_i = b''_i - l''_{i3}\, b''_3$, $i = 4$.

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
0	a'_{22}	a'_{23}	a'_{24}	b'_2
0	0	a''_{33}	a''_{34}	b''_3
0	0	0	a'''_{44}	b'''_4

Review: A = LU

- Actual storage scheme

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
l_{21}	a'_{22}	a'_{23}	a'_{24}	b'_2
l_{31}	l'_{32}	a''_{33}	a''_{34}	b''_3
l_{41}	l'_{42}	l'_{43}	a'''_{44}	b'''_4

Review: Complexity of Gaussian Elimination

- Elimination: $O(n^3)$
- Substitution: $O(n^2)$

Today

- Error estimation
- Improving the naïve approach
  - Gaussian elimination with ***partial pivoting*** (PA = LU factorization)

Error estimation

true solution: $\underline{x}$
approximate solution: $\underline{x}_a$

- Two questions regarding the accuracy of $\underline{x}_a$ as an approximation to the solution of the linear system of equations $A\underline{x} = \underline{b}$.

1. First we investigate what we can derive from the residual $\underline{r}_a = \underline{b} - A\underline{x}_a$. Note that $\underline{r} = \underline{b} - A\underline{x} = \underline{0}$
2. Then, how sensitive is the solution to the perturbations in the initial data? That is, what is the effect of errors in the initial data ($\underline{b}$, A) on the solution $\underline{x}$?

Infinity norm

- The **infinity norm**, or the **maximum norm**, of the vector $\underline{x} = [x_1, \ldots, x_n]^T$ is

$$\left\|\underline{x}\right\|_\infty = \max \; |x_i|, \qquad i = 1, \ldots, n.$$

- The infinity norm of x = [3, 2, -8, 1, 4, -2, -9, -4]^T is ?

Infinity norm

- The **matrix (absolute row sum) norm** of an *n* x *n* matrix A is

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|.$$

- Example

$$A = \begin{bmatrix} 1 & 1 \\ 1.0001 & -1 \end{bmatrix}$$

$$\|A\|_\infty = \ 2.0001$$

More definitions

true solution: $\underline{x}$
approximate solution: $\underline{x}_a$

- **Residual** (note: a vector)

  $\underline{b} - A\underline{x}_a$

- **Backward error**

  $\left\|\underline{b} - A\underline{x}_a\right\|_\infty$

- **Forward error**

  $\left\|\underline{x} - \underline{x}_a\right\|_\infty$

Example

- Consider the linear system:

$$x_1 + x_2 = 2$$
$$1.0001x_1 + x_2 = 2.0001$$

- The solution $\underline{x} = [1, 1]^T$

- Consider the approximate solution $\underline{x}_a = [-1, 3.0001]^T$

$$x_1 + x_2 = 2$$
$$1.0001x_1 + x_2 = 2.0001$$

- $\underline{x} = [1,\ 1]^T$
- $\underline{x}_a = [-1,\ 3.0001]^T$
- The **backward error** is:

$$\left\|\underline{r}_a\right\|_\infty = \left\|\underline{b} - A\underline{x}_a\right\|_\infty = \left\|\begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3.0001 \end{bmatrix}\right\|_\infty$$

$$= \left\|\begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 2.0001 \\ 2 \end{bmatrix}\right\|_\infty = \left\|\begin{bmatrix} -0.0001 \\ 0.0001 \end{bmatrix}\right\|_\infty$$

$$= 0.0001$$ 小

$$x_1 + x_2 = 2$$
$$1.0001x_1 + x_2 = 2.0001$$

- $\underline{x} = [1,\ 1]^T$
- $\underline{x}_a = [-1,\ 3.0001]^T$
- The **forward error** is:

$$\|\underline{x} - \underline{x}_a\|_\infty = \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3.0001 \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} 2 \\ -2.0001 \end{bmatrix} \right\|_\infty$$

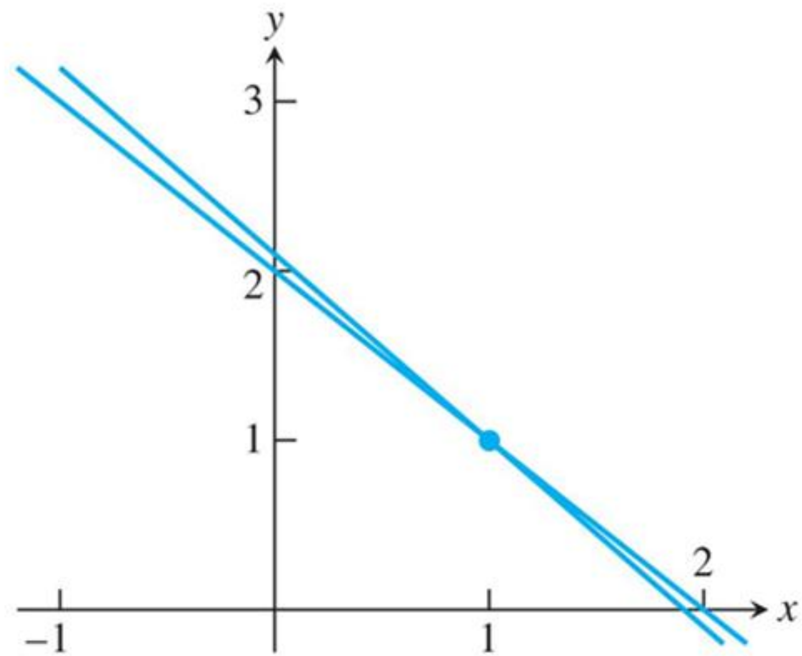
$$= 2.0001$$

大

What does this mean?

backward error 小 forward error 大

- Even though the “approximate solution” is relatively far from the exact solution, it nearly lies on both lines!



The **error magnification factor**

$$\text{error magnification factor} = \frac{\text{relative forward error}}{\text{relative backward error}} = \frac{\dfrac{||x - x_a||_\infty}{||x||_\infty}}{\dfrac{||r||_\infty}{||b||_\infty}}$$

- Example: $\underline{x} = [1, 1]^T$, $\underline{x}_a = [-1, 3.0001]^T$, $\underline{b} = [2, 2.0001]^T$

relative forward error: 2.0001/1 = 2.0001

relative backward error: 0.0001/2.0001 = 0.00005

error magnification factor : 2.0001/0.00005 = **40004.0001**

The **condition number**

- The condition number of a square matrix A, **cond**(A), is the maximum possible error magnification factor for solving A$\underline{x}$ = $\underline{b}$, over all right-hand sides $\underline{b}$.
- The condition number of the *n* x *n* matrix A is

$$\text{cond}(A) = ||A|| \cdot ||A^{-1}||.$$

$$A \times A^{-1} = A^{-1} \times A = \mathbf{I}$$

Example

$$x_1 + x_2 = 2$$
$$1.0001x_1 + x_2 = 2.0001$$

$$A = \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \qquad \|A\|_\infty = 2.0001$$

$$A^{-1} = \begin{bmatrix} -10000 & 10000 \\ 10001 & -10000 \end{bmatrix} \qquad \|A^{-1}\|_\infty = 20001$$

The condition number of A is

cond(A) = 2.0001 * 20001 = **40004.0001**

So, how does the residual $\hat{\mathbf{r}} := \mathbf{b} - A\hat{\mathbf{x}}$ affect the error $\mathbf{z} := \hat{\mathbf{x}} - \mathbf{x}$?

$$A\mathbf{z} = A(\hat{\mathbf{x}} - \mathbf{x}) = A\hat{\mathbf{x}} - \mathbf{b} = -\hat{\mathbf{r}}.$$

$$\|\mathbf{b}\| = \|A\mathbf{x}\| \le \|A\|\,\|\mathbf{x}\|, \quad \rightarrow \frac{\|\mathbf{b}\|}{\|A\|} \le \|\mathbf{x}\|$$

$$\|\mathbf{z}\| = \|-A^{-1}\hat{\mathbf{r}}\| \le \|A^{-1}\|\,\|\hat{\mathbf{r}}\|$$

$$\frac{\|\mathbf{z}\|}{\|\mathbf{x}\|} \le \frac{\|A^{-1}\|\|\hat{\mathbf{r}}\|}{\|\mathbf{b}\|/\|A\|} \le \|A\|\,\|A^{-1}\|\,\frac{\|\hat{\mathbf{r}}\|}{\|\mathbf{b}\|}$$

relative forward error

relative backward error

Summary of error estimation

$$cond(A) \cdot \epsilon$$

$$\text{error magnification factor} = \frac{\text{relative forward error}}{\text{relative backward error}} = \text{cond(A)}$$

$$\epsilon$$

$$= \|A\| \times \|A^{-1}\|$$

Today

- Error estimation
- Improving the naïve approach
  - Gaussian elimination with partial pivoting (PA = LU factorization)

Cases where the naïve Gaussian elimination may fail?

- When meeting a zero multiple…

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

- Could be solved by exchanging row 1 and row 2

x_1	x_2	1
0	1	4
1	1	7

$\Rightarrow$

x_1	x_2	1
1	1	7
0	1	4

- It is rare to hit a precisely zero pivot, but common to hit a very small one.

Swamping

- Consider the system:

$$\begin{aligned} 10^{-20}x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 4. \end{aligned}$$

- Exact solution

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow[(2)-(1)*10^{20}]{} \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2-10^{20} & 4-10^{20} \end{array}\right]$$

$$\begin{bmatrix} 10^{-20} & 1 & | & 1 \\ 1 & 2 & | & 4 \end{bmatrix} \xrightarrow[(2)-(1)*10^{20}]{} \begin{bmatrix} 10^{-20} & 1 & | & 1 \\ 0 & 2-10^{20} & | & 4-10^{20} \end{bmatrix}$$

$$(2-10^{20})x_2 = 4-10^{20} \longrightarrow x_2 = \frac{4-10^{20}}{2-10^{20}};$$

$$10^{-20}x_1 + \frac{4-10^{20}}{2-10^{20}} = 1$$

$$x_1 = 10^{20}\left(1 - \frac{4-10^{20}}{2-10^{20}}\right)$$

$$x_1 = \frac{-2\times 10^{20}}{2-10^{20}}.$$

$$[x_1, x_2] = \left[\frac{2\times 10^{20}}{10^{20}-2}, \frac{4-10^{20}}{2-10^{20}}\right] \approx [2, 1].$$

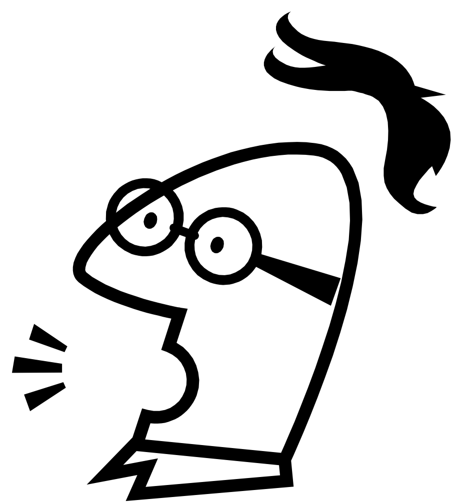
$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow[(2)\,-\,(1)*10^{20}]{} \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2-10^{20} & 4-10^{20} \end{array}\right]$$

- IEEE double precision
  - $2 - 10^{20} = -10^{20}$
  - $4 - 10^{20} = -10^{20}$

$$-10^{20}x_2 = -10^{20} \longrightarrow x_2 = 1.$$

$$10^{-20}x_1 + 1 = 1,$$

$$[x_1, x_2] = [0, 1].$$



$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 10^{-20} & 1 & 1 \end{array}\right] \xrightarrow[(2)-(1)*10^{-20}]{} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1-2\times 10^{-20} & 1-4\times 10^{-20} \end{array}\right]$$

- IEEE double precision, after row exchange
  - $2 - 10^{20} = -10^{20}$
  - $4 - 10^{20} = -10^{20}$

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_2 &= 1 \end{aligned}$$

$[x_1, x_2] = [2,1]$

The difference?

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow[\text{(2) – (1)*}10^{20}]{} \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2 - 10^{20} & 4 - 10^{20} \end{array}\right]$$

"Swamp" the bottom equation!
Two independent equations → two copies of the top equation

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 10^{-20} & 1 & 1 \end{array}\right] \xrightarrow[\text{(2) – (1)*}10^{-20}]{} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 - 2 \times 10^{-20} & 1 - 4 \times 10^{-20} \end{array}\right]$$

Remedy for swamping (and zero pivoting)

- Multipliers in Gaussian elimination should be kept as small as possible to avoid swamping.
- **Partial pivoting**
  - Forces the absolute value of multipliers to be no larger than 1

Partial pivoting

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
a_{21}	a_{22}	a_{23}	a_{24}	b_2
a_{31}	a_{32}	a_{33}	a_{34}	b_3
a_{41}	a_{42}	a_{43}	a_{44}	b_4

- Searches for the maximal element in modulus: The index p of the pivot in the k-th step of GE is determined by

$$|a_{pk}^{(k-1)}| = \max_{i \geq k} |a_{ik}^{(k-1)}|$$

- If $p > k$ then rows p and k are exchanged.
- This strategy implies that $|l_{ik}| \leq 1$.

Example: Partial pivoting

- Solve the linear system:

$$
\begin{aligned}
x_1 - x_2 + 3x_3 &= -3 \\
-x_1 - 2x_3 &= 1 \\
2x_1 + 2x_2 + 4x_3 &= 0.
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c} 1 & -1 & 3 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 2 & 4 & 0 \end{array}\right]
\qquad
\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]
$$

$$
\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]
\qquad
\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right]
$$

Example: Partial pivoting (cont.)

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & -2 & 1 & -3 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\frac{1}{2}x_3 = -\frac{1}{2}$$

$$-2x_2 + x_3 = -3$$

$$2x_1 + 2x_2 + 4x_3 = 0,$$

$$\boxed{x = [1, 1, -1]}$$

Permutation matrix

- A permutation matrix is a n x n matrix consisting of all zeros, except for a single 1 in every row and column.
- Is the identity matrix a permutation matrix?
- How many 3x3 permutation matrices?

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

Fundamental theorem of Permutation matrices

- Let *P* be the *n* x *n* permutation matrix formed by a particular set of row exchanges applied to the identity matrix. Then, for any *n* x *n* matrix *A*, *PA* is the matrix obtained by applying exactly the same set of row exchanges to *A*.
- Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

PA = LU factorization

- Find the PA=LU factorization of $A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \longrightarrow \text{exchange rows 1 and 2} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{array}{c} \text{subtract } \frac{1}{2} \times \text{row 1} \\ \text{from row 2} \end{array} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ 1 & 3 & 1 \end{bmatrix} \longrightarrow \begin{array}{c} \text{subtract } \frac{1}{4} \times \text{row 1} \\ \text{from row 3} \end{array} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ \frac{1}{4} & 2 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \text{exchange rows 2 and 3} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -1 & 7 \end{bmatrix}$$

$$\longrightarrow \begin{array}{c} \text{subtract } -\frac{1}{2} \times \text{row 2} \\ \text{from row 3} \end{array} \longrightarrow \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 8 \end{bmatrix}$$

$$\underset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}}$$

Solving $A\underline{x} = \underline{b}$ (using $PA = LU$)

- $A\underline{x} = \underline{b}$
- $PA\underline{x} = P\underline{b}$
- $LU\underline{x} = P\underline{b}$
- $L(U\underline{x}) = P\underline{b}$, $U\underline{x} = \underline{c}$
- Solve

  $L\underline{c} = P\underline{b}$ for $\underline{c}$

  $U\underline{x} = \underline{c}$ for $\underline{x}$

Example: PA = LU for solving A$\underline{x}$ = $\underline{b}$

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}}_{U}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}}_{U}$$

Solve

L<u>c</u> = P<u>b</u> for <u>c</u>

U<u>x</u> = <u>c</u> for <u>x</u>

1. L<u>c</u> = P<u>b</u> for <u>c</u>

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix}$$

get <u>c</u> = $[0, 6, 8]^T$

2. U<u>x</u> = <u>c</u> for <u>x</u>

$$\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 8 \end{bmatrix}$$

get <u>x</u> = $[-1, 2, 1]^T$

程式練習

And, please upload your program on moodle.

- Use **PA = LU** factorization with pivoting to solve the linear system

$$\begin{pmatrix} 4.0 & 2.0 & -1.0 & 3.0 \\ 3.0 & -4.0 & 2.0 & 5.0 \\ -2.0 & 6.0 & -5.0 & -2.0 \\ 5.0 & 1.0 & 6.0 & -3.0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16.9 \\ -14.0 \\ 25.0 \\ 9.4 \end{pmatrix}$$

- Please output
  - the permutation matrix P
  - the factorization matrices L and U
  - the solution $\underline{x}$ (bonus 加分, if completed in course)