CS4102 Algorithms

Nate Brunelle

Spring 2018

Today’s Keywords

- Graphs
- MaxFlow/MinCut
- Ford-Fulkerson
- Edmunds-Karp

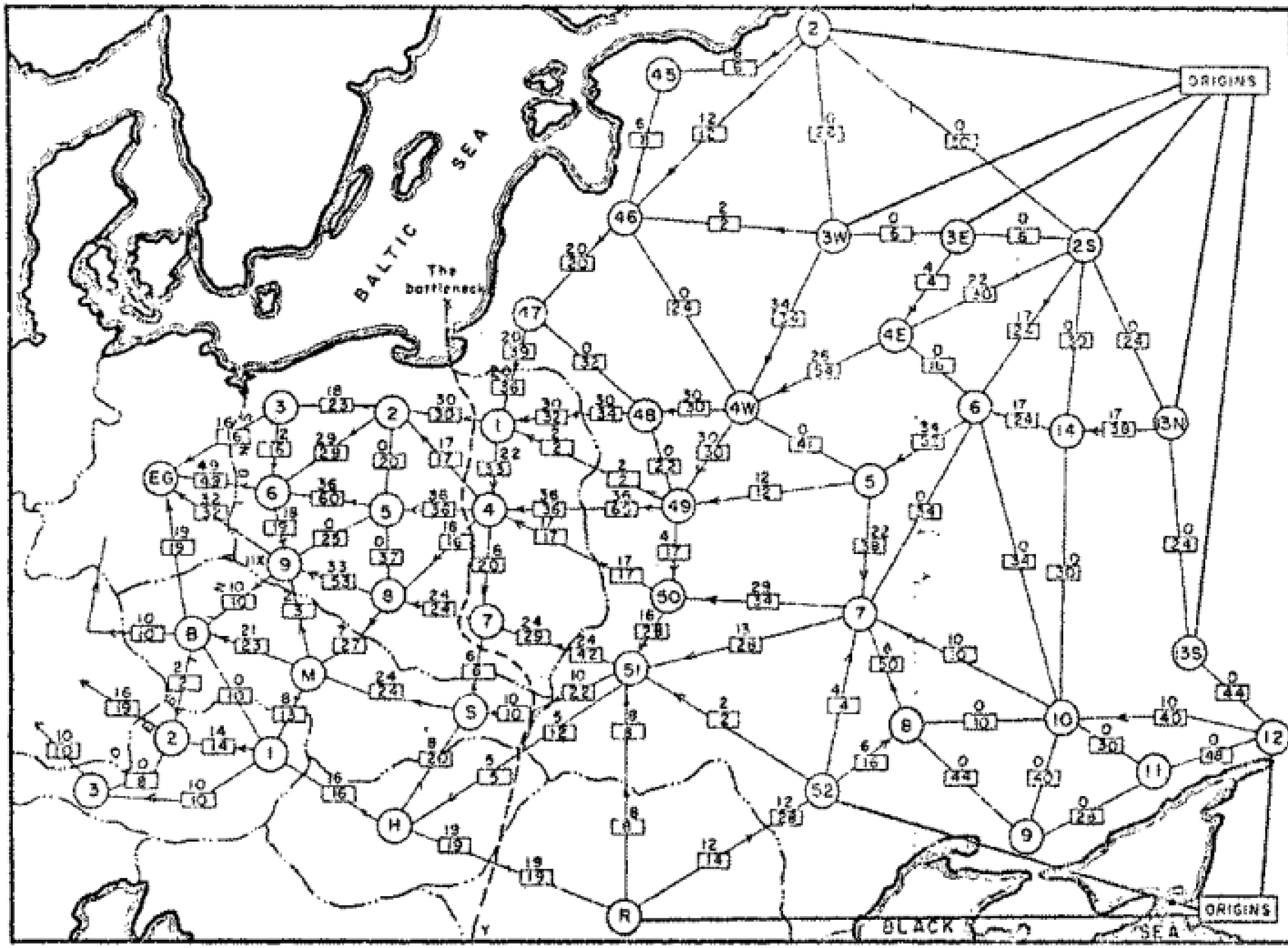
CLRS Readings

- Chapter 25
- Chapter 26

Homeworks

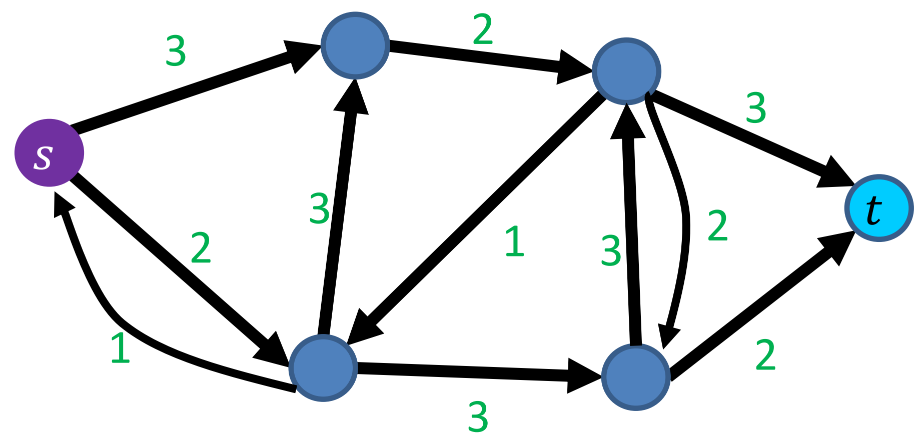
- HW7 Due Saturday 4/21 at 11pm
  - Written (use LaTeX)
  - Graphs

Max Flow / Min Cut



Railway map of Western USSR, 1955

Flow Network



Graph $G = (V, E)$

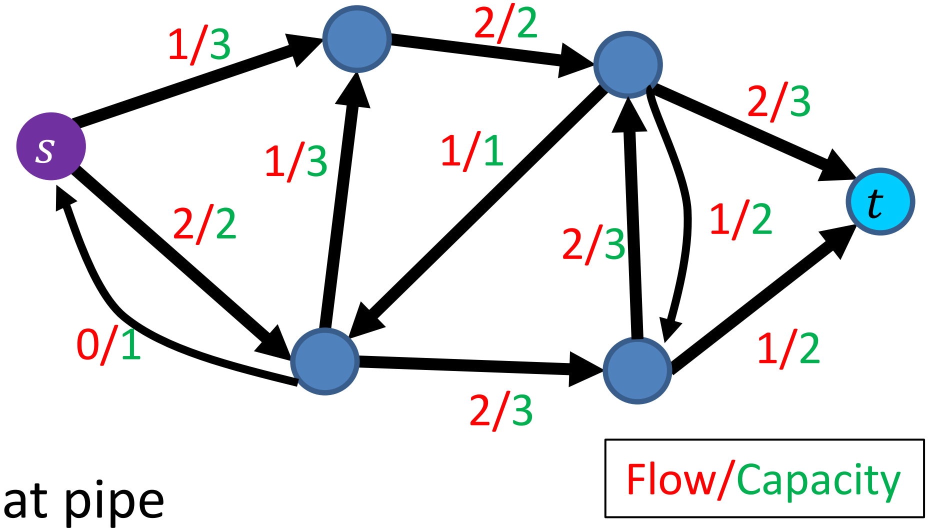
Source node $s \in V$

Sink node $t \in V$

Edge Capacities $c(e) \in$ Positive Real numbers

Max flow intuition: If s is a faucet, t is a drain, and s connects to t through a network of pipes with given capacities, what is the maximum amount of water which can flow from the faucet to the drain?

Flow



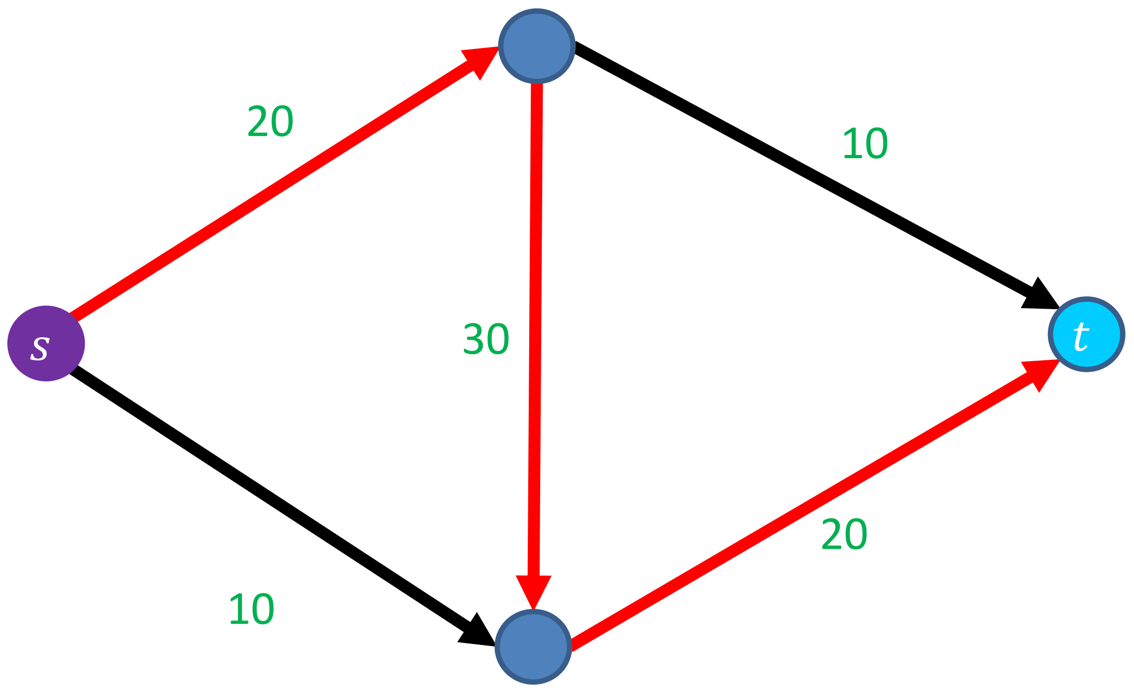
- Assignment of values to edges
  - $f(e) = n$
  - Amount of water going through that pipe
- Capacity constraint
  - $f(e) \leq c(e)$
  - Flow cannot exceed capacity
- Flow constraint
  - $\forall v \in V - \{s, t\}, inflow(v) = outflow(v)$
  - $inflow(v) = \sum_{x \in V} f(v, x)$
  - $outflow(v) = \sum_{x \in V} f(x, v)$
  - Water going in must match water coming out
- Flow of G: $|f| = outflow(s) - inflow(s)$ 3 in example above
  - Net outflow of s

Max Flow

- Of all valid flows through the graph, find the one which maximizes:
  - $|f| = outflow(s) - inflow(s)$

Greedy doesn't work

Saturate Highest Flow Path First



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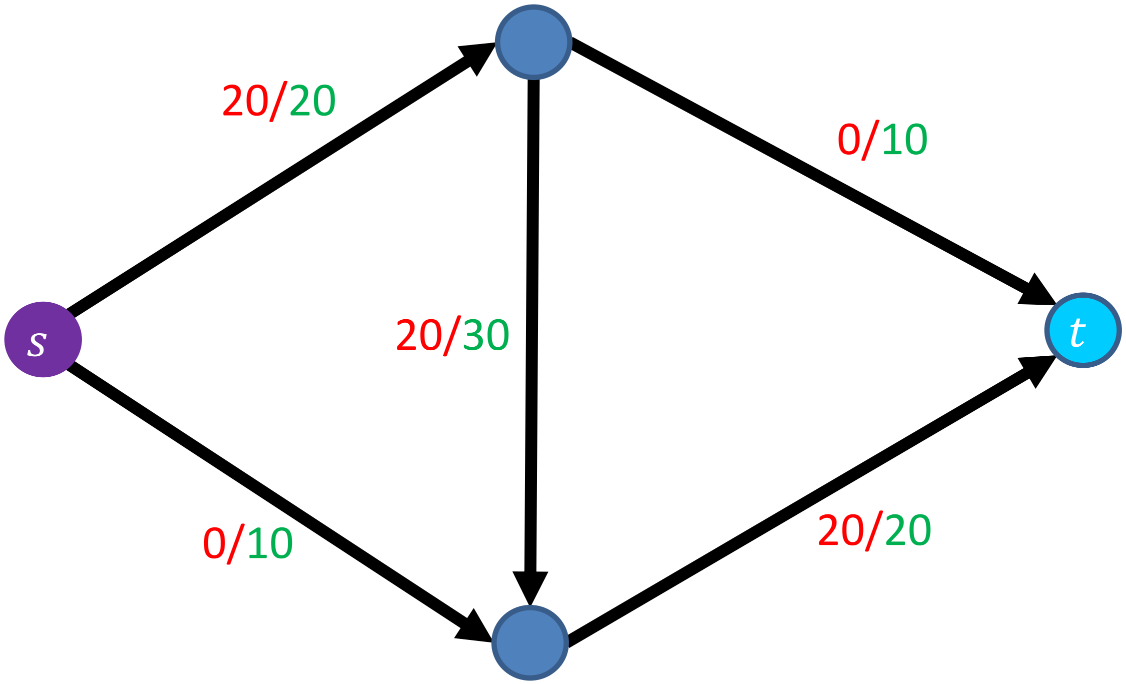
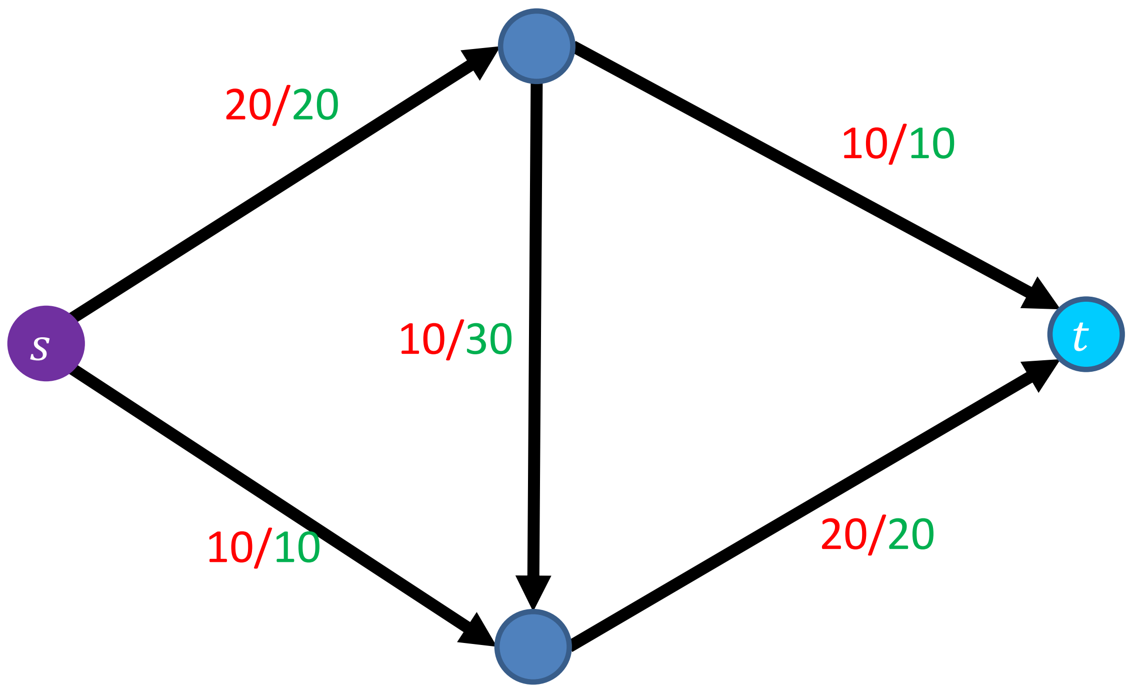


Overall Flow: $|f| = 20$

Greedy doesn't work

Better Solution

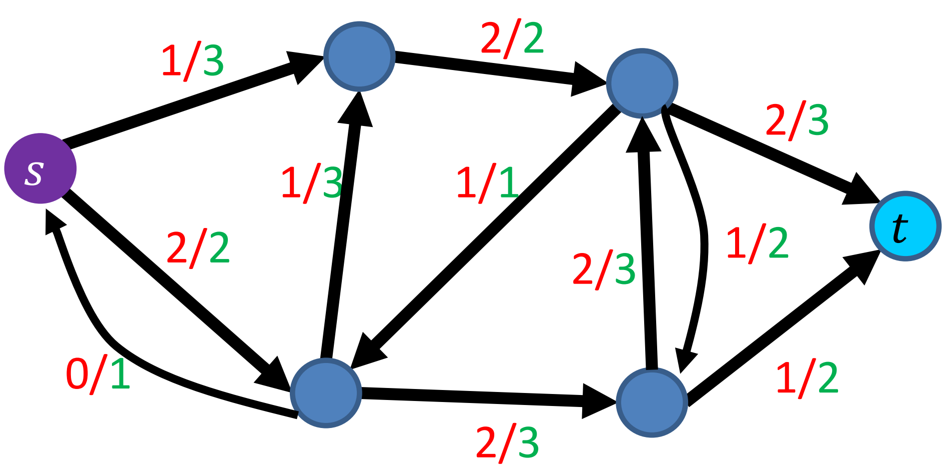


Overall Flow: $|f| = 30$

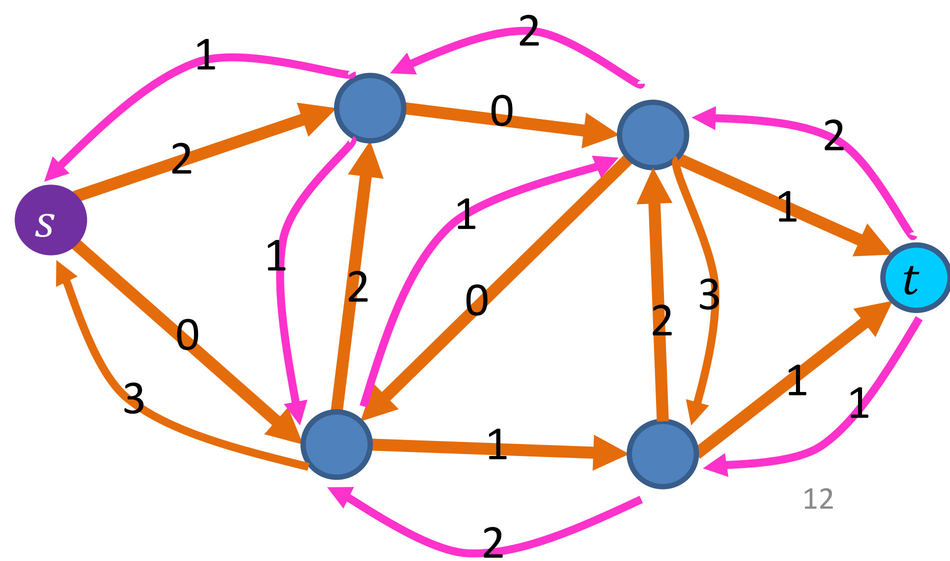
Residual Graph G_f

- Keep track of net available flow along each edge
- "Forward edges": weight is equal to available flow along that edge in the flow graph
  - $w(e) = c(e) - f(e)$
- "Back edges": weight is equal to flow along that edge in the flow graph
  - $w(e) = f(e)$

Flow Graph G

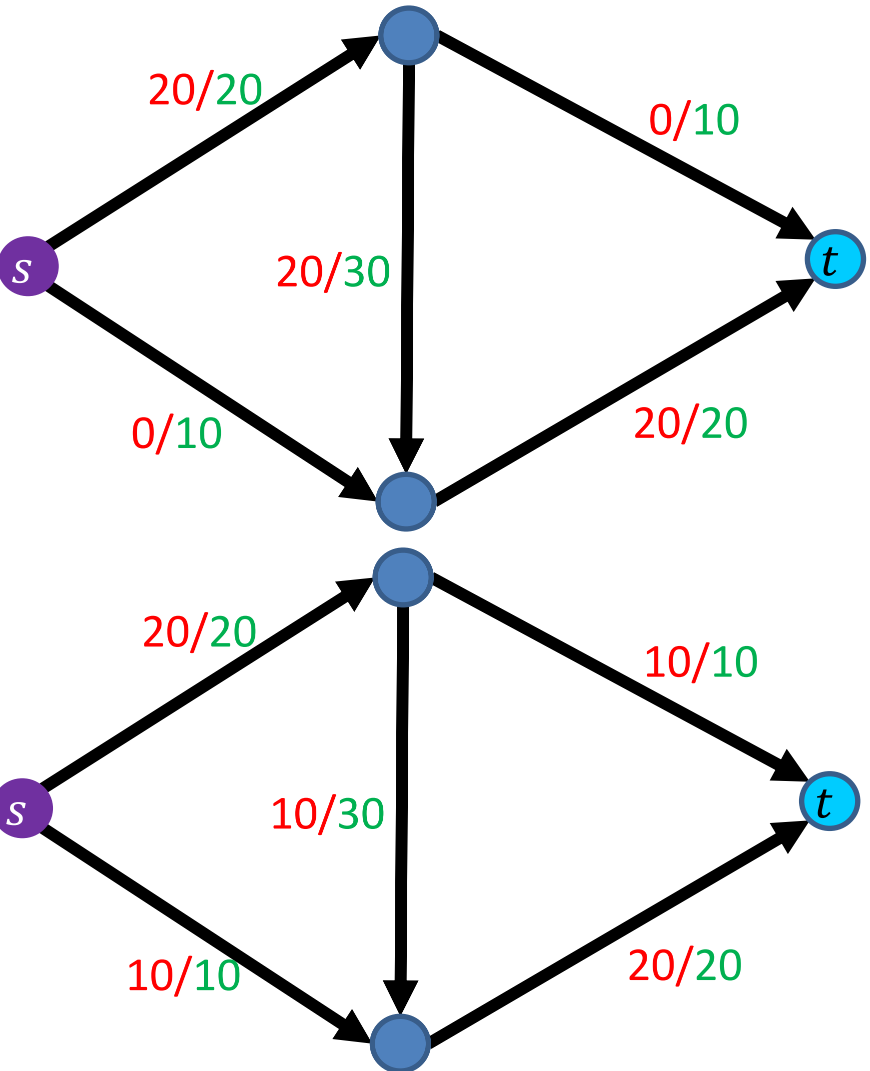


Residual Graph G_f

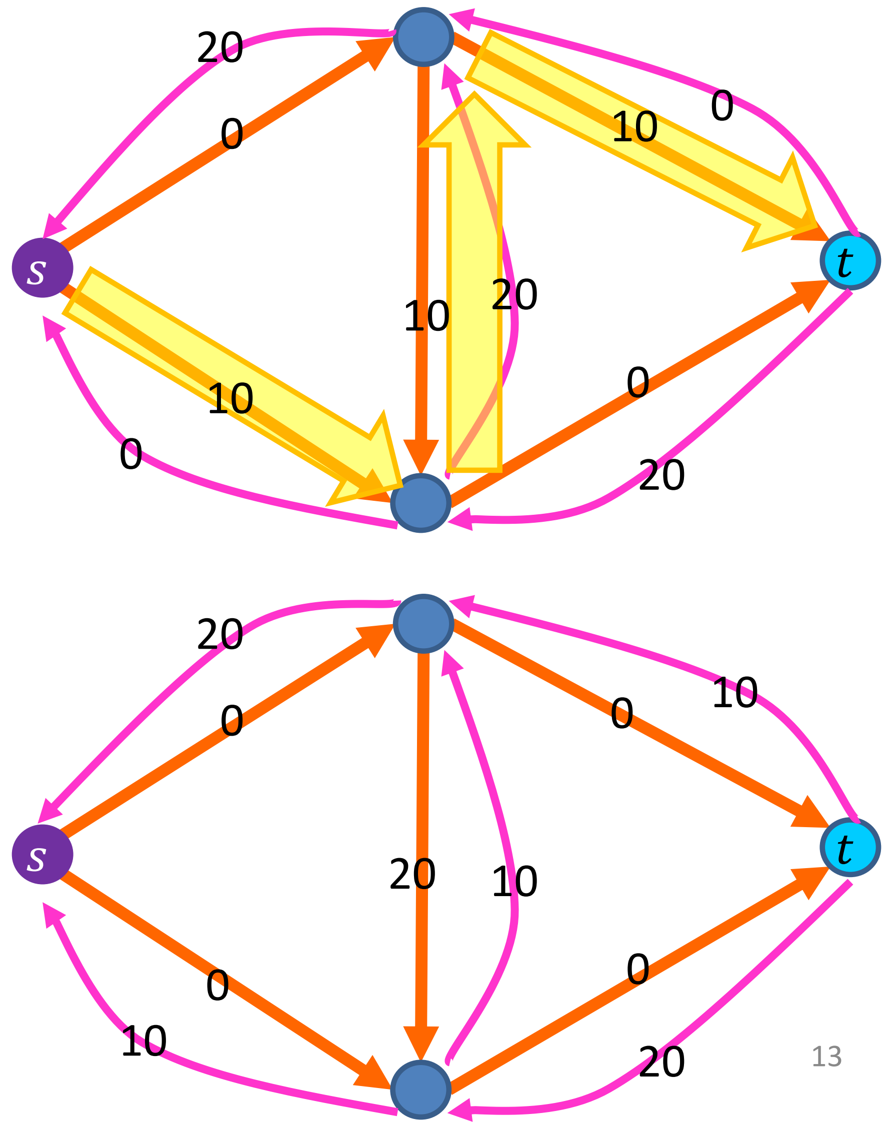


Residual Graphs Example

Flow Graph



Residual Graph



Ford-Fulkerson

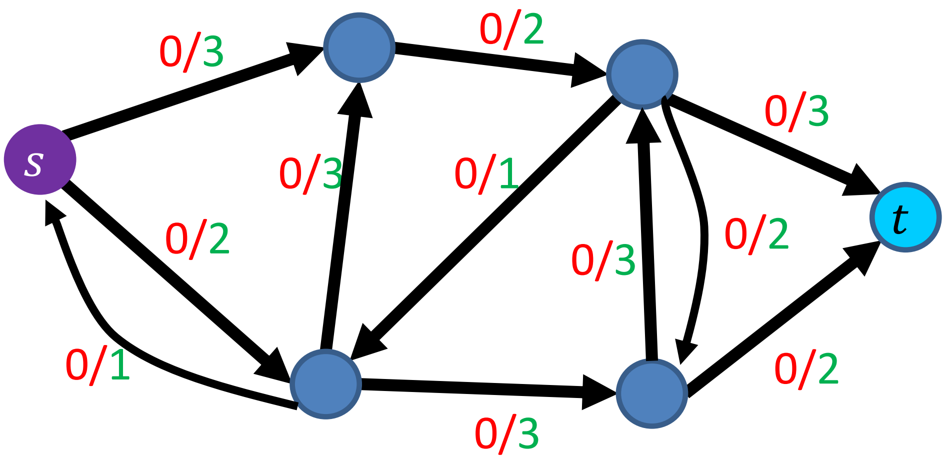
- Augmenting Path: a path of positive-weight edges from s to t in the residual graph

- Algorithm: Repeatedly add the flow of any augmenting path

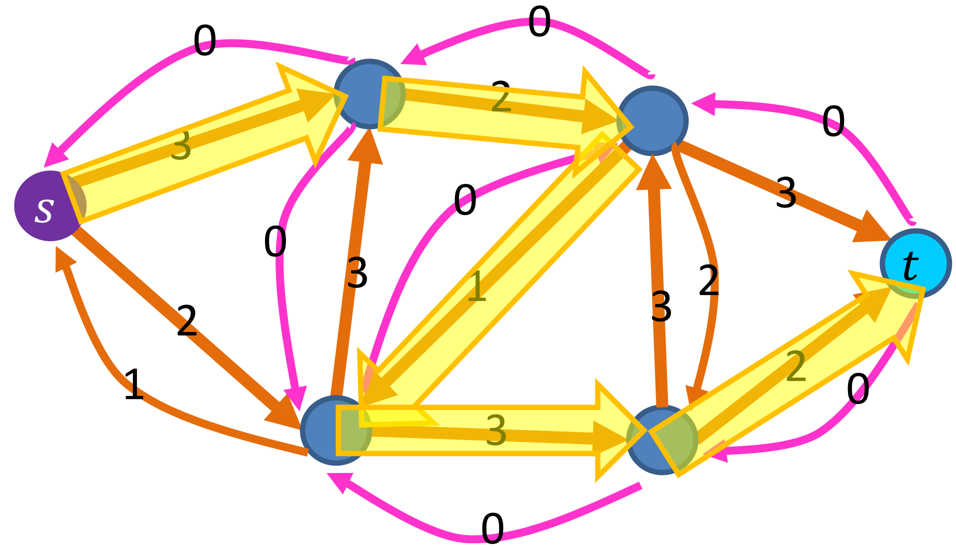
$\forall(u,v) \in E$ Initialize $f(u,v) = 0$
While there is an augmenting path p in G_f
  let $f = \min_{u,v \in p} c_f(u,v)$
  add f to the flow of each edge in p

Ford Fulkerson: example

Flow Graph G



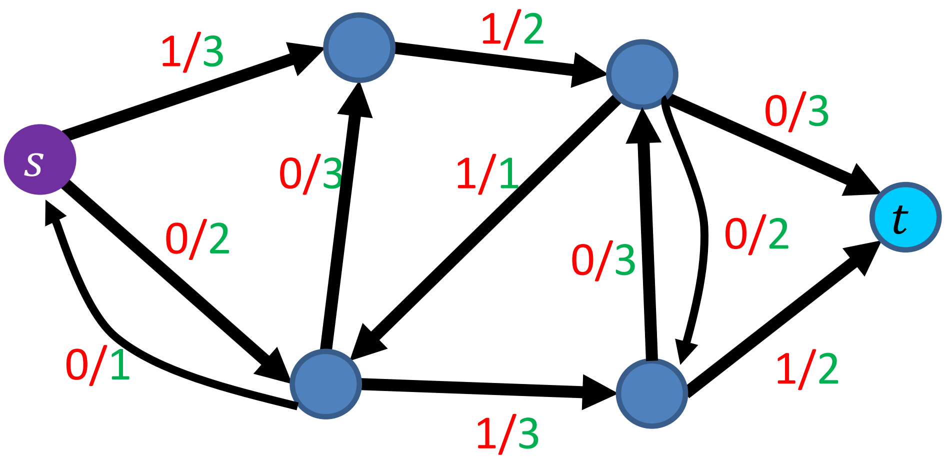
Residual Graph G_f



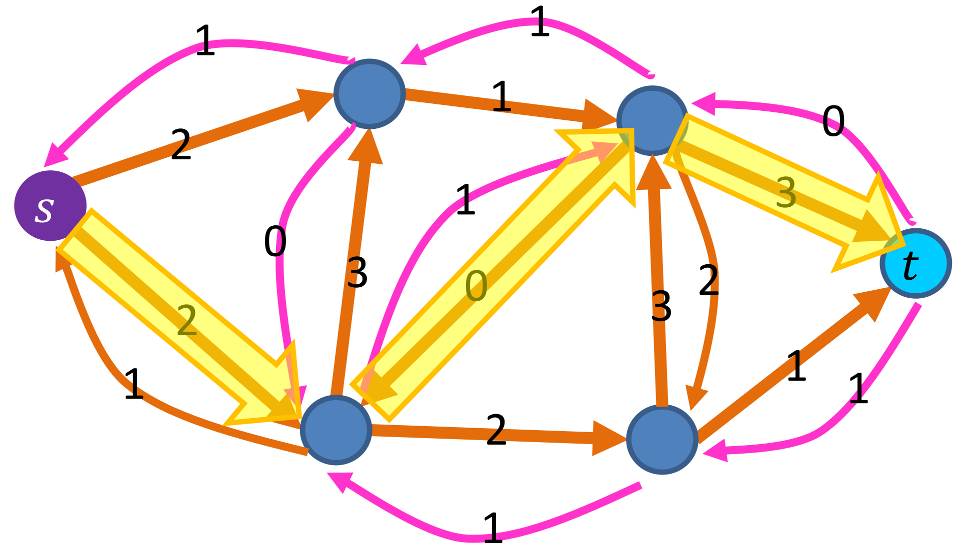
Add flow of 1 to this path

Ford Fulkerson: example

Flow Graph G



Residual Graph G_f



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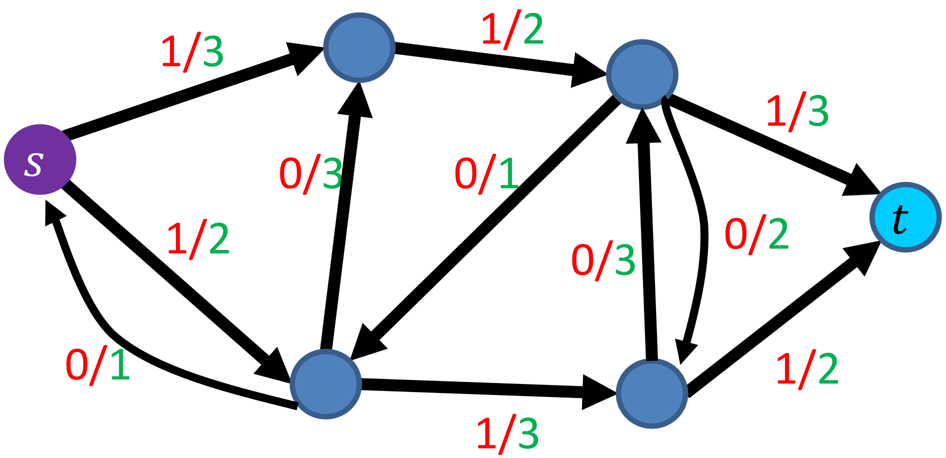
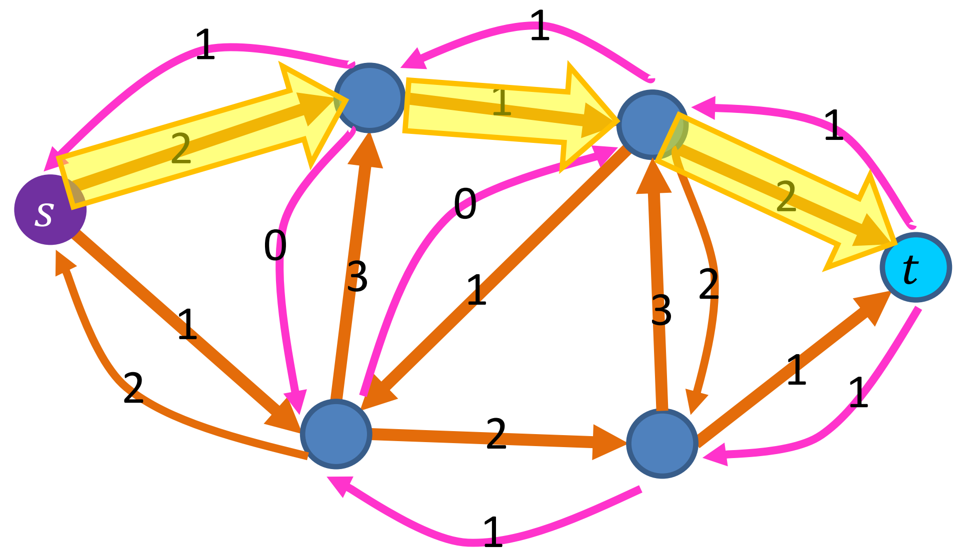
Ford Fulkerson: example

Flow Graph G



Residual Graph G_f



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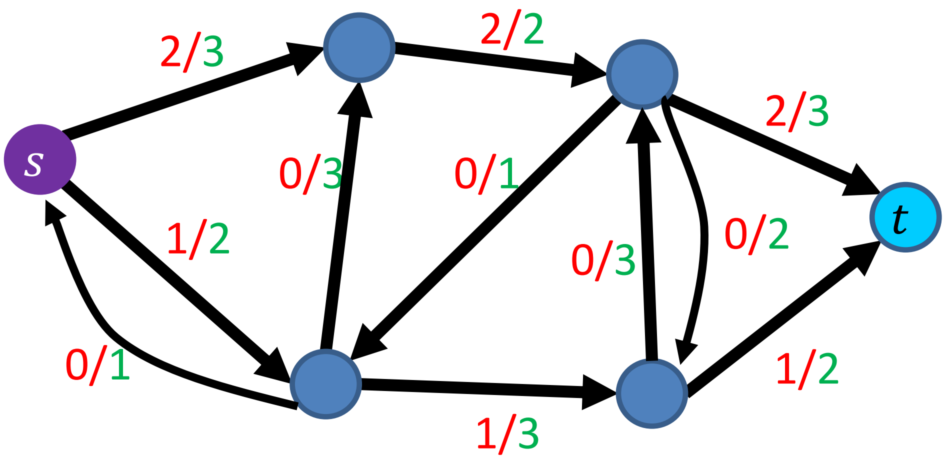
Ford Fulkerson: example

Flow Graph G



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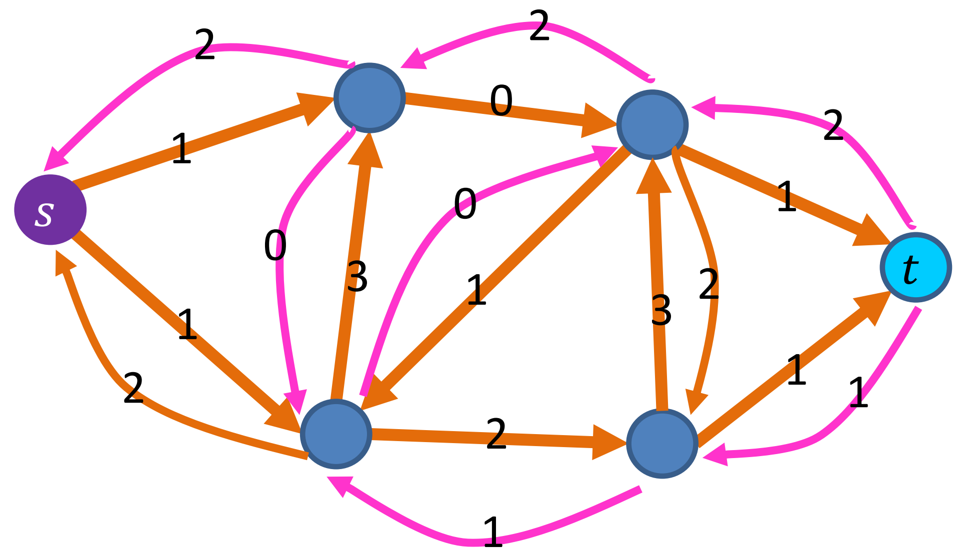

Ford-Fulkerson: Run Time

- Augmenting Path: a path of positive-weight edges from s to t in the residual graph
- Algorithm: Repeatedly add the flow of any augmenting path

$\forall(u,v) \in E$ Initialize $f(u,v) = 0$
While there is an augmenting path p in G_f
    let $f = \min_{u,v \in p} c_f(u,v)$
    add f to the flow of each edge in p

Time to find an augmenting path: BFS: $\Theta(V+E)$

$\Theta(E \cdot |f|)$

Number of iterations of While loop: $|f|$

Why might we loop $|f|$ times?

$\forall (u,v) \in E$ Initialize $f(u,v) = 0$

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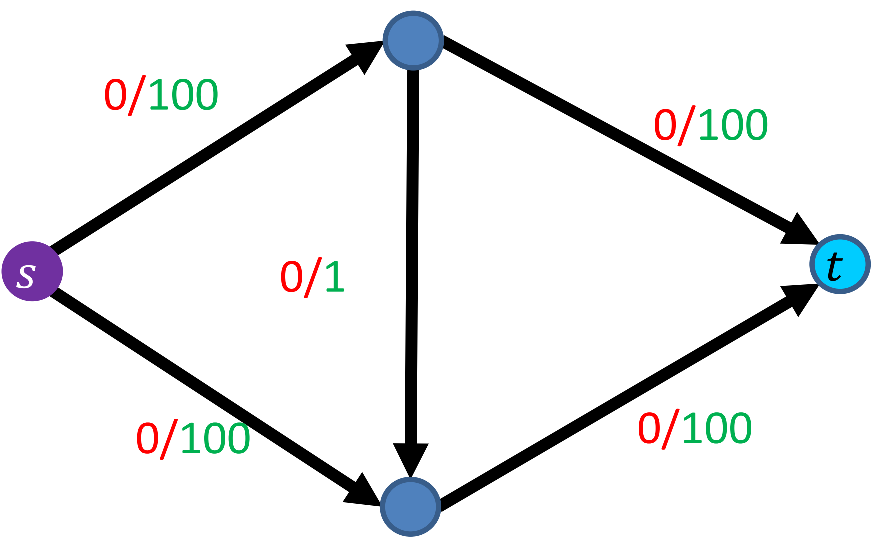

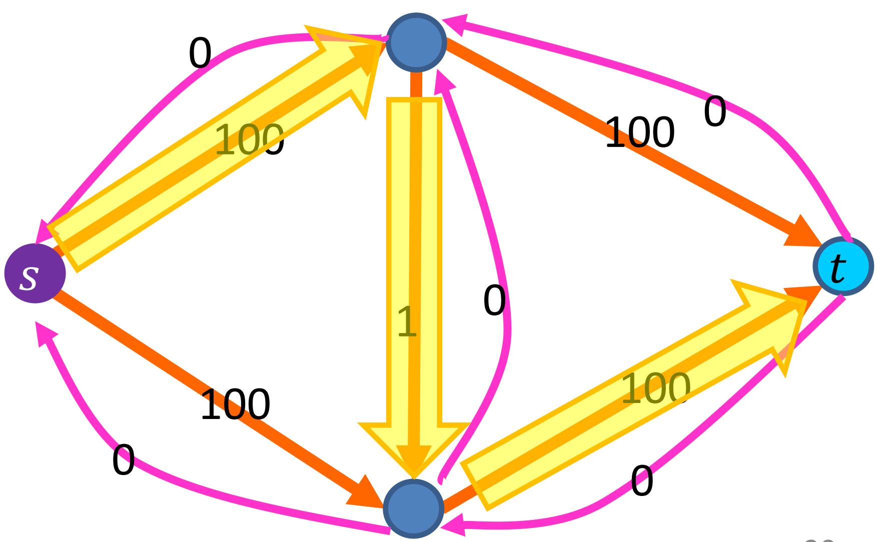

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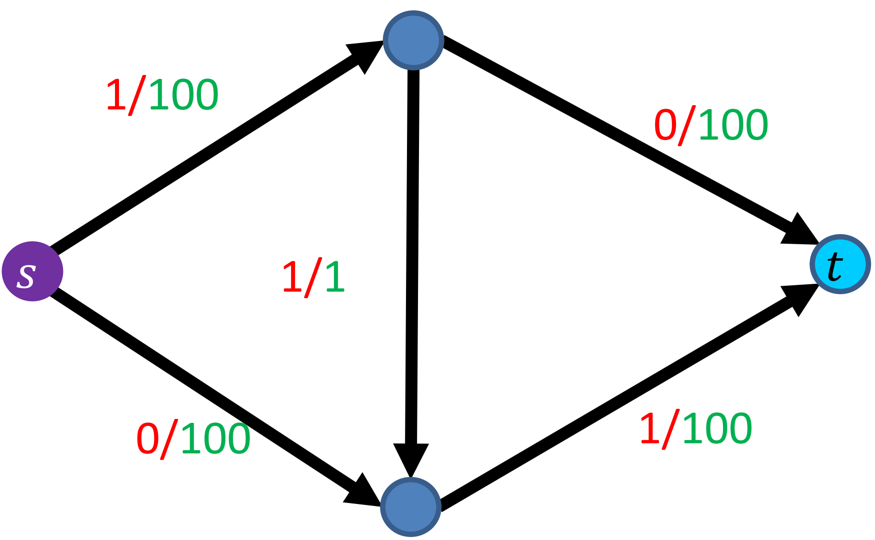

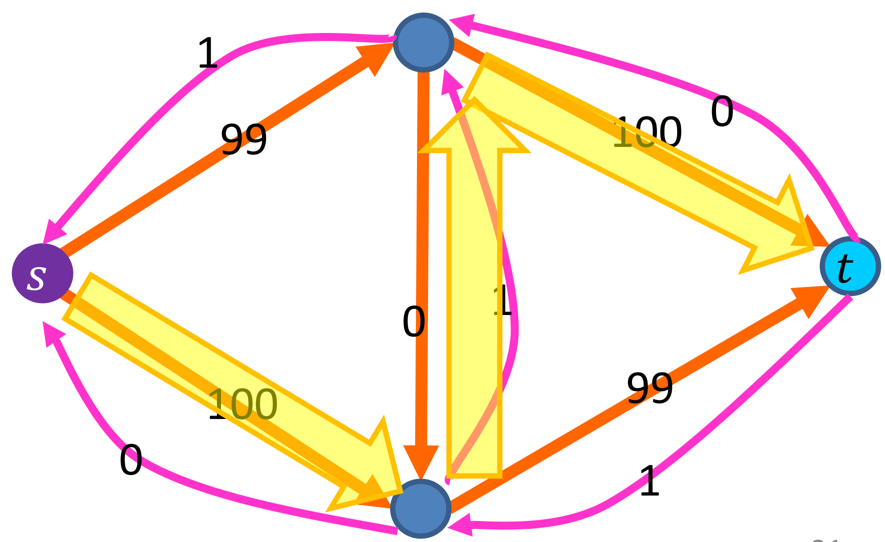

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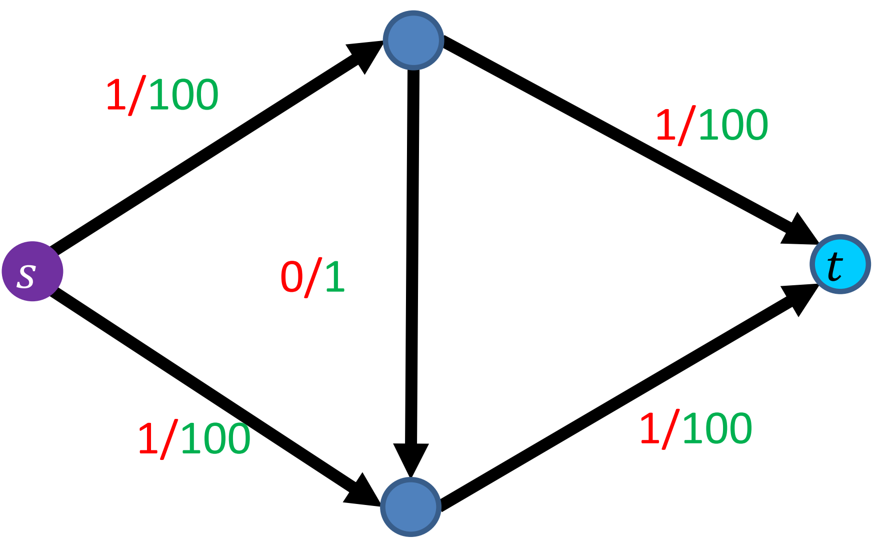

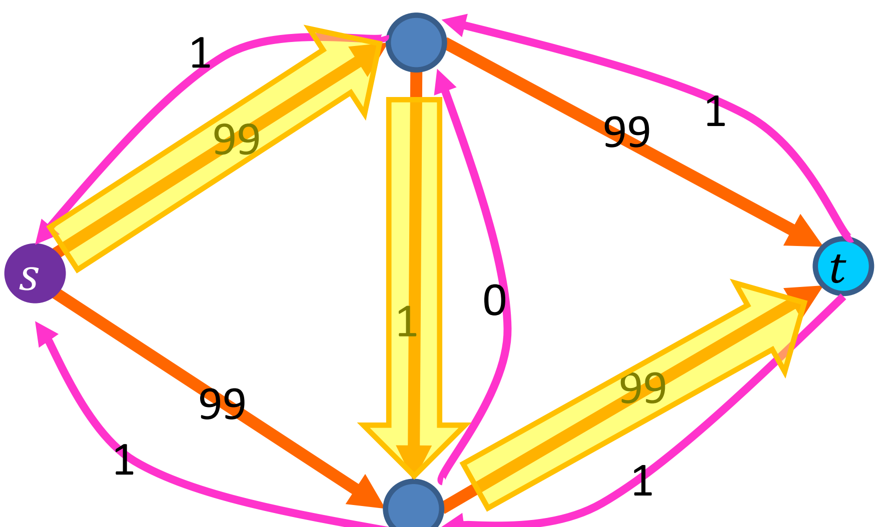

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$\forall (u,v) \in E$ Initialize $f(u,v) = 0$

While there is an augmenting path p in G_f

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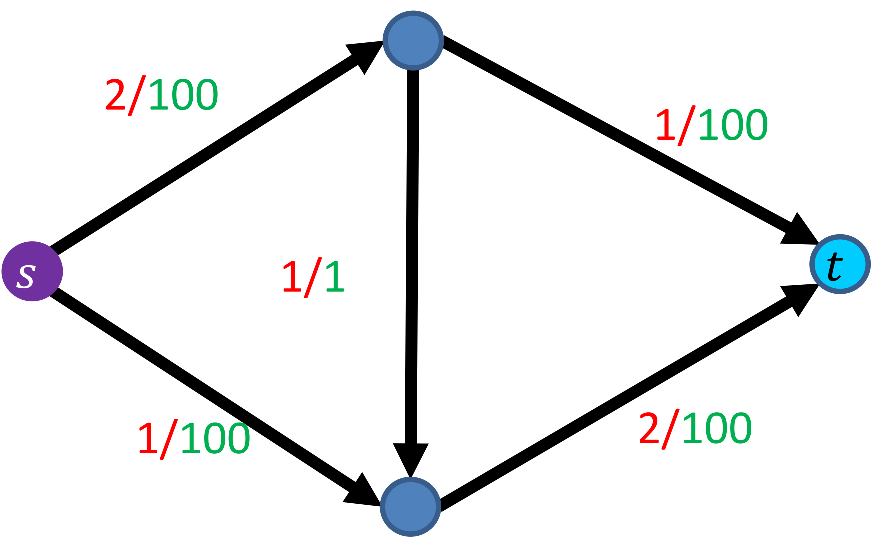
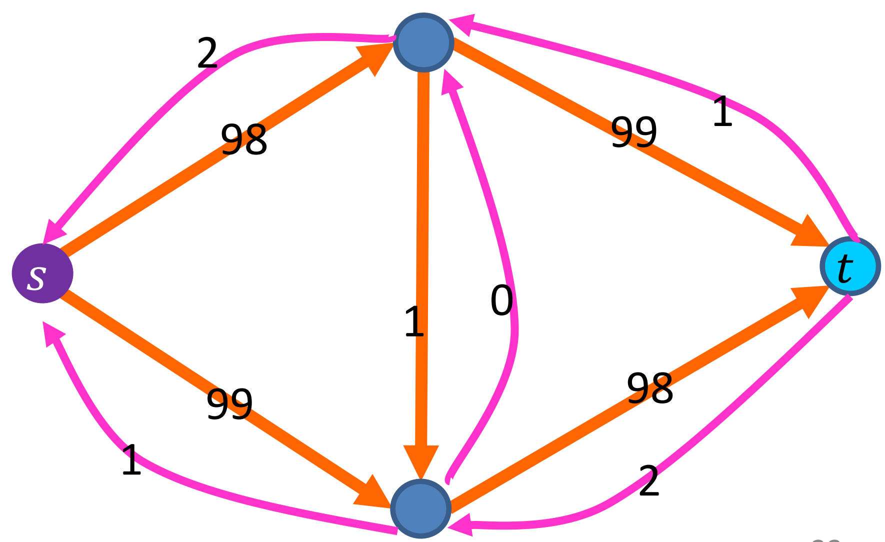
  add f to the flow of each edge in p

Each time we increase flow by 1

Loop runs 100 times





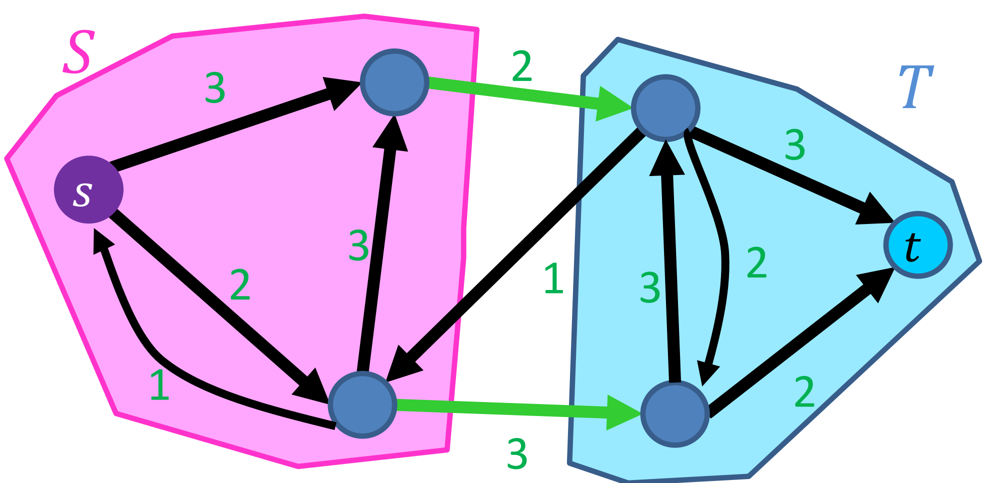
Can We Avoid this?

- Edmunds-Karp Algorithm
- $\Theta(\min(E|f|, VE^2))$
- Choose augmenting path with fewest edges

$\forall(u,v) \in E$ Initialize $f(u,v) = 0$
While there is an augmenting path in G_f
    let p be the shortest augmenting path
    let $f = \min_{u,v \in p} c_f(u,v)$
    add f to the flow of each edge in p

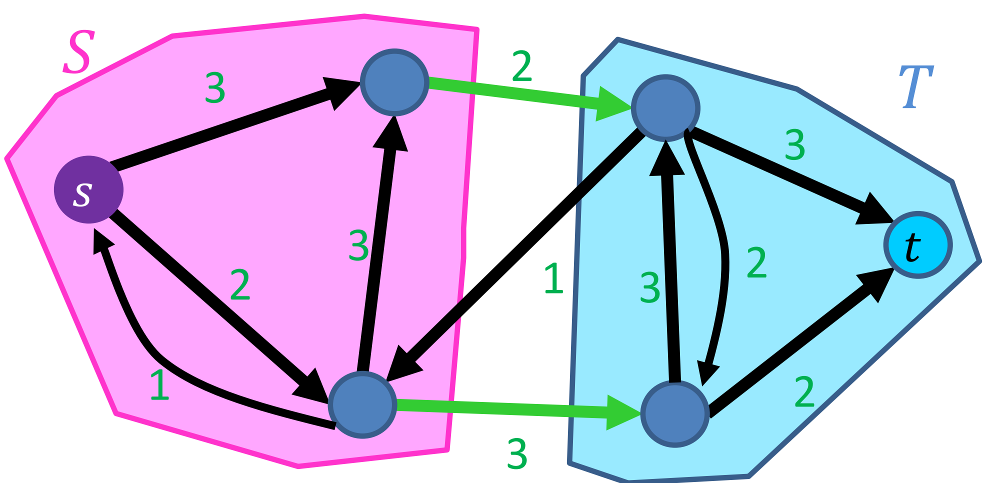
Showing Correctness of Ford-Fulkerson

- Consider cuts which separate s and t
  - Let $s \in S$, $t \in T$, s.t. $V = S \cup T$
- Cost of cut $(S, T) = ||S, T||$
  - Sum **capacities** of edges which go from S to T
  - This example: 5



Maxflow≤MinCut

- Max flow upper bounded by any cut separating s and t
- Why? "Conservation of flow"
  - All flow exiting s must eventually get to t
  - To get from s to t, all "tanks" must cross the cut
- Conclusion: If we find the minimum-cost cut, we've found the maximum flow
  - $\max_f |f| \leq \min_{S,T} ||S,T||$

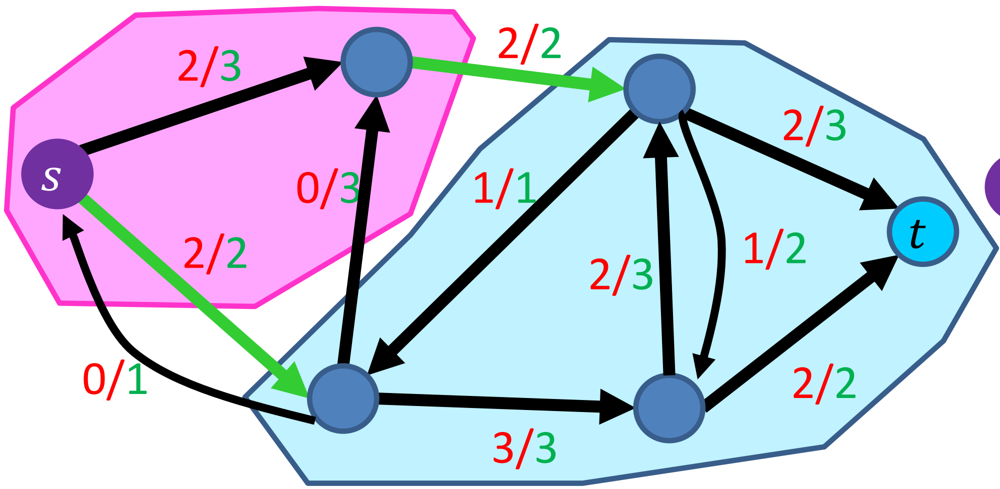


Maxflow/Mincut Theorem

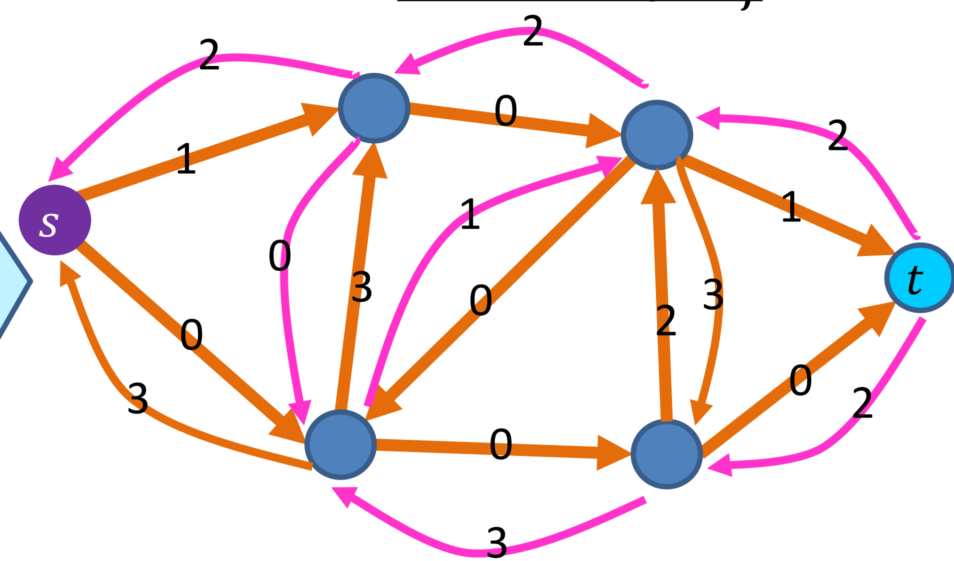
- To show Ford-Fulkerson is correct:
  - Show that when there are no more augmenting paths, there is a cut with cost equal to the flow
- Conclusion: the maximum flow through a network matches the minimum-cost cut
  - $\max_{f} |f| = \min_{S,T} ||S,T||$
- Duality
  - When we've maximized max flow, we've minimized min cut (and vice-versa), so we can check when we've found one by finding the other

Example: Maxflow/Mincut

Flow Graph G



Residual Graph G_f



$$|f| = 4$$

$$||S, T|| = 4$$

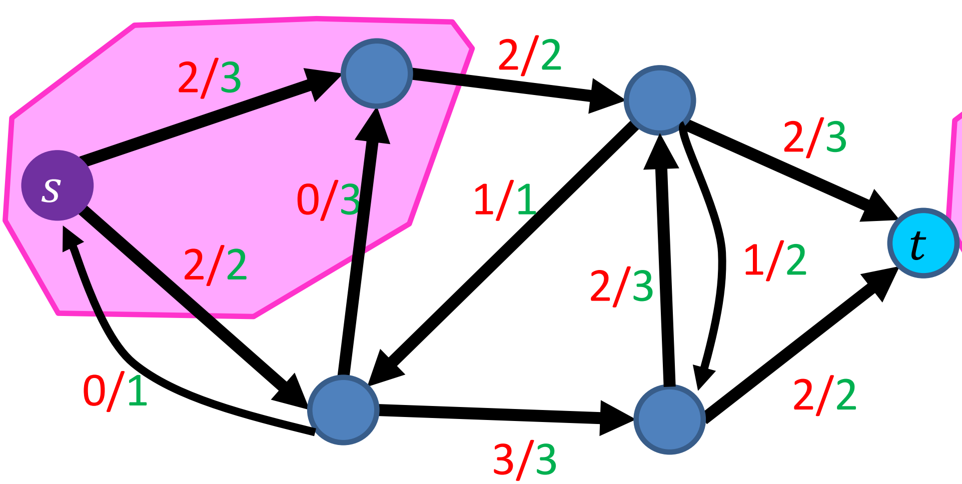
No Augmenting Paths

Idea: When there are no more augmenting paths, there exists a cut in the graph with cost matching the flow

Proof: Maxflow/Mincut Theorem

- If $|f|$ is a max flow, then G_f has no augmenting path
  - Otherwise, use that augmenting path to “push” more flow
- Define S = nodes reachable from source node s by positive-weight edges in the residual graph
  - $T = V - S$
  - S separates s , t (otherwise there’s an augmenting path)

Flow Graph G



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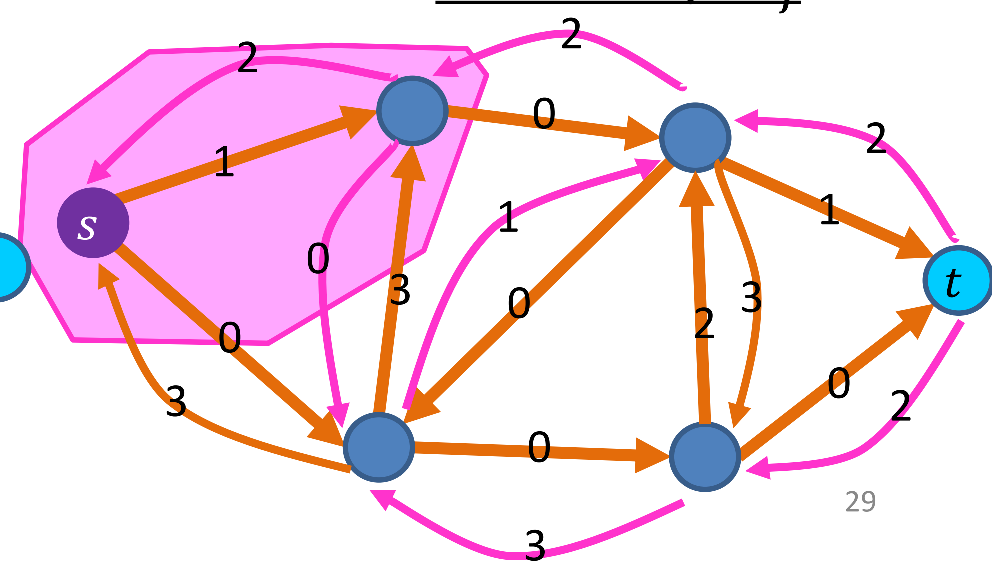
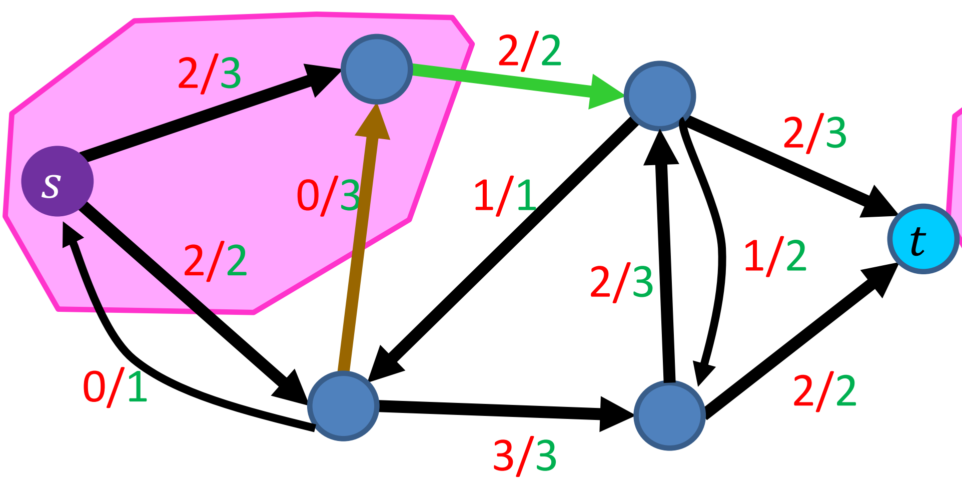


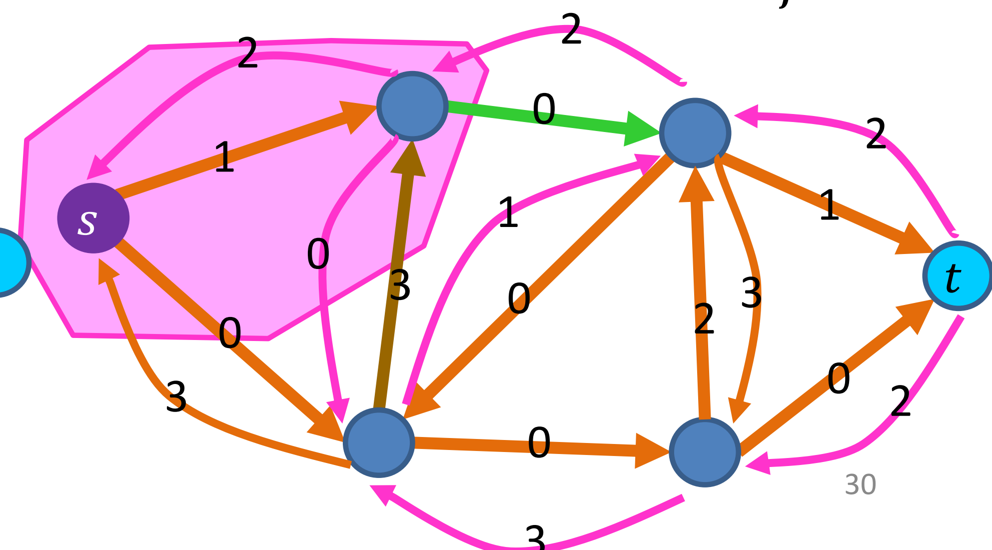
Proof: Maxflow/Mincut Theorem

- To show: $||S, T|| = |f|$
  - Weight of the cut matches the flow across the cut
- Consider edge (u, v) with $u \in S$, $v \in T$
  - $f(u, v) = c(u, v)$, because otherwise $w(u, v) > 0$ in G_f, which would mean $v \in S$
- Consider edge (y, x) with $y \in T$, $x \in S$
  - $f(y, x) = 0$, because otherwise the back edge $w(y, x) > 0$ in G_f, which would mean $x \in S$

Flow Graph G



Residual Graph G_f



Proof Summary

1. The flow $|f|$ of G is upper-bounded by the sum of capacities of edges crossing any cut separating source s and sink t

2. When Ford-Fulkerson Terminates, there are no more augmenting paths in G_f

3. When there are no more augmenting paths in G_f then we can define a cut $S =$ nodes reachable from source node s by positive-weight edges in the residual graph

4. The sum of edge capacities crossing this cut must match the flow of the graph

5. Therefore this flow is maximal

Other Maxflow algorithms

- **Ford-Fulkerson**
  - $\boldsymbol{\Theta(E|f|)}$
- **Edmonds-Karp**
  - $\boldsymbol{\Theta(E^2V)}$
- Push-Relable (Tarjan)
  - $\Theta(EV^2)$
- Faster Push-Relable (also Tarjan)
  - $\Theta(V^3)$