Homework 1.3

10 points. End term: 4-th week (ocober 22 - october 27, 2020)

1. For one of the following LP problems: solve the problem using the simplex method, then solve it again using the geometric approach and outline the progress of the simplex algorithm.

$$\textbf{(a)}\left\{\begin{array}{lrl} \min & z = 3x_1 + 9x_2 & \\ \text{s. t.} & -5x_1 + 2x_2 & \leqslant 30 \\ & -3x_1 + x_2 & \leqslant 12 \\ & x_1, x_2 \geqslant 0 & \end{array}\right. \qquad \textbf{(b)}\left\{\begin{array}{lrl} \min & z = -2x_1 - x_2 & \\ \text{s. t.} & 2x_1 + x_2 & \leqslant 4 \\ & 2x_1 + 3x_2 & \leqslant 3 \\ & 4x_1 + x_2 & \leqslant 5 \\ & x_1 + 5x_2 & \leqslant 1 \\ & x_1, x_2 \geqslant 0 & \end{array}\right.$$

2. The following tableau corresponds to an iteration of the simplex method:

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	0	-2	1	e	0	2	f
	1	g	0	-2	0	1	1
	0	0	0	h	1	4	3
z	0	a	0	b	c	3	d

Categorize the variables as basic and nonbasic and provide the current values of all the variables. Find conditions on the parameters $a, b, \dots, h$ so that the following statements are true.

(i) The current basis is optimal.

(ii) The current basis is the unique optimal basis.

(iii) The current basis is optimal but alternative optimal bases exists.

(iv) The problem is unbounded.

(v) The current solution will improve if x_4 is increased. When x_4 is entered into the basis, the change in the objective is zero.

3. Devise an implemetation (C, C++, C#, Java) for the simplex algorithm using the following pseudocode (the simplex tableau has $(m+1) \times (n+1)$ dimensions):

// if $t_{m+1,k} \geqslant 0$, $\forall k$, the simplex table is <u>optimal</u>: the current basic feasible solution is optimal.

while $(\exists\, t_{m+1,j} < 0,\ j \leqslant n)$ **{**

  let $1 \leqslant l \leqslant n$ **such that** $t_{m+1,l} < 0$**;**

  // l is the pivot column, i.e., on column l will be the <u>entering basic variable</u>

  if $(t_{h,l} \leqslant 0,\ \forall\, 1 \leqslant h \leqslant m)$

  // the <u>problem</u> has no optimum solution (although it has feasible solutions), i.e. the objective functions is <u>unbounded</u>

    return;

  let $1 \leqslant k \leqslant m$ **such that** $\dfrac{t_{k,n+1}}{t_{k,l}} = \min\left\{\dfrac{t_{h,n+1}}{t_{h,l}} \ :\ t_{h,l} > 0\right\}$

  // k is the pivot row, on line k will be the <u>leaving basic variable</u>

  for $i = \overline{1, m+1}$**,** $i \neq k$

    for $j = \overline{1, n+1}$**,** $j \neq l$

      $t_{i,j} \leftarrow \dfrac{t_{i,j}t_{k,l} - t_{i,l}t_{k,j}}{t_{k,l}}$**;** *// <u>pivoting rule</u>*

for $i = \overline{1, m+1}$**,** $i \neq k$
  $t_{i,l} \leftarrow 0$**;**
for $j = \overline{1, n+1}$**,** $j \neq l$
  $t_{k,j} \leftarrow \dfrac{t_{k,j}}{t_{k,l}}$**;**
$t_{k,l} \leftarrow 1$**;**
}

Formally the **Bland rule** (for cycle avoiding) is

- among all candidates for entering column, choose the one with the smallest index, i.e. replace

  let $1 \leqslant l \leqslant n$ **such that** $t_{m+1,l} < 0$**;**

  with

  let $1 \leqslant l \leqslant n$ **such that** $l = \min\{j \,:\, t_{m+1,j} < 0\}$**;**

- among all rows for which the minimum ratio $\min\left\{\dfrac{t_{h,n+1}}{t_{h,l}} \,:\, t_{h,l} > 0\right\}$ is the same, choose the row corresponding to the basic variable with the smallest index.

4. Run the above implemented algorithm on the following problems and verify your results using **Gurobi Solver**:

$$\textbf{(a)} \begin{cases} \min & z = -x_1 - x_2 \\ \text{s. t.} & -2x_1 + x_2 \leqslant 2 \\ & -x_1 + 2x_2 \leqslant 7 \\ & x_1 \leqslant 3 \\ & x_1, x_2 \geqslant 0 \end{cases} \qquad \textbf{(b)} \begin{cases} \min & z = -x_1 - 2x_2 \\ \text{s. t.} & -x_1 + x_2 \leqslant 2 \\ & -2x_1 + x_2 \leqslant 1 \\ & x_1, x_2 \geqslant 0 \end{cases}$$

$$\textbf{(c)} \begin{cases} \min & z = 3x_1 - 2x_2 - 4x_3 \\ \text{s. t.} & 4x_1 + 5x_2 - 2x_3 \leqslant 22 \\ & x_1 - 2x_2 + x_3 \leqslant 30 \\ & x_1, x_2, x_3 \geqslant 0 \end{cases}$$