Homework 1.2

10 points. End term: 4-th week (october 22 - october 28, 2020)

Solve three of the following problems.

1. Consider the LP problem $\max\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} \leqslant \mathbf{b}, \mathbf{x} \geqslant \mathbf{0}\}$. If $\mathbf{x}^1$ and $\mathbf{x}^2$ are feasible solution, then $\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$ is also a feasible solution. (The feasible region is a convex set.)

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let c be a real constant. Prove that the set $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leqslant c\}$ is convex. *Hint: Recall the definitions of convex set and convex function.*

3. Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be two feasible solutions to the problem $\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0}\}$, such that $\mathbf{c}^T\mathbf{x}^1 = \mathbf{c}^T\mathbf{x}^2 = \alpha$. Prove that any convex combination, $\mathbf{x}$, of $\mathbf{x}^1$ and $\mathbf{x}^2$ has the same objective function value: $\mathbf{c}^T\mathbf{x} = \alpha$.

4. Consider the set $\{\mathbf{x} \in \mathbb{R}^n : x_1 = \ldots = x_{n-1} = 0, 0 \leqslant x_n \leqslant 1\}$. Could this be the feasible region of a problem in standard form?

5*. (Carathéodory) Let $\mathcal{M} = \{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\}$ be a finite family o vectors from $\mathbb{R}^n$, and $\mathcal{S} = conv(\mathcal{M})$ its convex hull. Prove that any element of $\mathcal{S}$ can be expressed in the form $\sum_{i=1}^{k} \lambda_i \mathbf{x}^i$, where $\sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geqslant 0$, for all i, with at most $(n+1)$ of the coefficients λ_i being nonzero.

Solve one of the following problems.

6. Consider a linear program with a single constraint

$$\begin{array}{lrl} \min & z = c_1x_1 + c_2x_2 + \ldots c_nx_n & \\ \text{s. t.} & a_1x_1 + a_2x_2 + \ldots a_nx_n & \leqslant b \\ & x_1, x_2, \ldots, x_n \geqslant 0 & \end{array}$$

Under what conditions is the problem feasible? Develop a simple rule to determine an optimal solution, if one exists. *Hint: The case $n = 2$ should be suggestive enough.*

7. Let $\mathbf{x}$ be a feasible point for the constraints $\{\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0}\}$ that is not an extreme point. Prove that there exists a vector $\mathbf{y} \neq 0$, with the following properties: $\mathbf{A}\mathbf{y} = \mathbf{0}$, and $x_i = 0 \Rightarrow y_i = 0$. *Hint: Recall the definition of the basic feasible solution.*

For three of the following LP problems, write the systems in standard form, determine all the basic solutions (feasible and infeasible), and find optimal basic solutions (if any).

8. $$\left\{\begin{array}{lrcr} \max & z = x_1 + 4x_2 & & \\ \text{s. t.} & x_1 + 2x_2 & \leqslant & 12 \\ & 3x_1 + 4x_2 & \leqslant & 21 \\ & x_1, x_2 & \geqslant & 0 \end{array}\right.$$

9. $$\left\{\begin{array}{lrcr} \max & z = 2x_1 + 5x_2 & & \\ \text{s. t.} & 2x_1 + 3x_2 & \leqslant & 10 \\ & 5x_1 + x_2 & \leqslant & 12 \\ & x_1 + 5x_2 & \leqslant & 15 \\ & x_1, x_2 & \geqslant & 0 \end{array}\right.$$

10. $$\left\{\begin{array}{lrcr} \max & z = 2x_1 + 4x_2 & & \\ \text{s. t.} & -3x_1 + 2x_2 & \leqslant & 6 \\ & x_1 + 2x_2 & \geqslant & 2 \\ & x_1, x_2 & \geqslant & 0 \end{array}\right.$$

11. $$\left\{\begin{array}{lrcr} \max & z = 4x_1 + 3x_2 + x_3 & & \\ \text{s. t.} & x_1 + x_2 + 2x_3 & \leqslant & 4 \\ & 5x_1 + 3x_2 & \leqslant & 15 \\ & x_1, x_2, x_3 & \geqslant & 0 \end{array}\right.$$