Homework 2.2

8 points. End term: 8-th week (19-25 november 2020)

1. Find the dual problems for two the following LP problems and verify that the dual of the dual is the primal.

$$
(\mathbf{a})\left\{\begin{array}{lrl}
\min & z = 6x_1 + x_2 + x_3 & \\
\text{s. t.} & 2x_1 + 3x_2 - x_3 + x_4 & = 1 \\
 & 4x_1 + 3x_2 - 2x_3 & \geqslant 9 \\
 & 2x_1 + 3x_2 + 8x_3 & \leqslant 5 \\
 & x_1 \geqslant 0, x_2 \leqslant 0. &
\end{array}\right.
\qquad
(\mathbf{b})\left\{\begin{array}{lrl}
\min & z = 2x_1 - 9x_2 + 5x_3 - 6x_4 & \\
\text{s. t.} & 4x_1 + 3x_2 + 5x_3 + 8x_4 & \geqslant 24 \\
 & 2x_1 - 7x_2 - 4x_3 - 6x_4 & \geqslant 17 \\
 & x_1, x_2, x_3 \geqslant 0. &
\end{array}\right.
$$

$$
(\mathbf{c})\left\{\begin{array}{lrl}
\min & -x_1 + 2x_2 + x_3 & \\
\text{s. t.} & 2x_1 - x_2 + x_3 & \leqslant 6 \\
 & x_1 + x_2 - x_3 & \geqslant 3 \\
 & x_1 - 2x_2 + 3x_3 & = 5 \\
 & x_1, x_3 \geqslant 0. &
\end{array}\right.
\qquad
(\mathbf{d})\left\{\begin{array}{lrl}
\min & z = 6x_1 - 3x_2 - 2x_3 + 5x_4 & \\
\text{s. t.} & 4x_1 + 3x_2 - 8x_3 + 7x_4 & = 11 \\
 & 3x_1 + 2x_2 + 7x_3 + 6x_4 & \geqslant 23 \\
 & 7x_1 + 4x_2 + 3x_3 + 2x_4 & \leqslant 12 \\
 & x_1, x_2 \geqslant 0, x_3 \leqslant 0. &
\end{array}\right.
$$

2. Consider the following LP problem

$$
\left\{\begin{array}{lrl}
\max & z = 2x_1 + 4x_2 + 3x_3 + x_4 & \\
\text{s. t.} & 3x_1 + x_2 + x_3 + 4x_4 & \leqslant 12 \\
 & x_1 - 3x_2 + 2x_3 + 3x_4 & \leqslant 7 \\
 & 2x_1 + x_2 + 3x_3 - x_4 & \leqslant 10 \\
 & x_1, x_2, x_3, x_4 \geqslant 0 &
\end{array}\right.
$$

The solution $x_1 = 0, x_2 = 10.4, x_3 = 0, x_4 = 0.4$ has been proposed as an optimal solution of the LP. Build its dual problem and use complementary slackness to verify the optimality of the above solution.

Solve three of the following problems.

3. Can you find a linear program which is its own dual? (We will say that the two problems are the same if one can be obtained from the other merely by multiplying the objective, any of the constraints, or any of the variables by -1.)

4. Let $\mathbf{A}$ be a symmetric $n \times n$ matrix and $\mathbf{c} \in \mathbb{R}^n$. Consider the following LP problem

$$
\begin{array}{rl}
\text{minimize} & z = \mathbf{c}^T\mathbf{x} \\
\text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{c} \\
 & \mathbf{x} \geqslant \mathbf{0}.
\end{array}
$$

Prove that if $\mathbf{x}_*$ satisfies $\mathbf{A}\mathbf{x}_* = \mathbf{c}$ and $\mathbf{x}_* \geqslant \mathbf{0}$, then $\mathbf{x}_*$ is an optimal solution.

5. Give an example of a pair (primal and dual) of linear programming problems, both of which have multiple optimal solutions.

6. Consider the following two LP problems

$$
(P_1)\left\{\begin{array}{rl}
\text{minimize} & z = \mathbf{c}^T\mathbf{x} \\
\text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\
 & \mathbf{x} \geqslant \mathbf{0}.
\end{array}\right.
\qquad
(P_2)\left\{\begin{array}{rl}
\text{minimize} & z = \mathbf{c}^T\mathbf{x} \\
\text{subject to} & \mathbf{A}\mathbf{x} = \overline{\mathbf{b}} \\
 & \mathbf{x} \geqslant \mathbf{0}.
\end{array}\right.
$$

Let $\mathbf{x}_*$ be an optimal solution to (P_1), $\mathbf{y}_*$ an optimal solution to the dual of (P_1), and $\overline{\mathbf{x}}$ an optimal solution to (P_2), assumed to exist. Prove that $\mathbf{y}_*^T(\overline{\mathbf{b}} - \mathbf{b}) \leqslant \mathbf{c}^T(\overline{\mathbf{x}} - \mathbf{x}_*)$. *Hint: use the duality and a similar exercise from seminar 5.*

7. Consider an LP problem with a single constraint

$$\begin{array}{ll} \min & z = c_1x_1 + c_2x_2 + \dots c_nx_n \\ \text{s. t.} & \quad a_1x_1 + a_2x_2 + \dots a_nx_n \leqslant b \\ & \quad x_1, x_2, \dots, x_n \geqslant 0 \end{array}$$

Using duality develop a simple rule to determine an optimal solution, if the latter exists.