TOPOLOGICAL DATA ANALYSIS: THE GEOGRAPHY OF COUNTRIES

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Abstract. In this paper I analyze the question of whether or not the countries of the world are evenly divided into first and third world categories using techniques of persistent homology and standard clustering algorithms. Data on health and economics is drawn from the GapMinder project and analyzed alongside geographic data of countries. Preliminary evidence is found in support of the existence of some division between countries of the world.

1. Introduction

Counties are often dived into two categories: a *1st world*, comprised of developed western-style countries, and a *3rd world* comprised of less-developed countries. The Gapminder project attempts to challenge this distinction by showing the lack of correlation between such indicators as life expectancy and income per person.[1] The topology of this data can also provide insight into this question, particularly by incorporating the geography of countries alongside social indicators. Persistent homology reveals that there exist at least two connected components in the data, representing distinct groups of countries. These connected components are illuminated by k-means clustering algorithm, which provides an interesting comparison with zero-order homology as a clustering algorithm in its own right. This paper begins with a brief overview of homology and persistent homology before applying these techniques to data drawn from the GapMinder project.

2. Simplicial Homology

Generally speaking, homology is concerned with counting the holes of varying dimensions in topological spaces. Homology groups provide a rough approximation to the homotopy groups that is more intuitive and, as it turns out, easily computed via linear algebra.

Begin with a simplicial complex K, a space with a triangulation, and define the set of n-dimensional cycles Z_n in the triangulation. Now, let B_n be the set of all cycles that also represent the boundary of some $(n+1)$-simplex in K. Then the homology group is formed by modding the cycles out by the boundaries, leaving just those cycles that are holes.

Definition 2.1. [2] *The nth* homology group *of a simplicial complex K is the quotient space*

$$H_n(K) = \frac{Z_n}{B_n}$$

It is imporant to note that there is a rich underlying structure of linear algebra in these definitions. Cycles and boundaries may be thought of as the kernals and images of (linear) boundary maps that connect the spaces of subsets of n-simplicies in K. Then the problem of finding the the order of each homology group, which is the real quantity of interest, is simple linear algebra. These orders are known as the Betti numbers.

Definition 2.2. *The ith* Betti Number *of a simplicial complex K is the order of the ith homology group of K, e.g. $b_i(K) = ||H_i(K)||$.*

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For example, b_0 counts the number of 0-holes, which can be thought of as the number of connected components. Likewise b_1 counts the number of 1-holes, or topological circles, and b_2 counts the number of 2-holes, or trapped volumes, in the space.

3. Persistent Homology

The definitions above require one to know the topological space before calculating homology groups, but what if instead there is only data that may or may not represent a particular topological space? Given a set of data representing points in Euclidean space, a "point cloud", there is a natural simplicial complex construction known as the Czech Complex. It turns out that this construction quickly grows to be too large to be convenient in calculations, but there is a suitable approximation known as the Vietoris-Rips Complex that is less computationally expensive. I refer to (Ghrist) for a technical explanation of why this is the case.

Definition 3.1. [4] *Given a set of points in Euclidean space, the* Vietoris-Rips Complex R_ϵ *is the abstract simplicial complex whose k-simplices are determined by unordered (k+1)-tuples of points whose closed* $\epsilon/2$*-ball neighborhoods have a point of common intersection.*

In other words, a set of $k+1$ points forms a k simplex in R_e only if the intersection of the neighborhoods around each point is not empty. The size of these neighborhoods is controlled by the parameter ϵ, a characteristic distance often called the "filtration time" in this context. Then the question arises what is the optimal filtration time to capture the *true* topology of the data? Of course, the answer is that no such preferred parameter value exists.

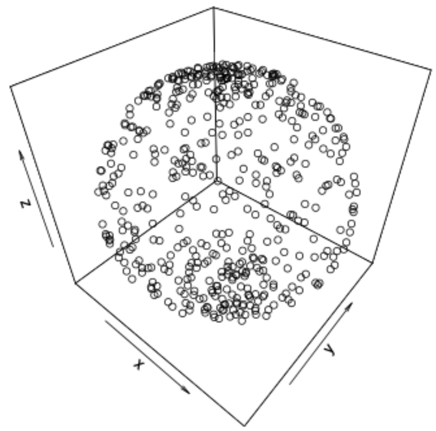
Instead of choosing a particular ϵ, we consider a simplicial complex chain, a series of nested R_ϵ with increasing filtration time. Then a feature is said to reflect the topology if it *persists* in multiple complexes over a significant range of ϵ. Rather than finding specific Betti numbers, we record the intervals over which elements of the homology groups persist. Barcodes are conventionally used to display these intervals, but a persistence diagram provides a more concise visualization by displaying each order homology in a single chart. Features located near the the diagonal are considered noise, as they vanish almost immediately after coming into existence. Consider the following example of the sphere.

Example 3.2. *The homology of the sphere is well known, and it provides a perfect example for our purposes. First, we randomly sample points from the sphere to generate the point cloud. There are a number of software packages with functions to calculate the homology intervals, such as* pHom *for R and* JavaPlex *for MatLab. Observe the 2-hole persisting until about* $\epsilon = 1$ *and the 0-hole persisting the full duration of the filtration. These agree with the Betti numbers* $b_0(S^2) = 1$ *and* $b_2(S^2) = 1$*. Note that every 1-hole vanishes soon after it comes into existence; these are noise in the data and do not contradict the real Betti number* $b_1(S^2) = 0$*.*

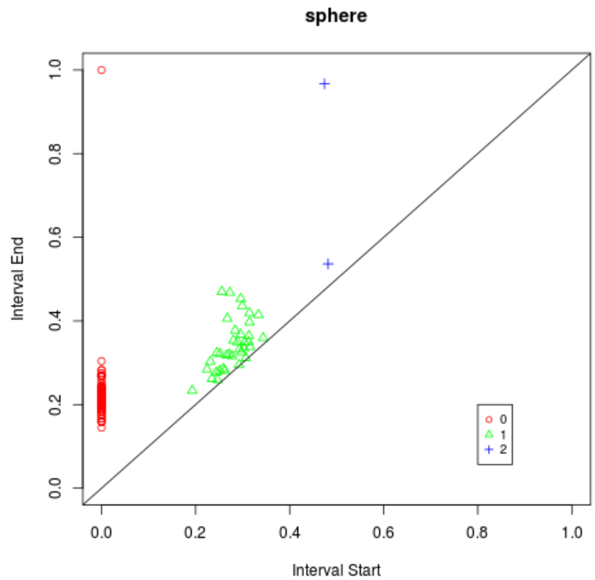
Now we are ready to examine the topology of the countries of the world. The point cloud is simply the set of centroids in cartesian coordinates. The strong 2-hole disappears since countries do not wrap around the globe evenly. Circles appear relatively lately, so one possibility is that they form the boundary of large bodies of water where the countries link up around - perhaps with island nations acting as a bridge across the southern part of the body. Unfortunatley, *phom*, our software of choice, does not permit the study of the simplicial complexes directly, so another approach is needed to descern which countries generate these circles.

4. TDA: Social Geography

Taking the persistent homology of country geography as a baseline, we then add social indicators as extra dimensions and observe the effects on the persistent homology. First, we require some notion of distnance in these social dimensions. To remedy the problem of noncommutative dimensions, we scale the data onto the interval $[-1, 1]$, where an advantageous value is closer to

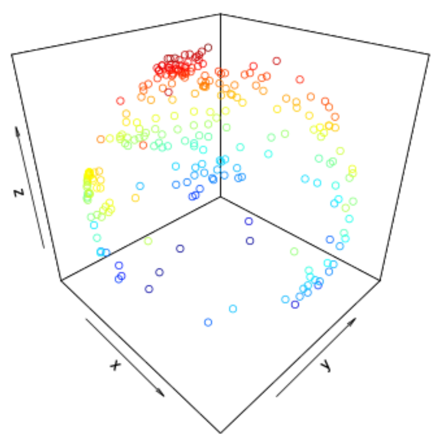


(A) Scatterplot of 500 points in S^2.

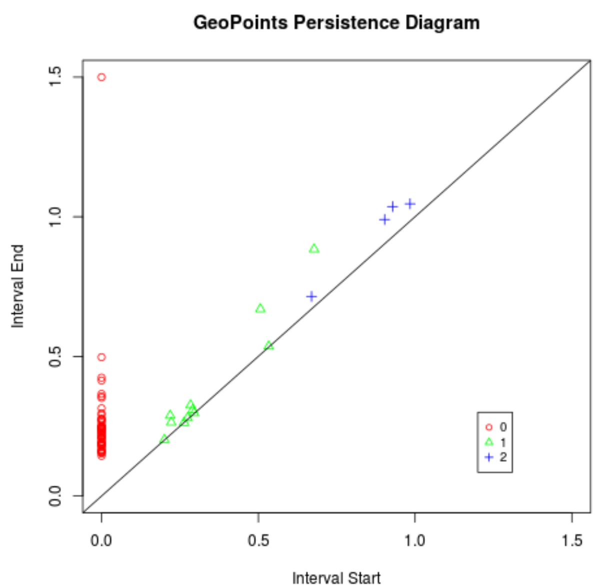


(B) Persistence diagram for the set of points.

FIGURE 1. The persistent homology of randomly selected points from the sphere.



(A) Scatterplot of Country centroids, a subset of S^2. Centroids colored by lattitude, oriented with the north pole in the usual position.



(B) Persistence diagram for country centroids.

FIGURE 2. The persistence homology of country geography.

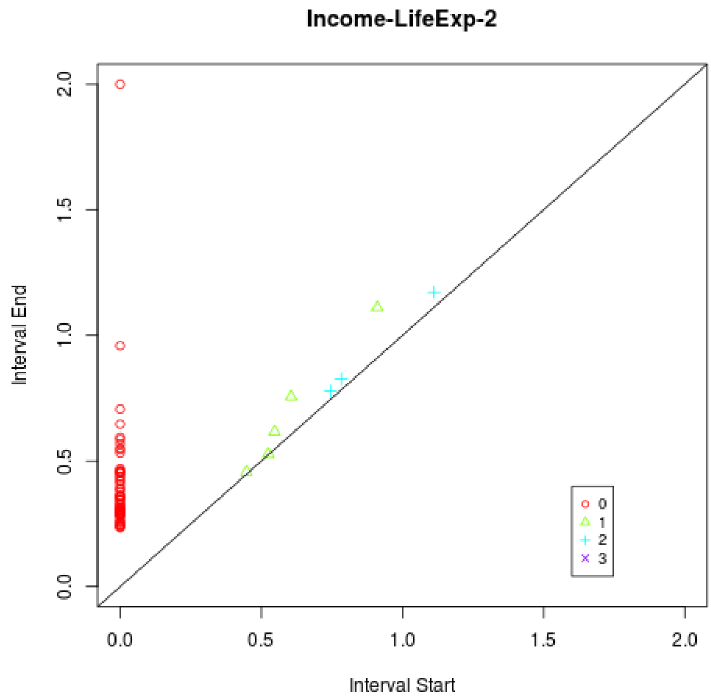


FIGURE 3. Persistance diagram of the data in five dimensions: geographic location, income per person, and life expectancy. Note the 0-hole that persists out to time 0.96, indicating a nontrival second connected component.

positive one, and a disadvantageous value is closer to negative one. In this way, a country with disadvantageous results will be closer in distance with other countries with disadvantageous results, and the same is true with positive results.

Thus distinct clusters, partitions, or connected-components (however we choose to describe such groupings) indicate a finer division of countries based on measurable indicators of quality. Combining this information with geographic data then sheds light on whether regions of countries reflect these divisions.

With addition of the indicators *Income Per Person* and *Life Expectancy*, an additional connected component arises in the form of a non-trivial $b_0 = 2$ over the parameter range $[0, 0.96]$. We see that all topological features washes out at about $\epsilon = 1.2$. The aforementioned second connected component may be considered significant since it persists over most of this range. Hence we observe initial evidence of a division of countries on geographic, health, and economic lines that did not exist before with just the geographic data. To understand this division, and discern how much water it holds, we examine the data by finding clusters with k-means, a standard clustering algorithm.

5. Finding Clusters of Countries

The k-means algorithm depends on a a method for dividing a set of points into regions known as Voronoi regions. This partitioning scheme is known as the Voronoi Diagram.

Definition 5.1. [7] *A* Voronoi Diagram *is a partitioning of a plane with n points into convex polygons such that each polygon contains exactly one generating point and every point in a given polygon is closer to its generating point than to any other.*

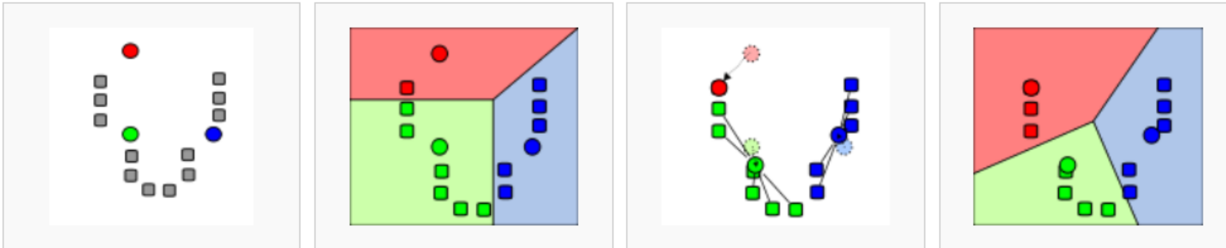


FIGURE 4. Diagram of k-means alogrithm. [6]

The Voronoi Diagram can be generalized to higher dimensions, but it requires making the boundary of the regions fuzzy in order to be computationally feasable.

Definition 5.2. k-means clustering *is a method for partitioning a data set into k clusters by the following algorithm:*

(1) *A set of k points $\{\alpha_i\}_0^k$ are assigned (often randomly) to represent the cluster centers.*
(2) *The Voronoi diagram partitions the data into sets $\{A_i\}_0^k$ about the cluster centers $\{\alpha_i\}_0^k$.*
(3) *Cluster centers are reassigned to the arithmetic mean of the points within the particular region, e.g. for some set A_i centered at α_i:*

$$\alpha_i' = \frac{1}{|A_i|} \sum_{x_j \in A_i} x_j$$

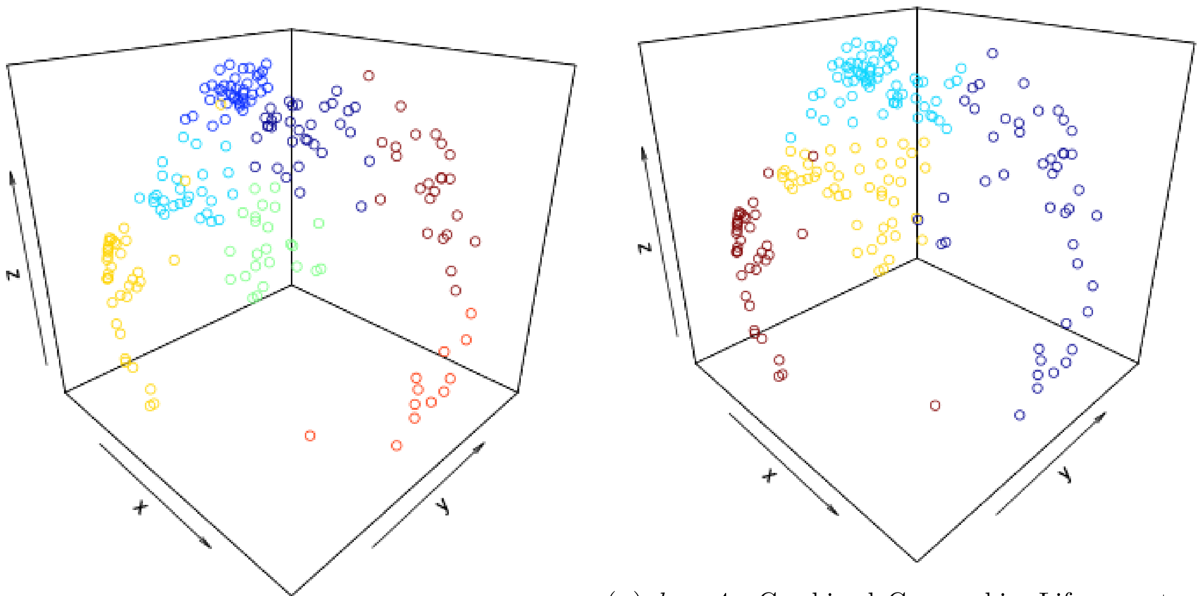
(4) *Steps (2)-(3) are repeated until the cluster centers converge.*

As with the initial persistent homology analysis, we first look at clusters based soley on geographic proximity. One drawback of the k-means is that it requires a predetermined choice of k, the number of clusters. The *pamk* function (R, *fpc* package) implements an algorithm that makes an intelligent guess at the number of clusters by comparing the results of multpile values of k. For soley geographic data, *pamk* finds seven distinct clusters that basically coincide with the continents. However, the regions collapse into four when data on Life Expectancy and Income per Person is incluced. Interstingly, but perhaps not surprisingly, northern and southern Africa are pulled toegether and made more distinct from European countries. On the other hand, Asian and countries of the South Pacific are brought together.

6. Discussion and further work

We have a natural tendancy to group countries in certain regions together not solely on geography but also on perceived social/polital trends. The results of preliminary TDA and clustering analysis show that this tendancy partly reflects reality. Further work is surely needed to see how far these distinctions run in the data, after all we have examined only two out of the hundreds of indicators available on the Gapminder site alone. While it is too early to make strong claims about the division of countries, it is notable how life expactancy and income per person - two indicators that represent the collusion of a wide swath of economic and health phenomena - seem to pull countries into certain geographic regions.

Why TDA? It may seem that these results could have been derived soley from an analysis of data clusters. The comparison itself is interesting. Can zero-order persistent homology serve as a clustering algorithm in its own right? Certain precautions must be taken. For example, the number of connect components of a simplicial complex based on point cloud data is highly sensitive to outliers - if a point lives very far away from its neighbors, then it will take a large filtration time to eliminate its undesireable contribution to b_0.



(A) $k = 7$. Clusters based on geographic data only.

(B) $k = 4$. Combined Geographic, Life expectancy, and Income per person data. See Appendix A for a list of countries by region.

FIGURE 5. Results of clustering by geographic data and the addition of social data.

Careful consideration must also be taken with how long a component should persist before considering it significant. For example, a data set may have a baseline density such that what we would consider distinct components are joined up prematurely. Sensitive to density in that it cannot distinguish features of uniform density unless they are spaced sufficiently far apart.

Despite these drawbacks, persistent homology does have some advantages. Unlike k-means, it does not require a predetermined choice in the number of clusters to look for. Also by restriciting yourself to particular ranges in filtration time you can examine clusters at desired densities, hopefully with the benefit of distinguishing fine from coarse structure. At the least, homology could put bounds on the ideal number of clusters to inform the use of algorithms such as k-means. This topic deserves more attention, since as TDA rises in usage it is only to our benefit to understand all possible uses of the data it gives.

Appendix A: Countries by Region

Region 1 (Dark Blue)	Region 2 (Light Blue)	Region 3 (Yellow)	Region 4 (Red)
Afghanistan	Albania	Angola	Antigua and Barbuda
Australia	Algeria	Benin	Argentina
Bangladesh	Armenia	Botswana	Aruba
Bhutan	Austria	Burkina Faso	Bahamas
Brunei	Azerbaijan	Burundi	Barbados
Cambodia	Bahrain	Cameroon	Belize
China	Belarus	Central African Rep.	Bolivia
Fiji	Belgium	Chad	Brazil
Guam	Bosnia and Herzegovina	Comoros	Cape Verde
Hong Kong, China	Bulgaria	Congo, Dem. Rep.	Chile
India	Canada	Congo, Rep.	Colombia
Indonesia	Croatia	Cote d'Ivoire	Costa Rica
Japan	Cyprus	Djibouti	Cuba
Kiribati	Czech Rep.	Equatorial Guinea	Dominican Rep.
Korea, Dem. Rep.	Denmark	Eritrea	Ecuador
Korea, Rep.	Egypt	Ethiopia	El Salvador
Kyrgyzstan	Estonia	Gabon	French Guiana
Laos	Finland	Gambia	French Polynesia
Macao, China	France	Ghana	Grenada
Malaysia	Georgia	Guinea	Guadeloupe
Maldives	Germany	Guinea-Bissau	Guatemala
Mauritius	Greece	Kenya	Guyana
Mayotte	Greenland	Lesotho	Haiti
Mongolia	Hungary	Liberia	Honduras
Nepal	Iceland	Madagascar	Jamaica
New Caledonia	Iran	Malawi	Martinique
New Zealand	Iraq	Mali	Mexico
Pakistan	Ireland	Mauritania	Nicaragua
Papua New Guinea	Israel	Mozambique	Panama
Philippines	Italy	Namibia	Paraguay
Reunion	Jordan	Niger	Peru
Russia	Kazakhstan	Nigeria	Puerto Rico
Samoa	Kuwait	Rwanda	Saint Lucia
Seychelles	Latvia	Sao Tome and Principe	Saint Vincent and the Grenadines
Singapore	Lebanon	Senegal	Suriname
Solomon Islands	Libya	Sierra Leone	Trinidad and Tobago
Sri Lanka	Lithuania	Somalia	United States
Taiwan	Luxembourg	South Africa	Uruguay
Tajikistan	Macedonia, FYR	South Sudan	Venezuela
Thailand	Malta	Sudan	
Timor-Leste	Moldova	Swaziland	
Tokelau	Montenegro	Tanzania	
Tonga	Morocco	Togo	
Vanuatu	Netherlands	Uganda	
Vietnam	Norway	Yemen, Rep.	
	Oman	Zambia	
	Poland	Zimbabwe	
	Portugal		
	Qatar		
	Romania		
	Saudi Arabia		
	Serbia		
	Slovak Republic		
	Slovenia		
	Spain		
	Sweden		
	Switzerland		
	Syria		
	Tunisia		
	Turkey		
	Turkmenistan		
	Ukraine		
	United Arab Emirates		
	United Kingdom		
	Uzbekistan		
	West Bank and Gaza		
	Western Sahara		

Figure 6. Result of clustering based on geographic data, income per person, and life expectancy. This table corresponds to plot (b) of Figure 5.

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