Normal Distribution

Andrey Chinnov, Sebastian Honermann, Carlos Zydorek

Case Studies
"Data Analytics"

Outline

1. Introduction
   - Normality as a requirement for statistical methods
   - Data Set Overview
2. Normality Testing
   - Graphical Methods for Normality Testing
     - Q-Q-Plots
     - Chi-Square Plot
   - Quantitative Methods for Normality Testing
     - Shapiro-Wilk Test
     - Pearson's Chi-Squared Test
     - Kolmogorov-Smirnov Test
3. Transformation to Normality
   - Box-Cox Transformation
   - Transformation Results Testing
4. Summary

Normality as a requirement for statistical methods

Data Set Overview

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Graphical Methods for Normality Testing

Q-Q-Plots

Graphical Methods for Normality Testing

Chi-Square Plot

Quantitative Methods for Normality Testing

Shapiro-Wilk Test

Quantitative Methods for Normality Testing

Pearson's Chi-Squared Test

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Kolmogorov-Smirnov Test

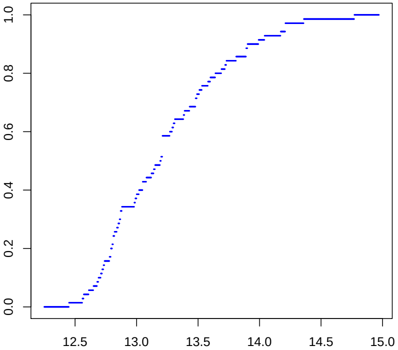
Let $x = (x_1, x_2, \dots, x_n)$ be a sample of unknown distribution $\mathbb{P}$.

Definition

$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{x_i \leq x\}}(x)$

- empirical c. d. f. , where

$$\mathbb{1}_{\{x_i \leq x\}}(x) = \begin{cases} 1 & \text{if } x_i \leq x \\ 0 & \text{otherwise.} \end{cases}$$



Glass Type 1, Natrium (Na)

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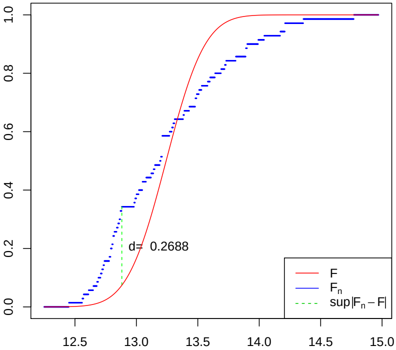
$$\bar{x} = \frac{1}{n}\sum_i x_i, \quad \sigma_x^2 = \frac{1}{n}(x_i - \bar{x})^2$$



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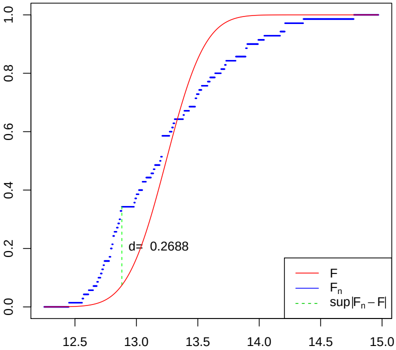
$d = \sup_{x\in\mathbb{R}} |F_n(x) - F(x)|$

- distance between them.



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- $\forall t > 0:$

$$P(D_n \leq t) \xrightarrow[n \to \infty]{} H(t) = 1 - 2 \sum_{i=1}^{\infty} (-1)^{i-1} \exp^{-2i^2 t^2}$$

Quantitative Methods for Normality Testing

Kolmogorov-Smirnov Test

The KS test uses the decision rule

$$\delta = \begin{cases} H_0 & : \quad D_n \leq c \\ H_1 & : \quad D_n > c \end{cases},$$

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$$\Longrightarrow \quad c \approx H_{1-\alpha}$$

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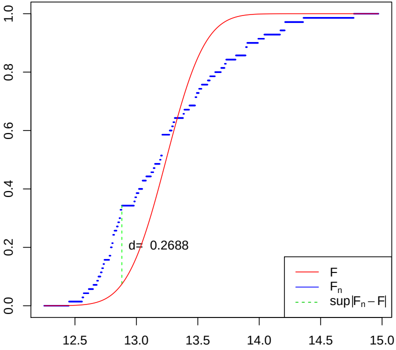

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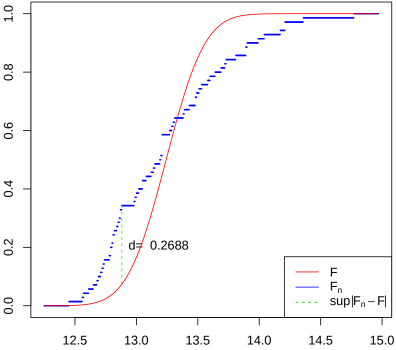
Example:

- $n = 70$



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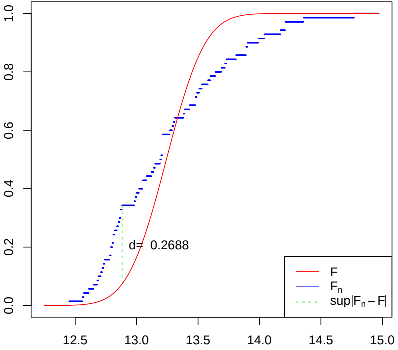

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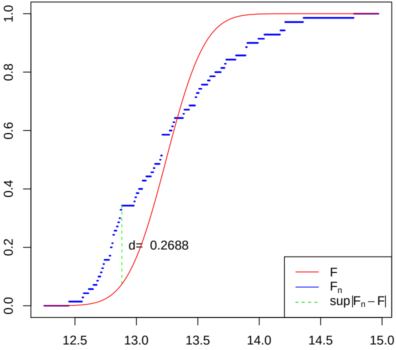
$$\Longrightarrow \quad c = H_{1-\alpha} = 1.6276$$



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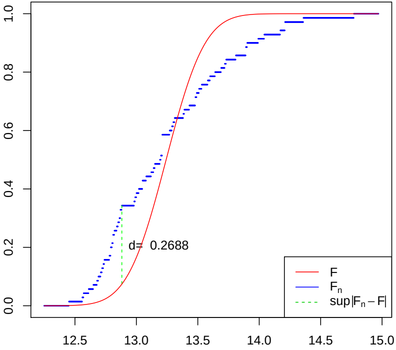
- $D_n > c \implies H_0$ rejected



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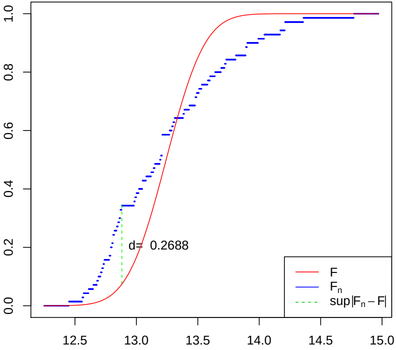

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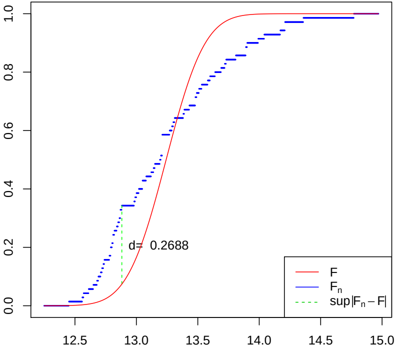
- $D_n > c \implies H_0$ rejected
- $\implies \mathbb{P} \neq \mathbb{P}_0$
- $\not\Longrightarrow$ data not normally distributed!!!



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Improved Kolmogorov-Smirnov Test

KS test is improved by solving the following optimization problem

$$KS(\mu, \sigma) = \sup_{x \in \mathbb{R}} |F_n(x) - F(x, \mu, \sigma)| \to \min .$$

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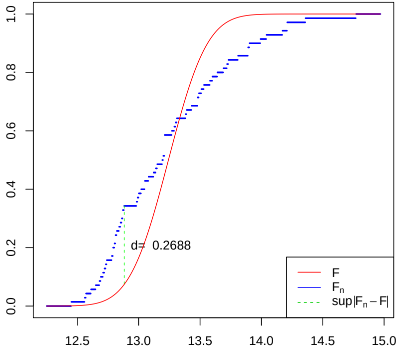

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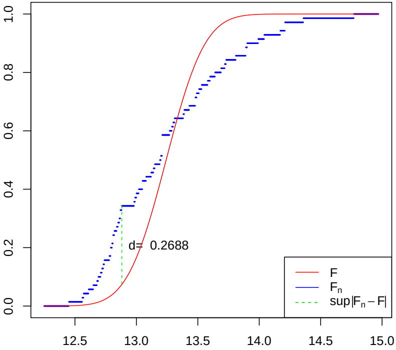
- Optimized vector of parameters
  $\hat{\mu} = 13.1770, \quad \hat{\sigma}^2 = 0.4682$



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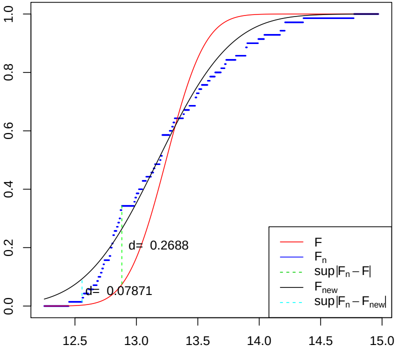

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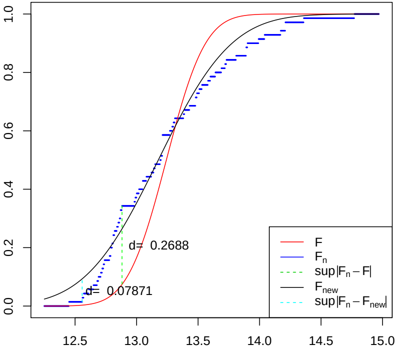

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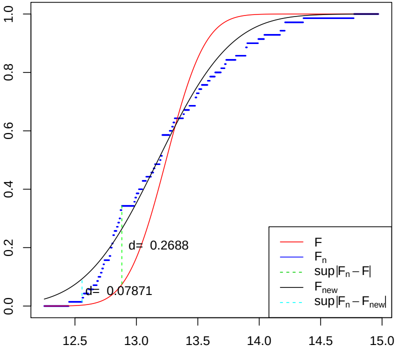

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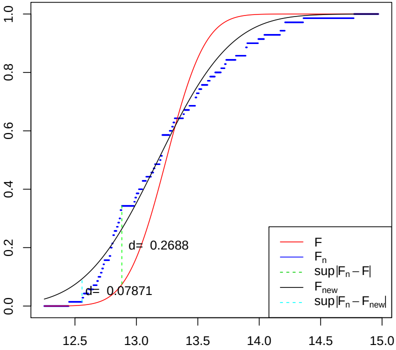

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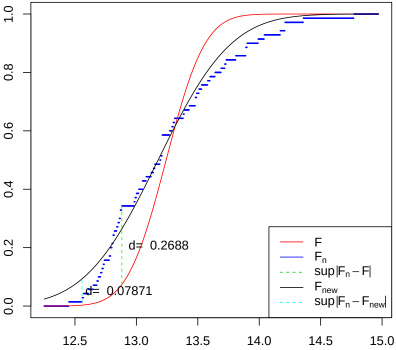
- $\implies$ data normally distributed!



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