





# Normality as a requirement for statistical methods

# Data Set Overview

- Glass data set from package `mlbench`
- sample of 214 observations
- 7 **types** of glass (but only 6 present in this sample)
- **Variables:** refractive index (RI) and 8 elements (Na, Mg, Al, Si, K, Ca, Ba, Fe)

## Definition

If data are not normally distributed, they can possibly be transformed by the parameterised power transformation

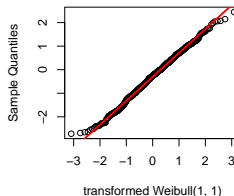
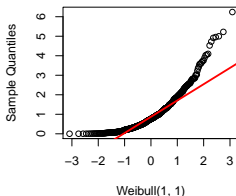
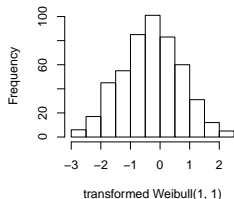
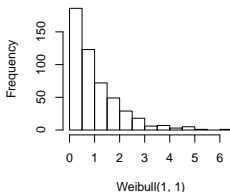
$$x^{(\lambda)} = \begin{cases} \frac{x^\lambda - 1}{\lambda} & \lambda \neq 0 \\ \ln(x) & \lambda = 0 \end{cases} \quad \text{for } x > 0$$

The optimal parameter  $\lambda$  for specific observations  $x_1, \dots, x_n$  can be obtained by a **maximum-likelihood** estimation, maximising the log likelihood

$$l(\lambda) = -\frac{n}{2} \ln \left[ \frac{1}{n} \sum_{j=1}^n (x_j^{(\lambda)} - \overline{x^{(\lambda)}})^2 \right] + (\lambda - 1) \sum_{j=1}^n \ln(x_j)$$

with  $\overline{x^{(\lambda)}} = \frac{1}{n} \sum_{j=1}^n x_j^{(\lambda)}$

## Transformation issues



## Transformation issues

Number of non-zero elements (x)	Frequency (y)
0	10
1	145
2	95
3	15
4	20
5	25
6	40
7	60
8	55
9	35
10	15
11	5
12	2

transformed Weibull(5, 1), Weibull(5, 4)

# Q-Q-Plots

**Sample:**

$$x = (x_1, x_2, \dots, x_n)$$

**Empirical quantiles:**

$$x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$$

**Theoretical quantiles:**

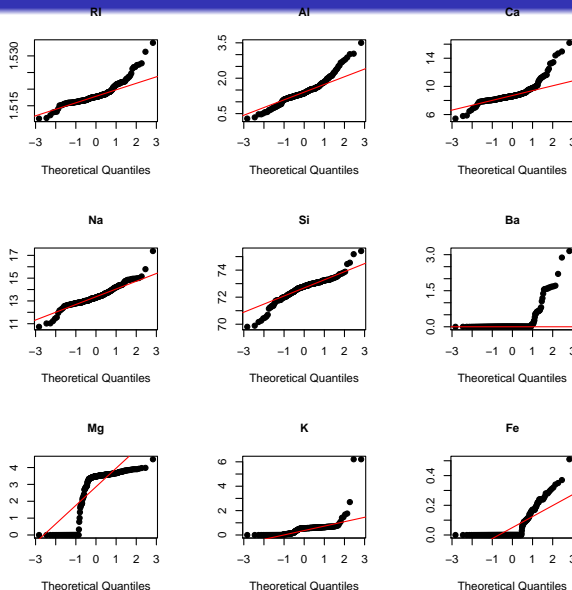
$$q_{(j)} = \Phi^{-1}(p_{(j)}),$$

where

$$p_{(j)} = \frac{j - \frac{1}{2}}{n},$$

$\Phi - N(0, 1)$  c. d. f. .

$\Rightarrow$  Plot  $x_{(i)}$  against  $q_{(i)}$

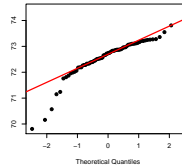




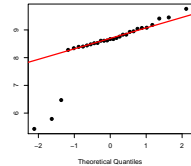
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- Variables could be normally distributed within the subclasses
- For some cases there appear to be a linear relationships
- For other cases a linear relationship is questionable
- In some subdatasets a linear relationship seems plausible, however  $n$  is very small

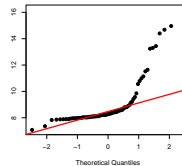
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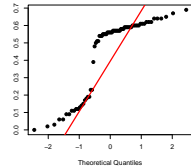
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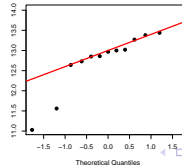
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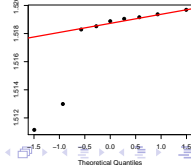
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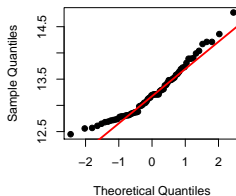


# Q-Q-Plots

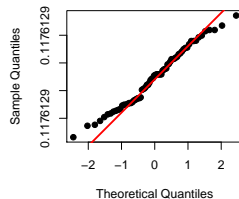
## Results of the Transformation of the Full Dataset :

- For some of the cases there seems to be a slight improvement
- For non-unimodal cases the transformation does not show significant improvements towards normality

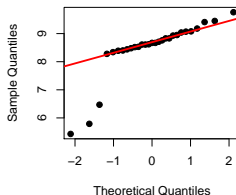
Na Glass Type 1



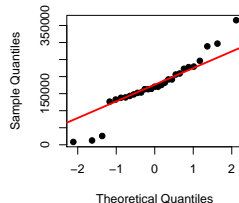
Na Glass Type 1 transformed



Ca Glass Type 7



Ca Glass Type 7 transformed

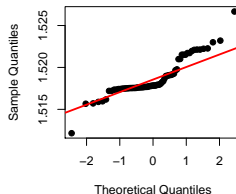


# Q-Q-Plots

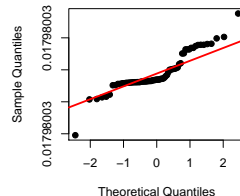
## Results of the Transformation of the Subdatasets :

- For unimodal cases the transformation shapes the distribution closer to normality
- For non-unimodal cases the transformation does not show significant improvements towards normality

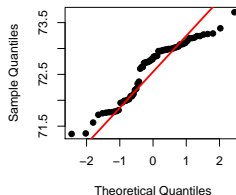
RI Glass Type 1



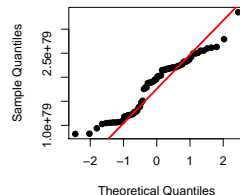
RI Glass Type 1 transformed



Si Glass Type 1



Si Glass Type 1 transformed



# Shapiro-Wilk Test

The test statistic  $W$  indicates the deviation of the observed quantile values from the assumed cumulative distribution function quantiles

$$W = \frac{\sum_{i=1}^n (a_i y_i)^2}{\sum_{i=1}^n (y_i - \bar{y})^2},$$

where

- $a_i$  denotes the normalised "best linear unbiased" coefficients,
- $y_i$  denotes the observations.

The critical value for  $W$  is obtained by the Monte Carlo Method  
 $\implies$   $p$ -value is calculated

**Important:** If a variable contains only zeros the Shapiro-Wilk test is not applicable, since the term in the denominator sums up to zero.

# Shapiro-Wilk Test

## Testing the Full Dataset :

Null hypothesis is rejected for all variables at a 1 % significance level

variable	test statistic	sig. level	critical value	p-value	rejected
RI	0.87	0.01	NA	1.0766713449726e-12	yes
Na	0.95	0.01	NA	3.4655430546966e-07	yes
Mg	0.7	0.01	NA	< 1.0e-15	yes
Al	0.94	0.01	NA	2.08315629600399e-07	yes
Si	0.92	0.01	NA	2.17503176825416e-09	yes
K	0.44	0.01	NA	< 1.0e-15	yes
Ca	0.79	0.01	NA	< 1.0e-15	yes
Ba	0.41	0.01	NA	< 1.0e-15	yes
Fe	0.65	0.01	NA	< 1.0e-15	yes

## After the Transformation :

The null hypothesis can be rejected for the four transformed variables

⇒ Possible

Explanation:

Combination of different distributions in the different glass types

## Test results of the Shapiro-Wilk test on the whole data sample

variable	test statistic	sig. level	critical value	p-value	rejected
RI	NA	NA	NA	NA	NA
Na	0.95	0.01	NA	8.75605777309153e-07	yes
Mg	NA	NA	NA	NA	NA
Al	0.97	0.01	NA	0.000244326513056066	yes
Si	0.93	0.01	NA	1.58998125691823e-08	yes
K	NA	NA	NA	NA	NA
Ca	0.89	0.01	NA	1.13880689831982e-11	yes
Ba	NA	NA	NA	NA	NA
Fe	NA	NA	NA	NA	NA

## Test results of the Shapiro-Wilk test on the whole transformed data sample

# Shapiro-Wilk Test

## Testing the

## Subdatasets

## Example – Glass

## Type 1 :

Null hypothesis is rejected for all variables at a 1 % significance level

variable	test statistic	sig. level	critical value	p-value	rejected
RI	0.88	0.01	NA	6.36192013015468e-06	yes
Na	0.95	0.01	NA	0.00459078607995831	yes
Mg	0.82	0.01	NA	8.02702432879544e-08	yes
Al	0.9	0.01	NA	5.42971629496434e-05	yes
Si	0.91	0.01	NA	0.000117060780025464	yes
K	0.77	0.01	NA	3.14049093233846e-09	yes
Ca	0.93	0.01	NA	0.00103561283726753	yes

Test results of the Shapiro-Wilk test on type 1 glass

## After the

## Transformation :

The null hypothesis cannot be rejected for 3 of the transformed variables

⇒ Apparently the transformation was successful

variable	test statistic	sig. level	critical value	p-value	rejected
RI	0.89	0.01	NA	1.62433657125306e-05	yes
Na	0.98	0.01	NA	0.353792914291578	no
Mg	0.83	0.01	NA	1.40023833110547e-07	yes
Al	0.96	0.01	NA	0.0459207068393172	no
Si	0.94	0.01	NA	0.00269629206710463	yes
K	NA	NA	NA	NA	NA
Ca	0.97	0.01	NA	0.148237775100495	no

Test results of the Shapiro-Wilk test on the transformed type 1 glass

# Graphical Methods for Normality Testing

## Shapiro-Wilk Test

# Pearson's Chi-Squared Test

## Theoretical foundations

- Divide observations  $X_1, \dots, X_N$  into **pairwise disjoint classes**  $C_1, \dots, C_K$
- Common requirement: minimum class size of 5
- Compare **observed** class frequencies to **expected** theoretical class frequencies for a certain distribution

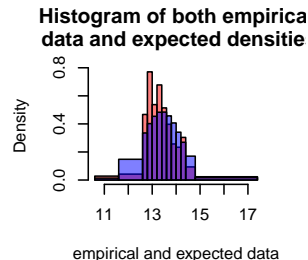
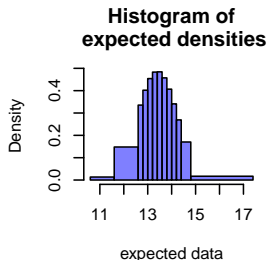
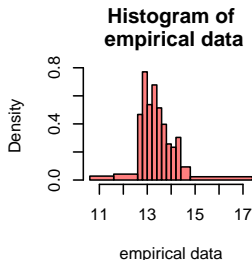
$$\text{test statistic: } \chi^2 = \sum_{k=1}^K \frac{(O_k - E_k)^2}{E_k}$$

- The test statistic is approximately  **$\chi^2$ -distributed** with  $K - 1$  degrees of freedom (minus one degree of freedom per estimated parameter)



# Pearson's Chi-Squared Test

## Theoretical foundations



# Pearson's Chi-Squared Test

## Theoretical foundations

- Test under the **null hypothesis** that the sample is drawn from a population with unknown distribution  $\mathbb{P}$  which is equal to the assumed distribution  $\mathbb{P}_0$ :

$$H_0 : \mathbb{P} = \mathbb{P}_0,$$

$$H_1 : \mathbb{P} \neq \mathbb{P}_0.$$

### Decision rule

$$\delta = \begin{cases} 1 & \text{if } \chi^2 > F^{-1}(1 - \alpha) \\ 0 & \text{otherwise} \end{cases} \quad \text{with } F = \chi^2_{K-1-p}$$

(significance level  $\alpha$ , number of estimated parameters  $p$ )

# Pearson's Chi-Squared Test

Test results for the whole sample

variable	test statistic	sig. level	critical value	p-value	rejected
Rl	64.95	0.01	13.28	2.64011035255862e-13	yes
Na	36.99	0.01	13.28	1.80797974702607e-07	yes
Mg	158.3	0.01	11.34	< 1.0e-15	yes
Al	27.2	0.01	9.21	1.24084046404516e-06	yes
Si	38.85	0.01	13.28	7.4876188027595e-08	yes
K	95.97	0.01	NA	NA	NA
Ca	131.13	0.01	13.28	< 1.0e-15	yes
Ba	31.37	0.01	NA	NA	NA
Fe	70.96	0.01	13.28	1.4210854715202e-14	yes

# Pearson's Chi-Squared Test

Test results for type 1 glass

variable	test statistic	sig. level	critical value	p-value	rejected
RI	28.01	0.01	9.21	8.26265138420545e-07	yes
Na	3.25	0.01	13.28	0.51688441877949	no
Mg	18.81	0.01	6.63	1.44068580684165e-05	yes
Al	23.55	0.01	11.34	3.10284613768141e-05	yes
Si	23.68	0.01	13.28	9.26014020323773e-05	yes
K	114.86	0.01	11.34	< 1.0e-15	yes
Ca	22.58	0.01	15.09	0.000405198755082603	yes
Fe	18.65	0.01	9.21	8.91413549507503e-05	yes

# Pearson's Chi-Squared Test

Test results for type 1 glass

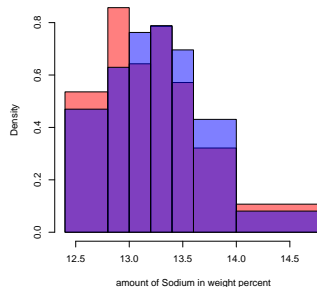
variable	test statistic	sig. level	critical value	p-value	rejected
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Mg	18.81	0.01	6.63	1.44068580684165e-05	yes
Al	23.55	0.01	11.34	3.10284613768141e-05	yes
Si	23.68	0.01	13.28	9.26014020323773e-05	yes
K	114.86	0.01	11.34	< 1.0e-15	yes
Ca	22.58	0.01	15.09	0.000405198755082603	yes
Fe	18.65	0.01	9.21	8.91413549507503e-05	yes

# Pearson's Chi-Squared Test

Test results for sodium of type 1 glass

class (interval)	frequencies	
	observed	expected
]12.4, 12.8]	15	13.15
]12.8, 13]	12	8.81
]13, 13.2]	9	10.68
]13.2, 13.4]	11	11.04
]13.4, 13.6]	8	9.74
]13.6, 14]	9	12.06
]14, 14.8]	6	4.52

Histogram of observed and expected densities



# Pearson's Chi-Squared Test

Test results for transformed type 1 glass

variable	test statistic	sig. level	critical value	p-value	rejected
RI	27.81	0.01	6.63	1.33864150764218e-07	yes
Na	1.59	0.01	13.28	0.810360513797024	no
Mg	17.87	0.01	NA	NA	NA
Al	6.41	0.01	11.34	0.093110657016404	no
Si	16.87	0.01	13.28	0.00205136639513992	yes
K	NA	0.01	NA	NA	NA
Ca	3.35	0.01	11.34	0.341234021909645	no
Fe	NA	0.01	NA	NA	NA

# Kolmogorov-Smirnov Test

## Preliminaries

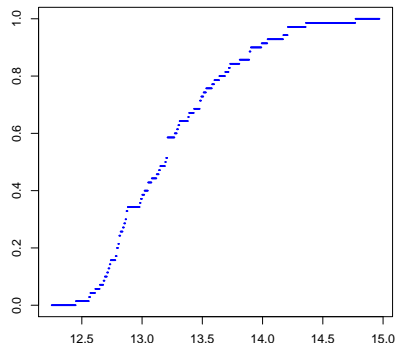
Let  $x = (x_1, x_2, \dots, x_n)$  be a sample of unknown distribution  $\mathbb{P}$ .

### Definition

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{x_i \leq x\}}(x)$$

- **empirical** c. d. f. , where

$$\mathbb{1}_{\{x_i \leq x\}}(x) = \begin{cases} 1 & \text{if } x_i \leq x \\ 0 & \text{otherwise.} \end{cases}$$



Glass Type 1, Sodium (Na)



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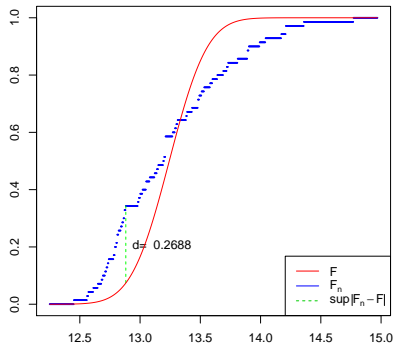
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$F(x)$  - theoretical normal c. d. f.

with

$$\bar{x} = \frac{1}{n} \sum_i x_i, \quad \sigma_x^2 = \frac{1}{n} (x_i - \bar{x})^2$$



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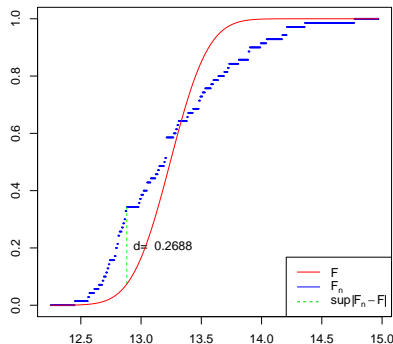
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with

$$\bar{x} = \frac{1}{n} \sum_i x_i, \quad \sigma_x^2 = \frac{1}{n} (x_i - \bar{x})^2$$

$$d = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)|$$

- distance between them.



Glass Type 1, Sodium (Na)

# Kolmogorov-Smirnov Test

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$$\begin{aligned} H_0 &: \mathbb{P} = \mathbb{P}_0, \\ H_1 &: \mathbb{P} \neq \mathbb{P}_0. \end{aligned}$$

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$$D_n = \sqrt{n} \cdot \sup_{x \in \mathbb{R}} |F_n(x) - F(x)|.$$

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Properties of  $D_n$  in case  $H_0$  is **TRUE**:

- Distribution of  $\hat{D}_n := (D_1, D_2, \dots, D_n)$  does not depend on  $F$

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 $\implies$  **tabulated**

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 $\implies$  **tabulated**
- $\forall t > 0$ :

$$P(D_n \leq t) \xrightarrow{n \rightarrow \infty} H(t) = 1 - 2 \sum_{i=1}^{\infty} (-1)^{i-1} \exp^{-2i^2 t^2}$$



# Kolmogorov-Smirnov Test

The KS test uses the decision rule

$$\delta = \begin{cases} H_0 & : D_n \leq c \\ H_1 & : D_n > c \end{cases},$$

where  $c$  - critical value

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$$\alpha = P(\delta \neq H_0 | H_0)$$

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where  $c$  - critical value that depends on a significance level  $\alpha$ :

$$\alpha = P(\delta \neq H_0 | H_0) = P(D_n > c | H_0)$$

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where  $c$  - critical value that depends on a significance level  $\alpha$ :

$$\alpha = P(\delta \neq H_0 | H_0) = P(D_n > c | H_0) = 1 - P(D_n \leq c | H_0)$$

# Kolmogorov-Smirnov Test

The KS test uses the decision rule

$$\delta = \begin{cases} H_0 & : D_n \leq c \\ H_1 & : D_n > c \end{cases},$$

where  $c$  - critical value that  
depends on a significance level  $\alpha$ :

$$\alpha = P(\delta \neq H_0 | H_0) = P(D_n > c | H_0) = 1 - P(D_n \leq c | H_0) \approx 1 - H(c).$$

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$$\implies c \approx H_{1-\alpha}$$

# Kolmogorov-Smirnov Test

The KS test uses the decision rule for a given significance level  $\alpha$

$$\delta = \begin{cases} H_0 & : D_n \leq H_{1-\alpha} \\ H_1 & : D_n > H_{1-\alpha} \end{cases}, \quad H(t) = 1 - 2 \sum_{i=1}^{\infty} (-1)^{i-1} \exp^{-2i^2 t^2}$$

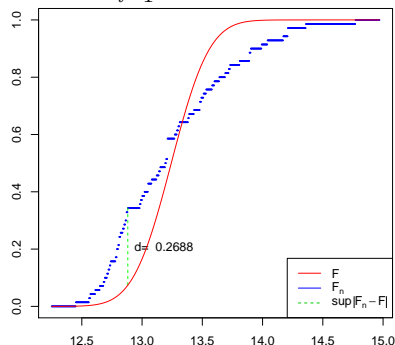


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**Example:**



Glass Type 1, Sodium (Na)

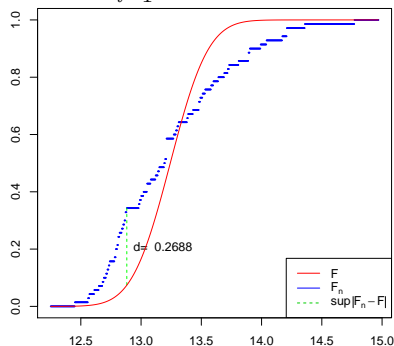
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**Example:**

●  $n = 70$



Glass Type 1, Sodium (Na)

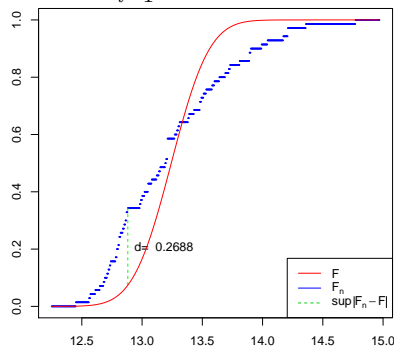
# Kolmogorov-Smirnov Test

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$$\delta = \begin{cases} H_0 & : D_n \leq H_{1-\alpha} \\ H_1 & : D_n > H_{1-\alpha} \end{cases}, \quad H(t) = 1 - 2 \sum_{i=1}^{\infty} (-1)^{i-1} \exp^{-2i^2 t^2}$$

## Example:

- $n = 70$
- $D_n = \sqrt{n} \sup |F_n - F| = 2.2493$



Glass Type 1, Sodium (Na)

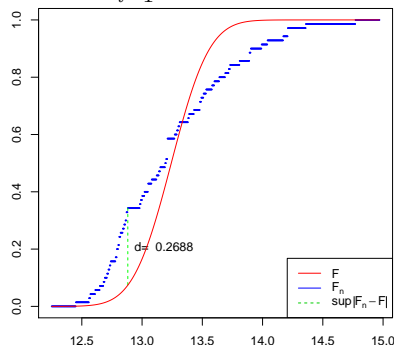
# Kolmogorov-Smirnov Test

The KS test uses the decision rule for a given significance level  $\alpha$

$$\delta = \begin{cases} H_0 & : D_n \leq H_{1-\alpha} \\ H_1 & : D_n > H_{1-\alpha} \end{cases}, \quad H(t) = 1 - 2 \sum_{i=1}^{\infty} (-1)^{i-1} \exp^{-2i^2 t^2}$$

## Example:

- $n = 70$
- $D_n = \sqrt{n} \sup |F_n - F| = 2.2493$
- $\alpha = 0.01$   
 $\implies c = H_{1-\alpha} = 1.6276$



Glass Type 1, Sodium (Na)

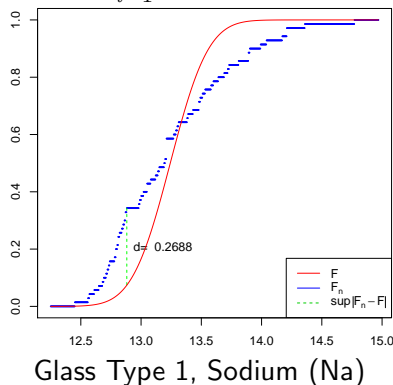
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$$\delta = \begin{cases} H_0 & : D_n \leq H_{1-\alpha} \\ H_1 & : D_n > H_{1-\alpha} \end{cases}, \quad H(t) = 1 - 2 \sum_{i=1}^{\infty} (-1)^{i-1} \exp^{-2i^2 t^2}$$

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- $D_n > c \implies H_0$  **rejected**



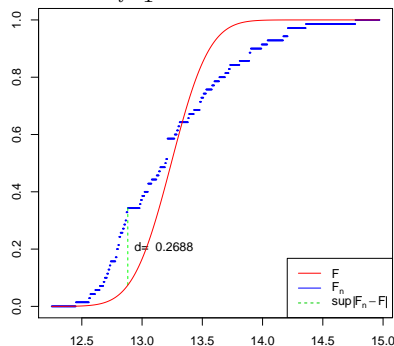
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- $\implies \mathbb{P} \neq \mathbb{P}_0$



Glass Type 1, Sodium (Na)

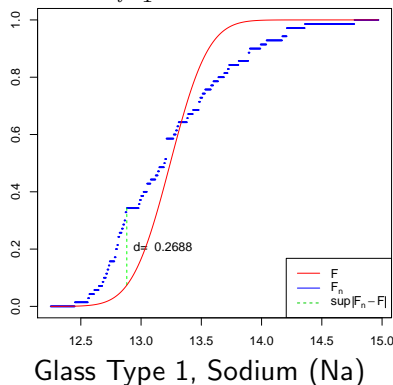
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 $\implies c = H_{1-\alpha} = 1.6276$
- $D_n > c \implies H_0$  **rejected**
- $\implies \mathbb{P} \neq \mathbb{P}_0$
- $\not\Rightarrow$  **data not normally distributed!!!**



# Kolmogorov-Smirnov Test

## Improvement

KS test is improved by solving the following optimization problem

$$KS(\mu, \sigma) = \sup_{x \in \mathbb{R}} |F_n(x) - F(x, \mu, \sigma)| \rightarrow \min.$$

*R* code used:



# Kolmogorov-Smirnov Test

## Improvement

KS test is improved by solving the following optimization problem

$$KS(\mu, \sigma) = \sup_{x \in \mathbb{R}} |F_n(x) - F(x, \mu, \sigma)| \rightarrow \min.$$

*R* code used:

```
c(mean(dat), var(dat))
```

```
[1] 13.2422857 0.2493019
```

```
#optim is a predefined R function in stats package
```

```
#default method of optimization is Nelder and Mead
```

```
result = optim(c(mean(dat), var(dat)), KS)
```

```
result$par
```

```
[1] 13.1769501 0.4682486
```

```
result$value
```

```
[1] 0.07870673
```

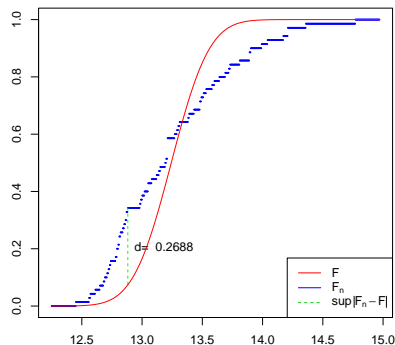
# Kolmogorov-Smirnov Test

## Improvement

KS test is improved by solving the following optimization problem

$$KS(\mu, \sigma) = \sup_{x \in \mathbb{R}} |F_n(x) - F(x, \mu, \sigma)| \rightarrow \min.$$

- Initial vector of parameters  
 $\mu = 13.2423, \quad \sigma^2 = 0.2493$
- Optimized vector of parameters  
 $\hat{\mu} = 13.1770, \quad \hat{\sigma}^2 = 0.4682$



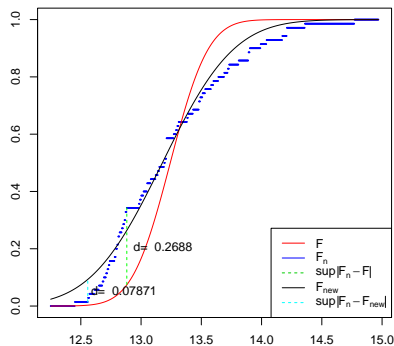
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Glass Type 1, Sodium (Na)

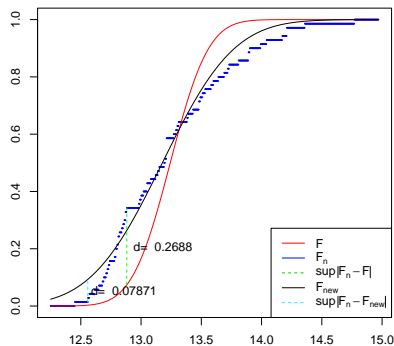
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- $D_n = \sqrt{n} \sup |F_n - F_{new}| = 0.6585$



Glass Type 1, Sodium (Na)

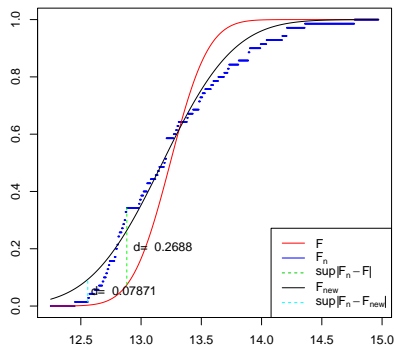
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- $c = 1.6276$



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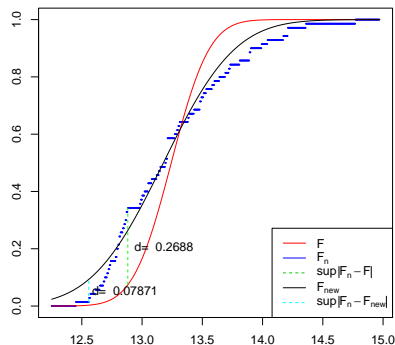
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- $c = 1.6276$
- $D_n < c \implies H_0$  **not rejected**



Glass Type 1, Sodium (Na)

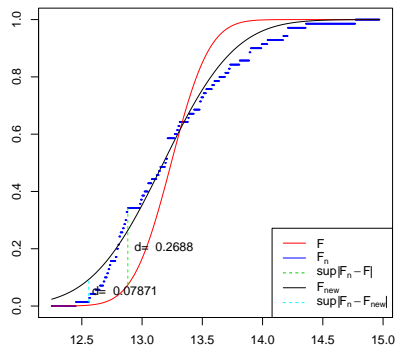
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Glass Type 1, Sodium (Na)

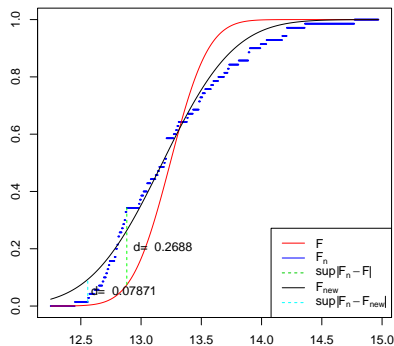
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- $c = 1.6276$
- $D_n < c \implies H_0$  **not rejected**
- $\implies \mathbb{P} = \mathbb{P}_0$
- $\implies$  **data normally distributed!**



Glass Type 1, Sodium (Na)



# Kolmogorov-Smirnov Test

Results of the improved test on the whole data set

Results of Improved KS test on the whole data set:

variable	test statistic	sig. level	critical value	p-value	rejected
RI	1.34	0.01	1.63	0.0561963016778131	no
Na	0.87	0.01	1.63	0.43825271603342	no
Mg	2.94	0.01	1.63	6.18457917100912e-08	yes
Al	0.84	0.01	1.63	0.474757887353829	no
Si	0.96	0.01	1.63	0.314710019077325	no
K	2.14	0.01	1.63	0.000212776619708754	yes
Ca	1.33	0.01	1.63	0.057710602872685	no
Ba	2.60	0.01	1.63	2.75476085742632e-06	yes
Fe	4.68	0.01	1.63	< 1.0e-15	yes

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- for 5 variables  $H_0$  is not rejected (RI,Na,Al,Si,Ca)

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- for 5 variables  $H_0$  is not rejected (RI,Na,Al,Si,Ca)
- for 4 variables  $H_0$  is rejected (Mg,K,Ba,Fe)

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- for 5 variables  $H_0$  is not rejected (RI,Na,Al,Si,Ca)
- for 4 variables  $H_0$  is rejected (Mg,K,Ba,Fe)
- The best statistics test value for Al

# Kolmogorov-Smirnov Test

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- for 5 variables  $H_0$  is not rejected (RI,Na,Al,Si,Ca)
- for 4 variables  $H_0$  is rejected (Mg,K,Ba,Fe)
- The best statistics test value for Al
- The worst statistic test value for Fe

# Kolmogorov-Smirnov Test

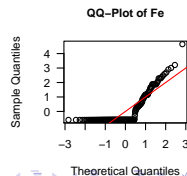
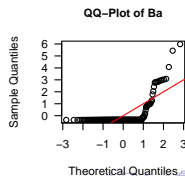
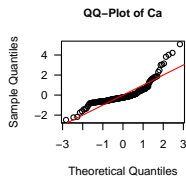
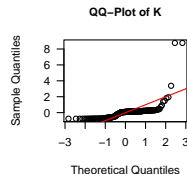
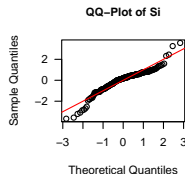
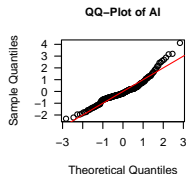
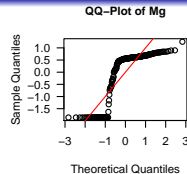
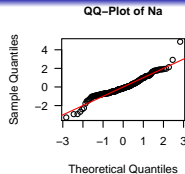
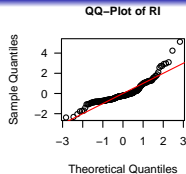
## Test Results:

variable	rejected
RI	no
Na	no
Mg	yes
Al	no
Si	no
K	yes
Ca	no
Ba	yes
Fe	yes

# Kolmogorov-Smirnov Test

## Test Results:

variable	rejected
Rl	no
Na	no
Mg	yes
Al	no
Si	no
K	yes
Ca	no
Ba	yes
Fe	yes



# Kolmogorov-Smirnov Test

## Results on the data subsets

variable	test statistic	sig. level	critical value	p-value	rejected
RI	1.31	0.01	1.63	0.0630043926883292	no
Na	0.66	0.01	1.63	0.77871853343362	no
Mg	0.49	0.01	1.63	0.967729719418776	no
Al	0.92	0.01	1.63	0.366854549713195	no
Si	1.06	0.01	1.63	0.208027646546284	no
K	1.73	0.01	1.63	0.00491847745617136	yes
Ca	0.84	0.01	1.63	0.48064266616439	no
Fe	2.65	0.01	1.63	1.63244296669252e-06	yes

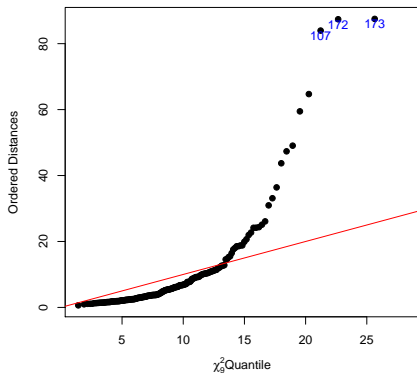
Test results of the improved KS test on type 1 glass

variable	test statistic	sig. level	critical value	p-value	rejected
RI	0.67	0.01	1.63	0.767837508224946	no
Na	0.52	0.01	1.63	0.951658259816235	no
Al	0.49	0.01	1.63	0.968495966845634	no
Si	0.56	0.01	1.63	0.915628110530136	no
K	1.43	0.01	1.63	0.0340401962712393	no
Ca	0.56	0.01	1.63	0.91558957374917	no
Ba	0.76	0.01	1.63	0.61288183743927	no

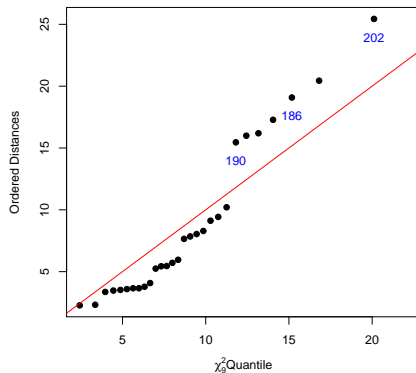
Test results of the improved KS test on type 7 glass



# Chi-Square Plot for Multivariate Normality



The whole data set



Type 7 Glass

# Plot of multivariate normal distribution

## Theoretical foundations

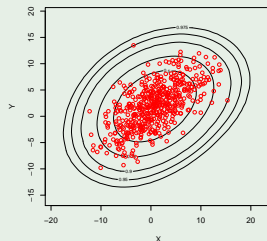
- Contour lines of the plot of a multivariate normal distribution are shaped **elliptically**
- Ellipsoids are centered at  $\mu : \{x : (x - \mu)' \Sigma^{-1}(x - \mu) = c^2\}$  with some constant  $c$ .

## Example

Multivariate normal distribution with sample size 500 and parameters

$$\mu = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

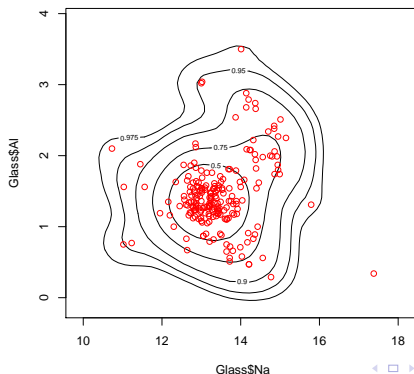
$$\Sigma = \begin{bmatrix} 27 & 15 \\ 15 & 18 \end{bmatrix}$$



# Plot of multivariate normal distribution

## Application

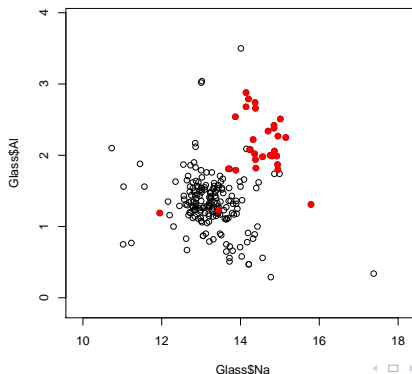
- Concerning the complete sample, the p-values for the variables Na and Al are highest among all used test methods.
- Plot data points and determine contour lines.



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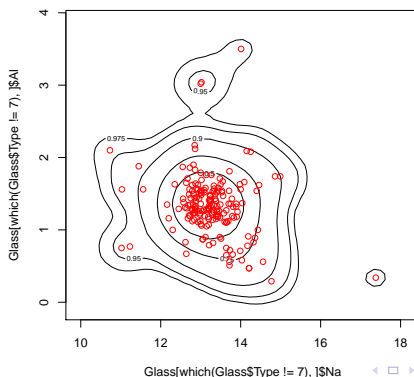
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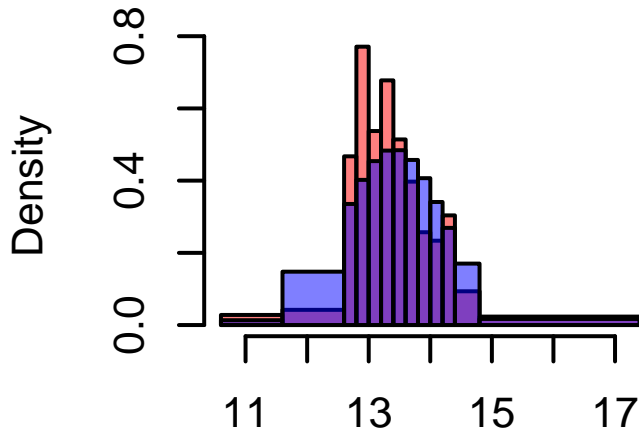
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# Pearson's Chi-Squared Test

## Theoretical foundations



empirical and expected data