Lecture 3: geometry of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ and Kontsevich formula

Disclaimer: these are *very* rough notes. Do not expect them to be exhaustive and/or detailed. There notes are intended more as a roadmap rather than a manual. Any comment is always welcome!

Guiding questions:

- What is the structure of the boundary of $\overline{M}_{0,n}$ and $\overline{M}_{0,n}(\mathbb{P}^r, d)$?
- How do these different moduli spaces relate one with each other?
- How can we use $\overline{M}_{0,n}(\mathbb{P}^r, d)$ to prove Kontsevich formula?

3.1. **The boundary of $\overline{M}_{0,n}$ and $\overline{M}_{0,n}(\mathbb{P}^2, d)$.** Recall that in the first lecture we constructed the proper, fine moduli scheme $\overline{M}_{0,n}$ of stable n-marked curves of genus 0. A natural object to study is the boundary $\partial\overline{M}_{0,n} := \overline{M}_{0,n} \setminus M_{0,n}$: what it its codimension? What are the irreducible components? And so on.

Example 3.1. Some particular cases:

(1) For $n = 4$, the boundary $\partial\overline{M}_{0,4}$ is equal to the three points $\{0\}$, $\{1\}$ and $\{\infty\}$. Therefore, the boundary has codimension 1 and it has 3 irreducible components.

Given a smooth 4-marked curve of genus zero, we can always assume that the first three markings correspond to 0, 1 and ∞.

Then we see that the stable curve corresponding to the boundary point $\{0\}$ in $\overline{M}_{0,4}$ is the one obtained as stable limit of a $\mathbb{P}^1$ where the fourth marking σ_4 is approaching the first one σ_1: this limit is a pair of $\mathbb{P}^1$ glued at one point such that the markings σ_1 and σ_4 are on one irreducible component, and the markings σ_2 and σ_3 are on the other one.

It is easy to see that the other boundary points correspond to a different recombination of the markings on the irreducible components.

(2) For $n = 5$, we have $\overline{M}_{0,5} \simeq \overline{U}_{0,4}$, where the latter is the universal family over $\overline{M}_{0,4}$, and $M_{0,5} \simeq (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{\times 2} \setminus \Delta$, where Δ is the diagonal. Recall that $\overline{U}_{0,4}$ was constructed blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ (see first lecture).

We deduce that the boundary has codimension 1 and its components are the proper transform $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$ for p in $\{0, 1, \infty\}$, the diagonal Δ and the three exceptional divisors, i.e. the boundary has 10 irreducible components.

What are the stable marked curves corresponding to the generic point of each component? To answer this question, observe that the two coordinates in $M_{0,5} \subset \overline{M}_{0,5}$ amount to the last two markings σ_4 and σ_5.

The generic point of $\{0\} \times \mathbb{P}^1$ is obtained as a limit of the family where the first coordinate of $M_{0,5} \subset \overline{M}_{0,5}$ converge to 0, i.e. σ_4 converges to σ_1: when they collide, a new component spring out, and σ_1, σ_4 are sent to this new component.

We deduce that the generic point associated to this boundary divisor is a curve with two components, with σ_1, σ_4 on one side and σ_2, σ_3 and σ_5 on the other.

A similar description holds for the boundary components $\{1\} \times \mathbb{P}^1$ and $\{\infty\} \times \mathbb{P}^1$.

Consider now the divisor $\mathbb{P}^1 \times \{0\}$: the curve associated to the generic point of this divisor can be obtained by degenerating a smooth marked curve where σ_5 converges to σ_1 (just take the generic point of $\overline{M}_{0,4}$, consider its fibre in $\overline{M}_{0,5}$ and you obtain such a degeneration).

The limit of this family is easy to describe: when σ_1 and σ_5 collide, a new component spring out and σ_1 and σ_5 are sent to this new component. In other terms, the curve associated to the generic point of $\mathbb{P}^1 \times \{0\}$ has two components with markings σ_1 and σ_5 on one side, and the remaining ones on the other side.

A similar description holds for the boundary components $\mathbb{P}^1 \times \{1\}$ and $\mathbb{P}^1 \times \{\infty\}$.

The generic point of the diagonal Δ can be obtained by making σ_4 and σ_5 coincide: the stable limit has two components with markings $\{\sigma_1, \sigma_2, \sigma_3\}$ on one side and $\{\sigma_4, \sigma_5\}$ on the other.

Finally, consider the exceptional divisor obtained after blowing-up at 0: its generic point can be obtained by making the generic point (σ_4, σ_5) of $M_{0,5}$ degenerate along a generic direction towards the point $(0, 0)$. Therefore, the corresponding curve will be the stable limit of the family of smooth 5-marked curves where both σ_4 and σ_5 are converging to σ_1: this limit will be the union of two $\mathbb{P}^1$ with markings $\{\sigma_2, \sigma_3\}$ one one component and the remaining ones on the other.

From the above examples we can easily argue how the story goes in the general case.

Proposition 3.2. *The generic point of an irreducible component of the boundary $\partial\overline{M}_{0,n}$ is a curve with two irreducible components and n-markings distributed on the two components in such a way to make the curve stable.*

Each boundary component has codimension 1 and it is smooth.

Given a partition $A \cup B$ of $\{1, 2, \ldots, n\}$ with $|A|, |B| \geq 2$, we call $D(A|B)$ the boundary component of $\overline{M}_{0,n}$ whose generic point correspond to the curve with the markings σ_a for a in A on one component, and the markings σ_b for b in B on the other component.

Sketch of proof. The fact that each of this component has codimension 1 can be seen as follows: we are free to put $|A| - 2$ markings on the first component (as we can always assume that two of them together with the node belong to the set $\{0, 1, \infty\}$).

Similarly, we are free to decide where to put $|B| - 2$ markings on the other component: hence the dimension of this locus is $|A| + |B| - 4 = n - 4$, hence it has codimesion 1 in $\overline{M}_{0,n}$.

Another way of seeing this is by means of the *gluing morphism*: this is the morphism

$$\overline{M}_{|A|+1} \times \overline{M}_{|B|+1} \longrightarrow \overline{M}_{0,n}$$

which sends a pair of points $([C, \underline{\sigma}], [C'\underline{\sigma}')$ (here $\underline{\sigma}$ denotes the collection of all markings) to the curve obtained by gluing the marking $\sigma_{|A|+1}$ of C with the marking $\sigma'_{|B|+1}$ of C', with sections given by $\sigma_1, \ldots, \sigma_{|A|}, \sigma'_1, \ldots, \sigma_{|B|+1}$ (to show that this morphism actually exists and it is well defined, one can either exploit the fact that all the schemes involved are fine moduli spaces, and construct appropriate families of curves that induce the desired morphism, or rather work directly with the moduli functors/stacks and construct a natural transformation of functors).

The gluing morphism constructed above gives an isomorphism of the domain with the boundary divisor $D(A|B)$. From this we immediately see that the a priori only-just-a-closed-subscheme $D(A|B)$ is an irreducible, smooth and codimension 1 closed subscheme. $\square$

From this we deduce that $\partial\overline{M}_{0,2n}$ has

$$\frac{1}{2}\binom{2n}{n}+\binom{2n}{n-1}+\cdots+\binom{2n}{2}$$

components, and $\partial\overline{M}_{0,2n+1}$ has

$$\binom{2n+1}{n}+\binom{2n+1}{n-1}+\cdots+\binom{2n}{2}$$

irreducible components.

Another important morphism between moduli spaces of stable marked curves is the *forgetful morphism*: this is a morphism

$$\overline{M}_{0,n}\longrightarrow\overline{M}_{0,n-m}$$

for $n-m\geq 3$ that sends a curve C with markings $\sigma_1,\ldots,\sigma_n$ to the same curve C but with markings $\sigma_1,\ldots,\sigma_{n-m}$.

It can be shown that the fibres of this morphism are reduced. In particular, the preimage of $D(A'|B')\subset\overline{M}_{0,n-m}$ in $\overline{M}_{0,n}$ is formed by the union of the boundary divisors $D(A|B)$ where $A'\subset A$ and $B'\subset B$.

The following case

$$\overline{M}_{0,n}\longrightarrow\overline{M}_{0,4}\simeq\mathbb{P}^1$$

is quite relevant because, due to the linear equivalence relation $D(A'|B')\sim D(A''|B'')$ for every partition $A'\cup B'=A''\cup B''=\{1,2,3,4\}$ (this is a simple consequence of the fact that any two points in $\mathbb{P}^1$ are linearly equivalent), we deduce that also their preimages in $\overline{M}_{0,n}$ must be linearly equivalent.

Lemma 3.3. *Let i,j,k,l be distinct elements in $\{1,2,\ldots,n\}$. Then in Mnbar we have:*

$$D(A\cup\{i,j\}|B\cup\{k,l\})\sim D(A\cup\{i,k\}|B\cup\{j,l\})$$

for any partition $A\cup B=\{1,\ldots,n\}\setminus\{i,j,k,l\}$.

We move now from $\overline{M}_{0,n}$ to $\overline{M}_{0,n}(\mathbb{P}^r,d)$. Recall that, even for $n\geq 3$, the scheme $\overline{M}_{0,n}(\mathbb{P}^r,d)$ is not smooth and it is not a fine moduli space. Nevertheless, it is still irreducible, it is normal and it has at most finite quotient singularities. It contains an open subscheme $\overline{M}_{0,n}(\mathbb{P}^r,d)^*$ which is smooth and that represents the subfunctor of *birational* stable maps. Consequently, over this scheme there exists a universal family of birational stable maps.

We will investigate the boundary of $\overline{M}_{0,n}(\mathbb{P}^r,d)$ using our accumulated knowledge of $\partial\overline{M}_{0,n}$ together with the *forget-the-map morphism*

$$F:\overline{M}_{0,n}(\mathbb{P}^r,d)\longrightarrow\overline{M}_{0,n}$$

that sends a stable map $(C\to\mathbb{P}^r)$ to the marked stable curve C^{st}: by C^{st} here we mean the canonical model of C, i.e. the image of the morphism induced by the linear series $|\omega_C(\sigma_1+\cdots+\sigma_n)|$ (in simpler terms, this is the morphism that contracts the unstable components of C [stability of the map $C\to S$ does not imply stability of C!]).

Remark 3.4. To construct the forget-the-map morphism, there are basically two ways: the first one amounts to show that locally such a morphism exists, using the local structure of $\overline{M}_{0,n}(\mathbb{P}^r,d)$. We haven't pursued the investigation of the local structure of $\overline{M}_{0,n}(\mathbb{P}^r,d)$ in these notes so far, so it does not make much sense to spend much words on this method.

The other possibility is to define a natural transformation

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)\longrightarrow\overline{\mathcal{M}}_{0,n}$$

which only remembers the family of curves out of the family of maps, after possibly stabilizing it.

Then we can compose this morphism of stacks with the morphism $\overline{\mathcal{M}}_{0,n} \to \overline{M}_{0,n}$ (which exists because $\overline{M}_{0,n}$ is a coarse moduli space). Again by definition of coarse moduli space, this morphism should factorize through $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}(\mathbb{P}^r, d)$, so that we end up with the desired morphism $\overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}$.

Lemma 3.5. *The forget-the-map morphism is flat of relative constant dimension.*

The Lemma above implies that the pullback of boundary divisors $D(A|B)$ of $\overline{M}_{0,n}$ are divisors, and by construction they will be in the boundary of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. How many irreducible components $F^{-1}D(A|B)$ has?

To answer this question, let ξ be a generic point of $F^{-1}D(A|B)$ (for those not familiar with the notion of generic point, simply think of a very small but dense open subscheme of $F^{-1}D(A|B)$). Over this there exists the generic curve $C|_\xi$ together with a map $\mu : C|_\xi \to \mathbb{P}^r$ of degree d.

The generic curve $C|_\xi$ is the curve over the generic point of $D(A|B)$, hence it has two irreducible components. Let d_i for $i = 1, 2$ be the degree of μ restricted to the i^{th} irreducible component. Then there exists an open subscheme in $F^{-1}D(A|B)$ where every point correspond to a map which has degree d_1 on one component and degree d_2 on the second one. It is immediate to check that the closure of this open subscheme cannot be the whole $F^{-1}D(A|B)$, as most of the morphisms where the degree of the map restricted to the first component is different from d_1 are not included.

Proposition 3.6. *The generic points of the irreducible components of $F^{-1}D(A|B)$ in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ correspond to morphisms $\mu : C \to \mathbb{P}^r$ of degree d where C is the generic curve of $D(A|B)$ (hence it has two irreducible components) and the restriction of μ to the components has degree respectively d_1 and d_2, with $d_1 + d_2 = d$*

The other boundary components of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ are those that surjects onto $\overline{M}_{0,n}$ via the forgetful morphism. The generic points of these divisors must correspond to morphisms $\mu : C \to \mathbb{P}^r$ where one of the two components of C is unstable, i.e. it has only one or zero markings.

For each of the two cases above, we deduce the existence of an irreducible component depending on the degree of the restrictions of the map. Be careful that the degree of the map on the unstable component must be > 0.

Proposition 3.7. *The irreducible components of $\partial\overline{M}_{0,n}(\mathbb{P}^r, d)$ are in bijection with the choices of partitions $A \cup B = \{1, \dots, n\}$ and $d_1 + d_2 = d$, with the additional condition that when $|A|$ or $|B|$ is ≤ 1, then respectively d_1 or d_2 must be > 0.*

As before, this discussion can be made more rigourous by properly exploiting the gluing morphisms

$$\overline{M}_{0,n+1}(\mathbb{P}^r, d) \times_{\mathbb{P}^r} \overline{M}_{0,m+1}(\mathbb{P}^r, e) \longrightarrow \overline{M}_{n+m}(\mathbb{P}^r, d + e)$$

As you might probably already had guessed, we will not be offering here such a detailed proof.

There is one last set of morphisms that we have to introduce before moving on to the next section, i.e. the *evaluation morphisms.* For $1 \leq i \leq n$ we have:

$$ev_i : \overline{M}_{0,n}(\mathbb{P}^r, d) \longrightarrow \mathbb{P}^r$$

which sends a stable map $(\mu : C \to \mathbb{P}^r)$ to the point $\mu(\sigma_i)$. The construction of this morphism can be obtained by constructing a natural transformation $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r$ (as usual, the latter is identified with its functor of points), which is defined just as above.

But the coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ has the property that every morphism from the stack to a scheme should factorize through it, hence we obtain the desired evaluation morphism.

Lemma 3.8. *The evaluation morhism $ev_i : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r$ is flat of relative constant dimension.*

3.2. **The Kontsevich formula.** In this section we sketch a proof for the Kontsevich formula. Recall:

Theorem 3.9. *The number N_d of degree d rational plane curves passing through $3d-1$ points in general position satisfies:*

$$N_d + \sum \binom{3d-4}{3d_A-1} d_A^2 N_{d_A} \cdot N_{d_B} \cdot d_A d_b = \sum \binom{3d-4}{3d_A-2} d_A N_{d_A} \cdot d_B N_{d_B} \cdot d_A d_B$$

where the sum is taken over the pairs (d_A, d_B) such that $d_A + d_B = d$ and both numbers are ≥ 1.

Consider the coarse moduli scheme $\overline{M} := \overline{M}_{0,3d}(\mathbb{P}^2, d)$. Fix $3d-2$ points $Q_1, \ldots, Q_{3d-2}$ in $\mathbb{P}^2$ and two distinct lines L_1 and L_2 such that the set of point formed by the Q_i together with $L_1 \cap L_2$ are in general position. Define:

$$Y := ev_1^{-1}(Q_1) \cap ev_2^{-1}(Q_2) \cap \cdots \cap ev_{3d-2}^{-1}(Q_{3d-2}) \cap ev_{3d-1}^{-1}(L_1) \cap ev_{3d}^{-1}(L_2)$$

Recall that evaluation morphisms are flat of relative constant dimension, hence the expected codimension of Y is $2 \cdot (3d-2) + 2 = 6d-2$. The next lemma tells us that this is the actual codimension of Y.

Lemma 3.10. *The scheme Y has the expected codimension, is wholly contained in $\overline{M}^*$ (the locus of maps with no non-trivial automorphisms) and intersects transversally the boundary components of $\overline{M}$.*

Let us comment on the last statement: the dimension of $\overline{M}$ is $3d-3+3(d+1)-1 = 6d-1$, from which we deduce that Y is a curve, so it makes perfect sense that we expect Y to intersect the boundary components. The Lemma assures us that it will do so transversally, hence in a finite number of points.

Consider the morphism $f : \overline{M} \to \overline{M}_{0,4}$ that forgets the map and only remembers $\sigma_1, \sigma_2, \sigma_{3d-1}$ and σ_{3d}. We have seen that this is a flat morphism too, and it implies that the (reducible) divisor $D(1,2|3d-1,3d) := f^{-1}D(\sigma_1,\sigma_2|\sigma_3,\sigma_4)$ is linearly equivalent to $D(1,3d-1|2,3d) := f^{-1}D(\sigma_1,\sigma_3|\sigma_2,\sigma_4)$. In particular we have:

$$|Y \cap D(1,2|3d-1,3d)| = |Y \cap D(1,3d-1|2,3d-4)| \tag{1}$$

By explicitly counting the intersection points on the two sides we will deduce the Kontsevich formula.

Remark 3.11. Ok, but *where* are you doing that intersection product, so to use the fact that linearly equivalent divisors intersect a curve in the same number of points? Here is a quick explanation for you: first, the variety $\overline{M}$ has only finite quotient singularities, hence it can be proved that it satisfies Poincaré duality or, equivalently, there is an intersection product in rational homology that it is dual to the cup product in cohomology (in general, there is a Poincaré duality for orbifolds).

If you prefer the language of stacks, the fact that $\overline{M}$ is the coarse moduli space for a smooth Deligne-Mumford stack implies that it inherits a product strucure on the rational Chow groups.

Whatever theory you choose, both of them satisfy the property that the intersection product in homology/Chow theory of two closed subschemes is equal to the schematic intersection whenever the latter is transversal and it is wholly supported in the smooth open locus.

We examine now all the possible intersection points of the left side of 1. From now on, the irreducible component A is the one containing $\sigma_{3d-1}, \sigma_{3d}$:

- Consider the locus where the map $\mu : C \to \mathbb{P}^r$ has degree 0 on the component containing σ_{3d-1} and σ_{3d}. Take an intersection point with Y: by construction $\mu(\sigma_{3d-1}) \in L_1$ and $\mu(\sigma_{3d}) \in L_2$, but the component get contracted! This implies that the component is contracted to $L_1 \cap L_2$, and none of the other markings can be on this component (otherwise one of the Q_i would coincide with $L_1 \cap L_2$).

  Therefore, $\mu(C)$ is a degree d rational curve passing through

  $$Q_1, \dots, Q_{3d-2}, L_1 \cap L_2$$

  On the other hand, given such a curve, there is only one stable map $\mu : C \to \mathbb{P}^2$ up to isomorphism whose image is the given curve, because the contracted component has exactly three special points.

  Henceforth, the cardinality of the intersection of Y with this divisor is N_d.

- Consider the locus where the map $\mu : C \to \mathbb{P}^2$ has degree d_A on the A component. If the number of markings distinct from σ_{3d-1} and σ_{3d} on this component is $> 3d_A - 1$, we would have that the image $\mu(C_A)$ is a degree d_A-curve passing through more than $3d_A-1$ fixed points, hence those points cannot be in general position, which contradicts our hypothesis on the set $\{Q_1, \dots, Q_{3d-2}, L_1 \cap L_2\}$.

  If we have less than $3d_A - 1$ markings distinct from $\sigma_{3d-1}, \sigma_{3d}$ on C_A, the image curve $\mu(C_B)$ passes through more than $3d_B - 1$ of the points $\{Q_1, \dots, Q_{3d-2}\}$, so again this contradicts the genericity assumption.

  When we have $3d_A - 1$ markings on C_A distinct from σ_{3d-1} and σ_{3d}, the image of $\mu(C_A)$ will be a degree d_A rational curve passing through $3d_A - 1$ points in $\{Q_3, \dots, Q_{3d-2}\}$, and this is ok, and $\mu(C_B)$ will be a degree d_B rational curve passing through $3d_B - 1$ points of $\{Q_1, \dots, Q_{3d-2}\}$.

  There are $\binom{3d-4}{3d_A-1}$ ways of choosing what markings among the spare ones should go on the A-component.

  Now we have to count how many stable maps $\mu : C \to \mathbb{P}^2$ there are such that $\mu(\sigma_i) = Q_i$ for $i \leq 3d - 1$, $\mu(\sigma_{3d-1} \in L_1$ and $\mu(\sigma_{3d}) \in L_2$, assuming that the markings on each component are now decided.

  We have to decide what are the images of the markings σ_{3d-1}, σ_{3d} and of the node. The point $\mu(\sigma_{3d-1})$ has to be in $L_1 \cap \mu(C_A)$, which by the Bezout theorem has cardinality d_A, and $\mu(\sigma_{3d})$ has to be in $L_2 \cap \mu(C_A)$, which has the same cardinality. Therefore, we have d_A^2 possibilities. The node of C is sent by μ to any of the intersection points of $\mu(C_A)$ and $\mu(C_B)$: there are $d_A d_B$ of them.

  Now we are only left with counting how many curves passes through the fixed points plus the ones that we have selected as images of σ_{3d-1}, σ_{3d} and of the node: there are exactly $N_{d_A} N_{d_B}$ of them.

We have concluded the computation of the right side of 1, so now we move to the left side:

- Take an element in the intersection $Y \cap D(\sigma_1, \sigma_{3d-1} | \sigma_2, \sigma_{3d})$ such that $d_A = 0$, where d_A is the degree of the restriction of μ to C_A: then the C_A is contracted to a point, and the fact that it is contained in Y would imply that the point Q_1 belongs to the line L_1, which contradicts the general position assumption.

We deduce that there are no points in $Y \cap D(\sigma_1, \sigma_{3d-1} | \sigma_2, \sigma_{3d})$ such that $d_A = 0$ or $d_B = 0$.

- Suppose $d_A > 0$ and let m_A be the number of markings on C_A. If $m_A > 3d_A$, then the image curve $\mu(C_A)$ is a degree d_A rational curve passing through more than $3d_A - 1$ points among the fixed ones, which contradicts the general position hypothesis.

  On the other hand, if $m_A < 3d_A$, a similar argument applied to the other component C_B gives the same conclusion. We deduce that the elements in $Y \cap D(\sigma_1, \sigma_{3d-1} | \sigma_2, \sigma_{3d})$ must have $3d_A$ markings on C_A and $3d_b$ markings on C_B.

- The number of ways of distributing $3d_A - 2$ markings among the $3d - 4$ markings (we are excluding here σ_1, σ_2, σ_{3d-1} and σ_{3d} which are already assigned to the components) is equal to $\binom{3d-4}{3d_A-2}$.

  The node can be sent to any of the intersection points of $\mu(C_A)$ with $\mu(C_B)$: there are $d_A d_B$ of them.

  The marking σ_{3d-1} can go to any of the points in $\mu(C_A) \cap L_1$, and similarly σ_{3d} can go to any of the points in $\mu(C_B) \cap L_2$: this gives $d_A d_B$ possibilities.

  Finally, once the points are choosen, there are exactly N_{d_A} possible images for $\mu(C_A)$ and N_{d_B} possible images for $\mu(C_B)$.

Putting all together, we deduce the Kontsevich formula.