Lecture 1: moduli of stable marked curves of genus 0

Disclaimer: these are *very* rough notes. Do not expect them to be exhaustive and/or detailed. These notes are intended more as a roadmap rather than a manual. Any comment is always welcome!

Guiding questions:

- What are moduli spaces?
- What are modular compactifications?
- How do we construct moduli spaces?

In this lecture, we sketch answers for the first two questions and we construct a specific moduli space, namely the moduli space of smooth/stable marked curves of genus zero.

1.1. **Introduction to moduli.** Let X be a scheme. Its functor of points is:

$$\begin{array}{rlcl} X(-): & (Sch)^{op} & \longrightarrow & (Set) \\ & S & \longmapsto & Hom(S,X) \end{array}$$

Example 1.1. We list some examples where the functor of points of a scheme is equivalent to another natural functor:

(1) $X \simeq \mathbb{A}^1 \Longrightarrow \mathbb{A}^1(-) : S \longmapsto H^0(S, \mathcal{O}_S)$.

(2) $X \simeq \mathbb{G}_m \Longrightarrow \mathbb{G}_m(-) : S \longmapsto H^0(S, \mathcal{O}_S)^*$ (invertible elements in $H^0(S, \mathcal{O}_S)$).

(3) $X \simeq GL_n \Longrightarrow GL_n(-) : S \longmapsto GL_n(H^0(S, \mathcal{O}_S))$ (invertible matrices with values in $H^0(S, \mathcal{O}_S)$).

(4) $X \simeq \mathbb{P}^n$, then we have:

$$\mathbb{P}^n(-) : S \longmapsto \left.\left\{ \begin{array}{c} L \text{ line bundles over } S \text{, defined up to torsion} \\ \text{by line bundles coming from } S \text{, together with} \\ \text{an embedding } L \hookrightarrow \mathcal{O}_S^{\oplus n+1} \text{ of vector bundles} \end{array} \right\} \middle/ \simeq \right.$$

(5) $X \simeq Gr(d,n)$, then we have:

$$Gr(d,n)(-) : S \longmapsto \left.\left\{ \begin{array}{c} F \text{ vector bundles of rank } d \text{ over } S \text{, defined} \\ \text{up to torsion by line bundles coming from } S \text{,} \\ \text{together with an embedding } F \hookrightarrow \mathcal{O}_S^{\oplus n+1} \end{array} \right\} \middle/ \simeq \right.$$

We can also go backward and, given a certain functor, we can ask ouselves if such a functor is the functor of points of some scheme.

Definition 1.2. A functor $\mathcal{F} : (Sch)^{op} \to (Set)$ is *represented* by a scheme F if there exists an equivalence of functors

$$\mathcal{F} \simeq F(-)$$

Moduli problems are stated using the language of functors: you start with some type of geometric objects you want to parametrize (e.g. closed subschemes, curves, surfaces, line bundles over a fixed base, vector bundles over a fixed base), you construct a functor that sends a scheme S to the set of families of these objects over S and you ask: is this functor representable?

When a functor $\mathcal{F}$ coming from a moduli problem is represented by a scheme F, we call F a fine moduli scheme for the moduli problem/functor $\mathcal{F}$.

Example 1.3. Here we collect some examples of moduli problems admitting a fine moduli space:

- Moduli problem of vector subspaces of fixed rank r inside an ambient vector space of dimension d. The associated fine moduli space is the grassmannian $Gr(d, n)$.

- Moduli problem of closed subschemes inside an ambient projective variety X (to correctly define what a family of closed subschemes is, you need the notion of flatness). The associated fine moduli scheme is the Hilbert scheme $Hilb_X$. You can also restrict to closed subschemes having a fixed Hilbert polynomial, so to get a finite type moduli scheme.

- Moduli problem of coherent subsheaves inside a fixed coherent sheaf $\mathcal{E}$ over a projective scheme X. The associated fine moduli space is the Quot scheme $Quot_{\mathcal{E}/X}$. As before, you can consider only subsheaves having a certain Hilbert polynomial, so to get a moduli scheme of finite type.

- Moduli problem of line bundles having degree 0. The associated fine moduli space is the Picard variety.

We introduce now the moduli problem of n-marked, genus 0 smooth curves:

$$\mathcal{M}_{0,n} : S \longmapsto \left\{ \begin{array}{c} \pi : X \to S \text{ proper and smooth morphism whose} \\ \text{geometric fibres are smooth genus 0 curves, together} \\ \text{with } n \text{ sections } \sigma_i : S \to X \text{ which do not intersect} \end{array} \right\}$$

For $n \geq 3$, set:

- $M_{0,3} := pt$.

- $M_{0,4} := \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

- $M_{0,n} := (M_{0,4})^{n-3} \setminus \cup\Delta_{ij}$, where Δ_{ij} is the diagonal in $M_{0,4} \times M_{0,4}$ embedded via the i^{th} and the j^{th} inclusion. The union is taken over all the $1 \leq i, j \leq n-3$ with $i < j$.

Then we have the following

Proposition 1.4. *For $n \geq 3$, the schemes $M_{0,n}$ are fine moduli spaces for the moduli functor $\mathcal{M}_{0,n}$.*

Sketch of proof. A key observation is the following: say that you have a moduli problem $\mathcal{F}$, a candidate fine moduli space F and a family $U \to F$ which is an element of $\mathcal{F}(F)$. Then proving that F is a fine moduli space for F is equivalent to showing that for every scheme S and every family $X \to S$ there exists a unique morphism $g : S \to F$ such that $g^*U \simeq X$. The family $U \to F$ is usually called a *universal* family.

Let $\pi : X \to S$ be a family of smooth curves of genus 0 together with n sections $\sigma_i : S \to X$. Then there exists an S-isomorphism $X \simeq \mathbb{P}^1 \times S$ such that $\sigma_n(S) \mapsto \{\infty\} \times S$, $\sigma_{n-1}(S) \mapsto \{1\} \times S$, $\sigma_{n-2}(S) \mapsto \{0\} \times S$. The remaining sections induces a morphism

$$\sigma_1 \times \cdots \times \sigma_{n-3} : S \longrightarrow M_{0,n}$$

Construct the universal family over $M_{0,n}$ by taking the product $M_{0,n} \times \mathbb{P}^1$, with sections given by the three usual sections $\{0\} \times S$, $\{1\} \times S$ and $\{\infty\} \times S$ and the diagonal sections δ_{ij} for $1 \leq i < j \leq n-3$. Then you can verify that the pullback of this family along $\sigma_1 \times \cdots \times \sigma_{n-3}$ is equal to the original family of marked curves. Uniqueness also follows easily. $\square$

1.2. **Compactification of moduli spaces.** The moduli space $M_{0,n}$ is not proper (properness is compactness for algebraic geometers who, poor bastards, have to deal with Zariski topologies). This can be seen directly by looking at the scheme, but can actually be verified *from the moduli functor itself.*

Recall the valuative criterion of properness (below, you can interpret the spectrum of a discrete valuation ring R as the analogue in algebraic geometry of a small disk, and the spectrum of the fraction field K as the analogue of the punctured disk):

Lemma 1.5. *A finite type morphism of schemes $X \to S$ is proper if and only if for every discrete valuation ring R with fraction field K, and every commutative diagram of solid arrows*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathrm{R}) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

there exists a unique dotted arrow that makes every subdiagram commute.

In particular, a scheme of finite type (that is, locally defined by a finite number of equations in a finite number of variables) is proper if and only if every $\mathrm{Spec}(K) \to X$ (think of it as a morphism from a small punctured disk into X) can be extended to a morphism $\mathrm{Spec}(R) \to X$ (in other terms, you can always extend the morphism from the punctured disk to the whole disk).

We can turn this criterion into a definition of properness. What we gain in this way is that now this definition applies also to functors.

More precisely, we interpret a morphism from (the functor of points of) $\mathrm{Spec}(K)$ to a moduli functor $\mathcal{F}$ as an element ξ in $\mathcal{F}(\mathrm{Spec}(K))$, a morphism $\mathrm{Spec}(R) \to \mathcal{F}$ as an element η in $\mathcal{F}(\mathrm{Spec}(R))$, and the commutativity of the diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "\xi"] \arrow[d, "f"'] & X \\ \mathrm{Spec}(R) \arrow[ur, "\eta"] & \end{tikzcd}$$

as the requirement $f^*\eta = \xi$. Then we have:

Definition 1.6. A functor $\mathcal{F} : (Sch)^{op} \to (Set)$ is *proper* if for every DVR R with fraction field K and every object ξ in $\mathcal{F}(\mathrm{Spec}(K))$, there exists a unique object η in $\mathcal{F}(\mathrm{Spec}(R))$ such that $f^*\eta = \xi$, where $f : \mathrm{Spec}(K) \to \mathrm{Spec}(R)$ is the inclusion of the generic point.

Remark 1.7. The definition above is ad-hoc for representable functor. Non representable functors can sometimes be represented by a *Deligne-Mumford stack*, after upgrading the functor to a pseudofunctor with values in groupoids (whatever). In this case, the definition of properness is a bit more subtle: we do not ask anymore for the existence of a lifting object η in $\mathcal{F}(\mathrm{Spec}(R))$ but rather the lifting object can exists only after passing to a ramified cover $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$.

We can now apply the definition of properness to the moduli functor $\mathcal{M}_{0,n}$. An object ξ in $\mathcal{M}_{0,n}(\mathrm{Spec}(K))$ is by definition a family of n-marked, smooth genus 0 curves over $\mathrm{Spec}(K)$ (again, think of it as a family over the punctured disk). We can identify this family with $\mathrm{Spec}(K) \times \mathbb{P}^1$. The only possible extension of the family is the trivial one $\mathrm{Spec}(R) \times \mathbb{P}^1$ (we keep using the trick of the existence of the three distinct sections to trivialize the family of smooth genus zero curves).

But now we can choose the sections so to make two of them intersect in the central fibre, from which we deduce that $\mathcal{M}_{0,n}$ is not proper.

A *modular compactification* of $M_{0,n}$ should then be a proper scheme that contains $M_{0,n}$ as a dense open subscheme and that still represents a moduli functor (in other terms, the boundary should be made up of points corresponding to degenerations, in a sense to be made precise, of n-marked, genus 0 smooth curves).

A first idea is to allow the sections to intersect, i.e. to define:

$$\widetilde{\mathcal{M}}_{0,n} : S \longmapsto \left\{ \begin{array}{c} \text{Families } X \to S \text{ of genus zero} \\ \text{curves with } n \text{ sections } \sigma_i : S \to X \\ \text{which can possibly intersect} \end{array} \right\}$$

This functor actually has $\mathcal{M}_{0,n}$ as subfunctor, but we soon realize that it is not a separated functor, i.e. does not satisfy the uniqueness part of the definition, as the next example shows.

Example 1.8. Let $R = \mathrm{Spec}(k[[t]])$, the spectrum of the ring of formal power series in t (think of it as a little disk with complex coordinate t). Its fraction field is $K = k((t))$, the field of invertible formal power series (the ones that has a non-zero term in degree 0). Consider the family $X_1^* \to \mathrm{Spec}(K)$ given by $\mathrm{Spec}(K) \times \mathbb{P}^1 \to \mathrm{Spec}(K)$ with sections $(0, 1, \infty, t)$. Let $X_2^* \to \mathrm{Spec}(K)$ be the family with sections $(0, t^{-1}, \infty, 1)$.

These two families are isomorphic, with isomorphism induced by the projective linear transformation $X \to X$, $Y \to tY$. But if we take the limits of the sections in the extended families $X_1 \to \mathrm{Spec}(R)$ and $X_2 \to \mathrm{Spec}(R)$, we end up with two non-isomorphic central fibres (non-isomorphic in the sense that there is no way to transform the limiting sections in the central fibre of X_1 into the limiting sections in the central fibre of X_2).

Therefore, the family over $\mathrm{Spec}(K)$ can be extended to a family over $\mathrm{Spec}(R)$ in at least two distinct ways.

The second idea to compactify $\mathcal{M}_{0,n}$ is still simple but more subtle. Instead of allowing the sections to coincide, we allow the smooth genus 0 curves to degenerate to singular curves.

In the example above, we can do the following: we blow up the family $X_1 \to \mathrm{Spec}(R)$ at the point 0 in the central fibre. In this way we get a family of genus 0 curves whose generic fibre is smooth and the central one is obtained by gluing two smooth genus 0 curves at one point.

We can then extend the section by taking its proper transform. We end up with a family of so called *stable n-marked, genus 0 curves.*

Given a (posibly reducible) curve, define its dual graph as the graph whose vertices correspond to irreducible components of the curve and whose edges correspond to intersection points of some irreducible components (and obviously join the vertices associated to those components).

Define a *tree of curves* as a (possibly reducible) curve whose dual graph is a tree, i.e. contains no cycles.

Definition 1.9. A stable n-marked curve of genus 0 is a tree of smooth, genus zero curves such that every irreducible component has at least three special points (a special point is either a node or a marking).

Then we can define the following functor:

$$\overline{\mathcal{M}}_{0,n} : S \longmapsto \left\{ \begin{array}{c} \text{Flat and proper morphism } \pi : X \to S \\ \text{together with } n \text{ sections } \sigma_i : S \to X \\ \text{such that the geometric fibres are} \\ \text{stable } n\text{-marked curves of genus 0} \end{array} \right\}$$

Proposition 1.10. *There exists a fine moduli space $\overline{M}_{0,n}$ for the moduli problem $\overline{\mathcal{M}}_{0,n}$.*

We do not give a proof here of this proposition, instead we sketch how to construct $\overline{M}_{0,n}$.

We start with $\overline{M}_{0,4}$: recall that $M_{0,4} \simeq \mathbb{P}^1 \setminus 0, 1, \infty$. We claim that $\overline{M}_{0,4} \simeq \mathbb{P}^1$. To prove this, we construct a universal family of stable 4-marked curves of genus zero.

Consider the trivial family $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ with four sections given by $\mathbb{P}^1 \times \{0\}$, $\mathbb{P}^1 \times \{1\}$, $\mathbb{P}^1 \times \{\infty\}$ and $\Delta_{\mathbb{P}^1}$. We can take the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the intersection points of the first three divisors with the diagonal, i.e. $\{0\} \times \{0\}$, $\{1\} \times \{1\}$ and $\{\infty\} \times \{\infty\}$.

The scheme $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$ is a family of stable, 4-marked curves of genus zero over $\mathbb{P}^1$, and it can actually be proved that it is a universal family over it, hence $\overline{M}_{0,4} := \mathbb{P}^1$ is a fine moduli space.

A nice feature of this construction is that can be applied again: we claim indeed that $\overline{M}_{0,5} \simeq \overline{U}_{0,4}$, where the latter is the universal family over $\overline{M}_{0,4}$ that we constructed above. To prove this, we can take the fiber product $\overline{U}_{0,4} \times_{\overline{M}_{0,4}} \overline{U}_{0,4}$ together with the four sections coming from the universal family $\overline{U}_{0,4} \to \overline{M}_{0,4}$, and add a new section given by the diagonal $\Delta_{\overline{U}_{0,4}}$.

After a series of blow-up, we eventually end up with a family $\overline{U}_{0,5} \to \overline{U}_{0,4}$ of stable, 5-marked curves of genus zero, that can be proved to be universal. This implies that $\overline{M}_{0,5} \simeq \overline{U}_{0,4}$.

This argument can be recursively applied, and we get that $\overline{M}_{0,n} \simeq \overline{U}_{0,n-1}$.