EXERCISES FOR THE COURSE
“ALGEBRAIC GEOMETRY” (SOSE 2023)
SESSION 7

What to do with these exercises

Try to solve them, of course. Understanding when is the right moment for leaving an exercise unfinished is part of the game. If you are stuck, either try to find a different angle to tackle the problem, or leave the exercise for the moment. You can come back later, or chew the problem during the day.

We will discuss together your efforts on Tuesday. The best thing to do would be to prepare a bunch of solutions/approaches to the problems below.

Exercises

In what follows we work over an algebraically closed field κ.

Exercise 1.

(1) Consider the rational morphism $\varphi : \mathbb{A}^{n+1} \dashrightarrow \mathbb{P}^n$ given by $(x_0, \ldots, x_n) \mapsto (x_0 : \ldots : x_n)$. Show that this morphism is regular outside of the origin but does not extend to a regular morphism on the whole $\mathbb{A}^{n+1}$.

(2) Let $\widetilde{\mathbb{A}^{n+1}}$ be the blow up at the origin of $\mathbb{A}^{n+1}$. Show that the rational morphism φ extends to a regular morphism $\widetilde{\varphi} : \widetilde{\mathbb{A}^{n+1}} \to \mathbb{P}^n$.

(3) Let $\mathbb{V}_{-1} \subset \mathbb{A}^{n+1} \times \mathbb{P}^n$ be the set of pairs $((x_0, \ldots, x_n), (y_0 : \ldots : y_n))$ such that $(x_0, \ldots, x_n)$ belongs to the unique line passing through the origin and $(y_0, \ldots, y_n)$. Show that this definition is well posed and that $\mathbb{V}_{-1}$ is a subvariety of $\mathbb{A}^{n+1} \times \mathbb{P}^n$ isomorphic to $\widetilde{\mathbb{A}^{n+1}}$.

(4) Show that the blow up at the origin of $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and that the latter is a quadric surface in $\mathbb{P}^3$.

Exercise 2. Let $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$ be the Veronese embedding defined in the previous exercises, and let Σ be the image of ν_2. We call Σ the Veronese surface.

(1) Find equations for the Veronese surface, using the exercises from the previous sessions if you like.

(2) Let $Y \subset \mathbb{P}^2$ be the Fermat cubic of equation $x_0^3 + x_1^3 + x_2^3 = 0$. Show that $\nu_2(Y)$ can be written as an intersection of nine quadrics.