Fundamentals of Statistical Modeling (VT21)

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Lab 3

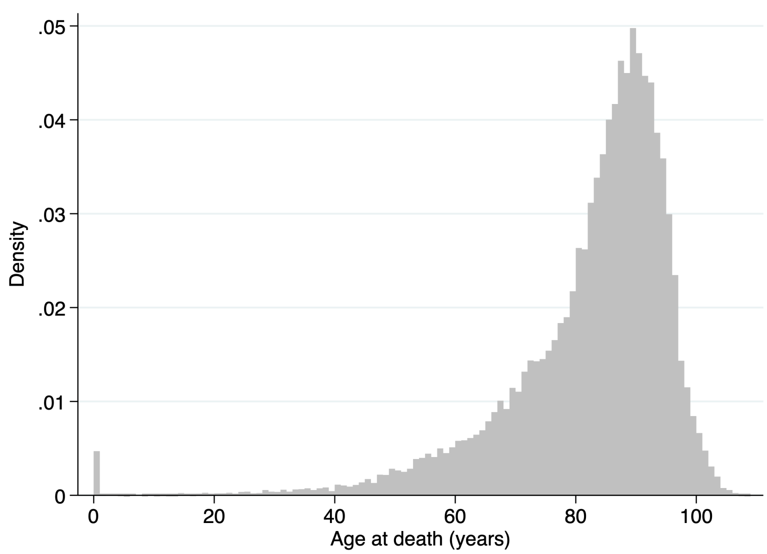
Load the dataset and the `mlci` command

```
. version 14
. use https://raw.githubusercontent.com/anddis/fsm/master/data/lab3_1.dta, clear
. run https://raw.githubusercontent.com/anddis/fsm/master/do/mlci.do
```

Exercise 1

I retrieved data on age at death among females in Switzerland in 2016 from `http://www.mortality.org` (variable `age`) ($n = 33,638$). There are no censored observations (we know the age at death for all individuals). Plot an histogram of age at death. What can we say about the distribution?

```
. hist age, width(1) name(p0, replace)
(bin=109, start=0, width=1)
. graph export p0.png, replace
(file p0.png written in PNG format)
```



Assume that $f(age)$ follows a generalized extreme values distribution. Estimate the parameters μ and σ. Constrain σ to be positive.

Remember: we're assuming that the variable age is Standard-Exponential–distributed after we apply the transform $G(y)$. The pdf of a Standard Exponential distribution is $f_{SE}(u) = \exp(-u)$.

```
. local G = "exp((age-{mu})/exp({theta}))"
. local g = "exp((age-{mu})/exp({theta}))/exp({theta})"
. local f = "exp(-`G´)*`g´"
```

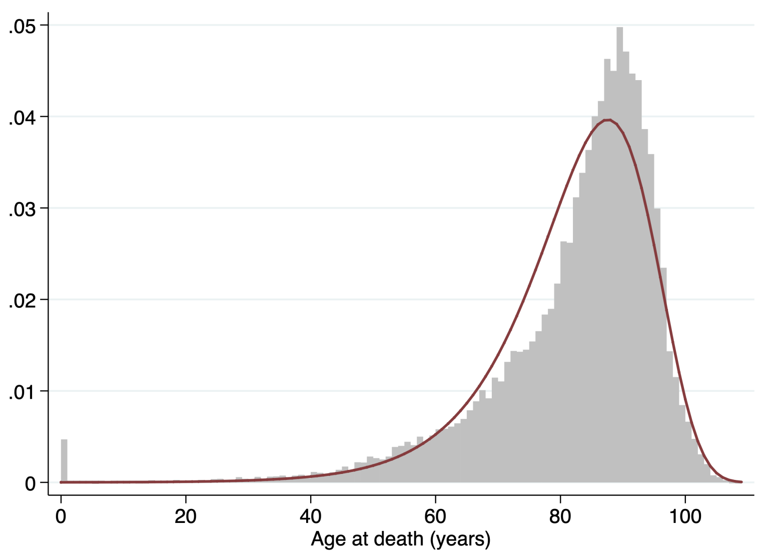
```
. mlexp(ln(`f´))
initial:       log likelihood =     -<inf>  (could not be evaluated)
feasible:      log likelihood = -370018.19
rescale:       log likelihood = -207462.71
rescale eq:    log likelihood =  -137283.6
Iteration 0:   log likelihood =  -137283.6
Iteration 1:   log likelihood = -130159.81
Iteration 2:   log likelihood = -129500.71
Iteration 3:   log likelihood = -129497.57
Iteration 4:   log likelihood = -129497.57
Maximum likelihood estimation
Log likelihood = -129497.57                     Number of obs     =     33,638
```

	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
/mu	87.58256	.052907	1655.40	0.000	87.47886	87.68625
/theta	2.227608	.0044218	503.78	0.000	2.218941	2.236275

```
. mlci exp /theta
 9.277647   95% CI: 9.197589, 9.358402
```

Plot the estimated density $\hat{f}(age)$ over the sample histogram

```
. gen fhat_age = exp(-exp((age-_b[/mu])/exp(_b[/theta])))*exp((age-_b[/mu])/exp(_b[/theta]))/exp(_b[/theta])
. tw (hist age, width(1)) (line fhat_age age, sort), name(p1, replace) legend(off)
. graph export p1.png, replace
(file p1.png written in PNG format)
```



Exercise 2

Inflate the probability of death during the first year of life ($age < 1$), while constraining it to be between 0 and 1. How do you interpret the coefficient η?

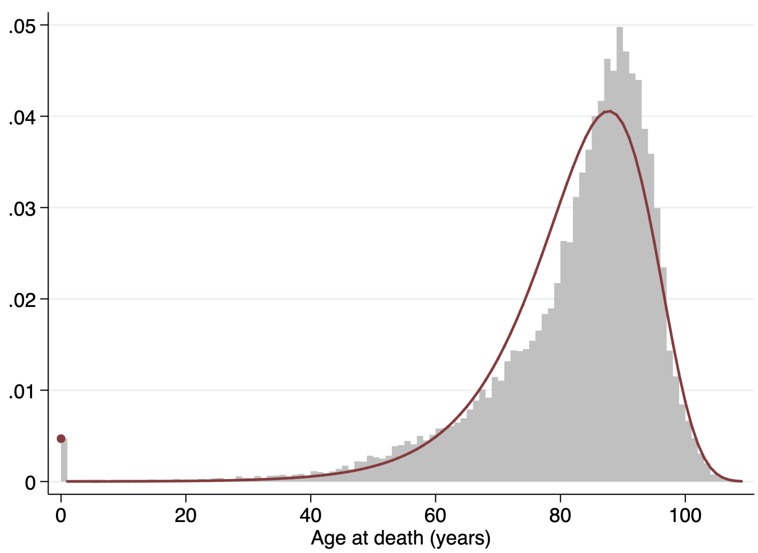
Note: we can probably improve the fit of this model by making it more flexible, for example using restricted cubic splines. This is described in the Extra material for Lab 3.

```
. local G = "exp((age-{mu})/exp({theta1}))"
. local g = "exp((age-{mu})/exp({theta1}))/exp({theta1})"
. local eta = "invlogit({theta2})"
. local f = "exp(-`G´)*`g´"
. mlexp ((age<1)*ln(`eta´) + (age>=1)*ln((1-`eta´)*`f´))
initial:       log likelihood =     -<inf>  (could not be evaluated)
feasible:      log likelihood = -703081.71
```

```
rescale:       log likelihood = -374140.51
rescale eq:    log likelihood = -136866.71
Iteration 0:   log likelihood = -136866.71
Iteration 1:   log likelihood = -129626.39
Iteration 2:   log likelihood =  -128639.2
Iteration 3:   log likelihood = -128638.98
Iteration 4:   log likelihood = -128638.98

Maximum likelihood estimation

Log likelihood = -128638.98                     Number of obs     =     33,638

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     /theta2 |  -5.356114   .0797433   -67.17   0.000    -5.512408    -5.19982
         /mu |   87.72222   .0516779  1697.48   0.000     87.62094    87.82351
     /theta1 |   2.200033   .0044046   499.49   0.000       2.1914    2.208666
------------------------------------------------------------------------------

. mlci exp /theta1
 9.025309   95% CI:  8.94773,  9.10356

. mlci invlogit /theta2
  .004697   95% CI: .0040201, .0054873
```

Plot the estimated density $\hat{f}(age)$ over the sample histogram

```
. gen fhat_age2 = invlogit(_b[/theta2])^(age<1) * ///
> ((1-invlogit(_b[/theta2]))* ///
> exp(-exp((age-_b[/mu])/exp(_b[/theta1])))*exp((age-_b[/mu])/exp(_b[/theta1]))/exp(_b[/theta1]))^(age>=1)

. tw (hist age, width(1)) (scatter fhat_age2 age if age<1, sort msiz(small) lc(maroon)) ///
> (line fhat_age2 age if age>=1, sort lc(maroon)), name(p2, replace) legend(off)

. graph export p2.png, replace
(file p2.png written in PNG format)
```

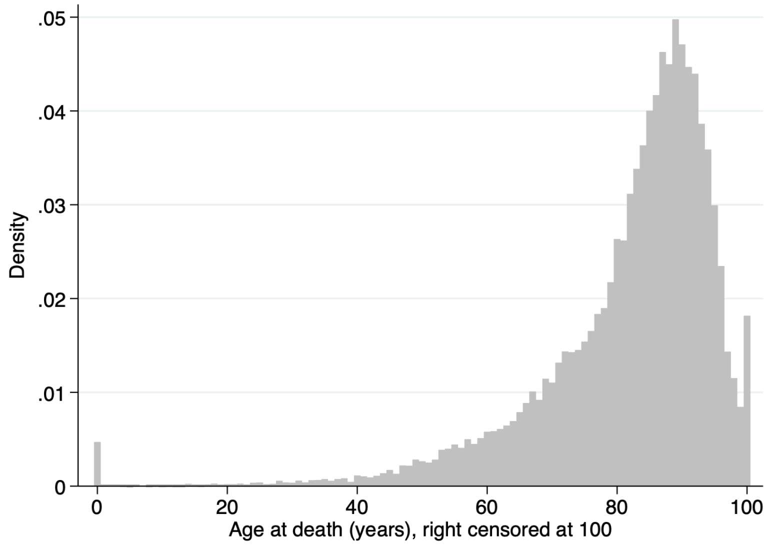


Exercise 3

Assume now that all ages above 100 years were recorded as 100 years (those ages are right-censored at 100 years) (variable `age100`).

Plot an histogram of age at death. Note the spike at $age = 100$ due to the censored observations.

```
. hist age100, discrete name(p00, replace)
(start=0, width=1)

. graph export p00.png, replace
(file p00.png written in PNG format)
```



Assume that $f(age)$ follows a generalized extreme values distribution. Estimate the parameters η, μ and σ. Constrain η to be between 0 and 1. Constrain σ to be positive. Take into account right-censoring in age-at-death. You'll need to generate an event/censoring indicator variable, first.

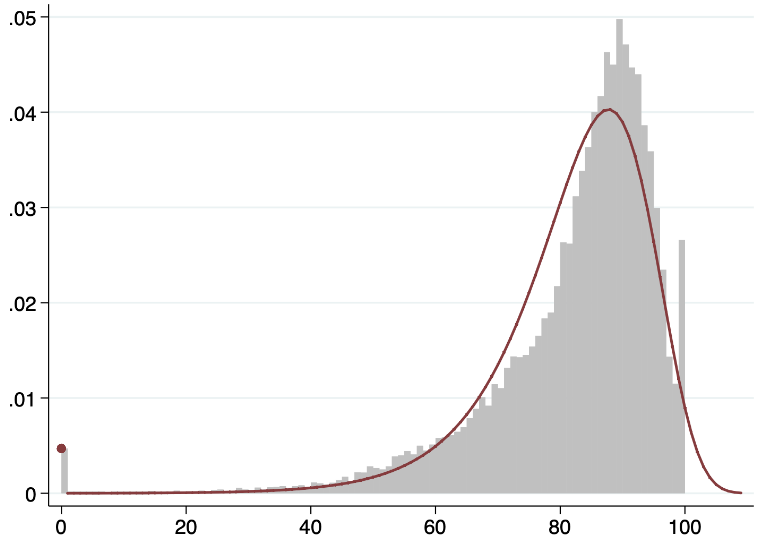
```
. gen d = (age < 100)

. local G = "exp((age100-{mu})/exp({theta1}))"

. local g = "exp((age100-{mu})/exp({theta1}))/exp({theta1})"

. local f = "exp(-`G´)*`g´"

. local S = "exp(-`G´)"

. local eta = "invlogit({theta2})"

. mlexp ((age<1)*ln(`eta´) + (age>=1)*ln((1-`eta´)*((`f´)^(d==1) * (`S´)^(d==0))))
initial:       log likelihood =     -<inf>  (could not be evaluated)
feasible:      log likelihood = -696974.21
rescale:       log likelihood = -371471.36
rescale eq:    log likelihood = -135999.31
Iteration 0:   log likelihood = -135999.31
Iteration 1:   log likelihood = -129310.49
Iteration 2:   log likelihood =  -127781.5
Iteration 3:   log likelihood = -127776.38
Iteration 4:   log likelihood = -127776.38

Maximum likelihood estimation

Log likelihood = -127776.38                     Number of obs     =     33,638
```

	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
/theta2	-5.356114	.0797433	-67.17	0.000	-5.512408	-5.19982
/mu	87.74612	.0521505	1682.56	0.000	87.6439	87.84833
/theta1	2.207031	.0045158	488.73	0.000	2.19818	2.215881

```
. mlci exp /theta1
 9.088688   95% CI:   9.0086, 9.169488

. mlci invlogit /theta2
  .004697   95% CI: .0040201, .0054873
```

Plot the estimated density $\hat{f}(age)$ over the sample histogram

```
. gen fhat_age3 = invlogit(_b[/theta2])^(age<1) * ///
> ((1-invlogit(_b[/theta2]))* ///
> exp(-exp((age-_b[/mu])/exp(_b[/theta1])))*exp((age-_b[/mu])/exp(_b[/theta1]))/exp(_b[/theta1]))^(age>=1)

. tw (hist age100, width(1)) (scatter fhat_age3 age if age<1, sort msize(small) lc(maroon)) ///
> (line fhat_age3 age if age>=1, sort lc(maroon)), name(p20, replace) legend(off)

. graph export p20.png, replace
(file p20.png written in PNG format)
```



Exercise 4

We know that age at death was actually recorded in integer years. The exact age at death is therefore unknown to us. We only know it happened between $\lfloor age \rfloor$ and $\lfloor age + 1 \rfloor$ years.

Estimate the parameters μ and σ. Constrain σ to be positive. Take into account interval-censoring and right-censoring at 100 years.

```
. gen age100_plus_1 = age100 + 1
. local Sy = "exp(-exp((age100-{mu})/exp({theta1})))"
. local Su = "exp(-exp((age100_plus_1-{mu})/exp({theta1})))"
. local eta = "invlogit({theta2})"
. mlexp ((age<1)*ln(`eta´) + (age>=1)*ln((1-`eta´)*(`Sy´-`Su´)^(d==1) * (`Sy´)^(d==0)))
initial:       log likelihood =     -<inf>  (could not be evaluated)
feasible:      log likelihood = -696974.21
rescale:       log likelihood = -371547.51
rescale eq:    log likelihood = -136937.58
Iteration 0:   log likelihood = -136937.58
Iteration 1:   log likelihood = -128566.68
Iteration 2:   log likelihood = -127661.04
Iteration 3:   log likelihood = -127655.04
Iteration 4:   log likelihood = -127655.04
Maximum likelihood estimation
Log likelihood = -127655.04                     Number of obs     =     33,638
```

	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
/theta2	-5.356118	.0797434	-67.17	0.000	-5.512412	-5.199824
/mu	88.227	.051902	1699.88	0.000	88.12528	88.32873
/theta1	2.201819	.0045281	486.25	0.000	2.192944	2.210694

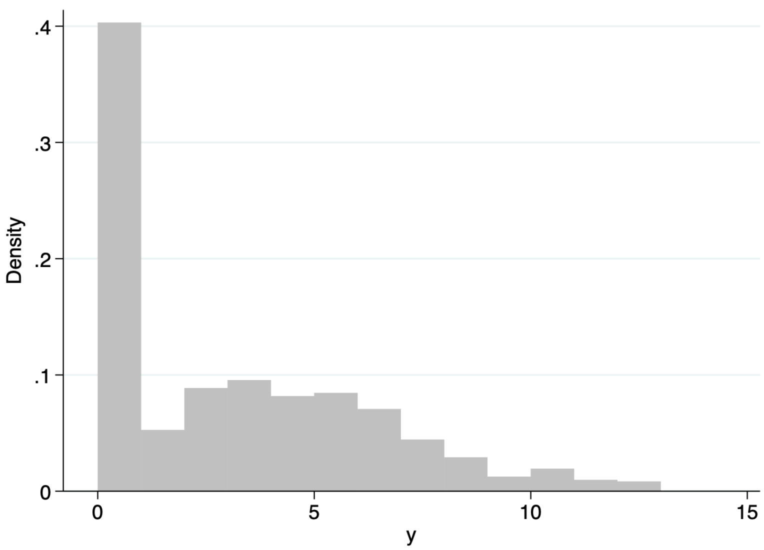
```
. mlci exp /theta1
 9.041443   95% CI: 8.961556, 9.122043
. mlci invlogit /theta2
  .004697   95% CI: .0040201, .0054873
```

Exercise 5

We measured how many times a random sample of 722 subjects were admitted to the hospital in 2016. Plot an histogram of the variable y.

```
. use https://raw.githubusercontent.com/anddis/fsm/master/data/lab3_2.dta, clear
.
```

```
. tab y

          y |      Freq.     Percent        Cum.
------------+-----------------------------------
          0 |        291       40.30       40.30
          1 |         38        5.26       45.57
          2 |         64        8.86       54.43
          3 |         69        9.56       63.99
          4 |         59        8.17       72.16
          5 |         61        8.45       80.61
          6 |         51        7.06       87.67
          7 |         32        4.43       92.11
          8 |         21        2.91       95.01
          9 |          9        1.25       96.26
         10 |         14        1.94       98.20
         11 |          7        0.97       99.17
         12 |          2        0.28       99.45
         13 |          4        0.55      100.00
------------+-----------------------------------
      Total |        722      100.00

. hist y, width(1) name(p000, replace)
(bin=13, start=0, width=1)

. graph export p000.png, replace
(file p000.png written in PNG format)
```



Assume that $f(y)$ follows a Bernoulli-Poisson Mixture model. It's similar to the Bernoulli-Negative-Binomial Mixture model, but the pmf is:

$$f_{BPM}(y) = (\beta + (1-\beta) * f_{\text{Poi}}(0))^{I(y=0)} \times ((1-\beta) * f_{\text{Poi}}(y))^{I(y>0)},$$

where $f_{\text{Poi}}(y)$ is the pmf of a Poisson distribution (`https://en.wikipedia.org/wiki/Poisson_distribution`) (see Stata's `poissonp()` function). Estimate the model's parameters. Remember to constrain the parameters to their parameter space.

```
. local beta = "invlogit({theta1})"

. local lambda = "exp({theta2})"

. local f = "(y==0)*ln(`beta´+(1-`beta´)*poissonp(`lambda´,0))+(y>0)*ln((1-`beta´)*poissonp(`lambda´,y))"

. mlexp (`f´)

initial:       log likelihood = -2898.2492
alternative:   log likelihood =  -2301.236
rescale:       log likelihood = -1893.7307
rescale eq:    log likelihood = -1701.2662
Iteration 0:   log likelihood = -1701.2662
Iteration 1:   log likelihood = -1533.4708
Iteration 2:   log likelihood = -1492.1875
Iteration 3:   log likelihood =  -1492.173
```

```
Iteration 4:   log likelihood =  -1492.173
Maximum likelihood estimation
Log likelihood =  -1492.173                    Number of obs     =        722
```

	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
/theta1	-.419094	.0771304	-5.43	0.000	-.5702669	-.2679211
/theta2	1.517249	.0230008	65.97	0.000	1.472168	1.56233

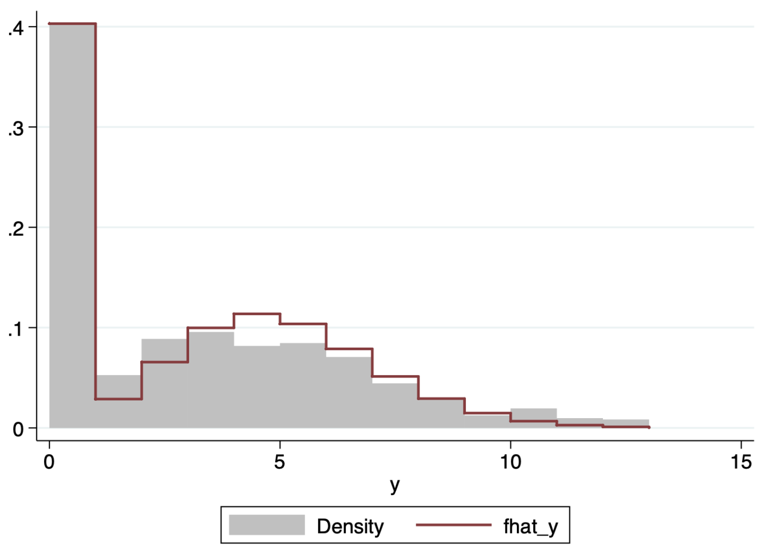
```
. mlci invlogit /theta1
 .3967336   95% CI: .3611752, .4334175
. mlci exp /theta2
 4.559664   95% CI: 4.358676, 4.769921
```

What's the probability that $Y = 0$ (0 hospital admissions) according to the model? And that $Y = 4$?

```
. di invlogit(_b[/theta1])+(1-invlogit(_b[/theta1]))*poissonp(exp(_b[/theta2]),0)
.40304709
. di (1-invlogit(_b[/theta1]))*poissonp(exp(_b[/theta2]),4)
.11370835
```

Plot the estimated density $\hat{f}(y)$ over the sample histogram

```
. gen fhat_y = exp((y==0)*ln(invlogit(_b[/theta1])+(1-invlogit(_b[/theta1]))*poissonp(exp(_b[/theta2]),0))+ ///
> (y>0)*ln((1-invlogit(_b[/theta1]))*poissonp(exp(_b[/theta2]),y)))
. tw (hist y, width(1)) (line fhat_y y, sort connect(J)), name(p3, replace)
. graph export p3.png, replace
(file p3.png written in PNG format)
```



Exercise 6

We consider Y the interval-censored version of a latent (unobserved) variable Y^*. Assume that Y^* follows a gamma distribution. Estimate its parameters. Again, inflate the probability that $Y = 0$.

```
. local beta = "invlogit({theta1})"
. local a = "exp({theta2})"
. local b = "exp({theta3})"
. local f = "(y==0)*ln(`beta´+(1-`beta´)*gammap(`a´,1/`b´))+(y>0)*ln((1-`beta´)*(gammap(`a´,(y+1)/`b´)-gammap(`a´,y/`b´)
> ))"
. mlexp (`f´)
initial:       log likelihood = -2541.5861
alternative:   log likelihood =  -1748.638
rescale:       log likelihood = -1726.2325
rescale eq:    log likelihood = -1499.4998
```

```
Iteration 0:   log likelihood = -1499.4998
Iteration 1:   log likelihood = -1485.5559
Iteration 2:   log likelihood = -1469.2134
Iteration 3:   log likelihood = -1468.9548
Iteration 4:   log likelihood = -1468.9545
Iteration 5:   log likelihood = -1468.9545
Maximum likelihood estimation
Log likelihood = -1468.9545                     Number of obs     =        722
```

	Coef.	Std. Err.	z	P>|z|	[95% Conf. Interval]	
/theta1	-.4280537	.0781964	-5.47	0.000	-.5813159	-.2747915
/theta2	1.254735	.0814365	15.41	0.000	1.095122	1.414347
/theta3	.363448	.0819923	4.43	0.000	.2027462	.5241499

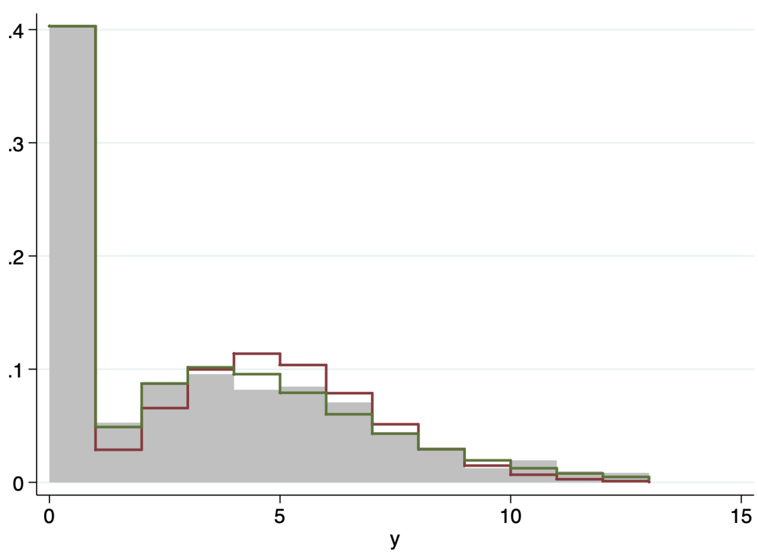
```
. mlci invlogit /theta1
 .3945912   95% CI: .3586299, .4317312
. mlci exp /theta2
 3.506908   95% CI: 2.989548,   4.1138
. mlci exp /theta3
  1.43828   95% CI: 1.224762, 1.689022
```

What's the probability that $Y = 0$ (0 hospital admissions) according to the model? And that $Y = 4$?

```
. di invlogit(_b[/theta1])+(1-invlogit(_b[/theta1]))*gammap(exp(_b[/theta2]),1/exp(_b[/theta3]))
.40304708
. di (1-invlogit(_b[/theta1]))*(gammap(exp(_b[/theta2]),5/exp(_b[/theta3]))-gammap(exp(_b[/theta2]),4/exp(_b[/theta3])))
.09560065
```

Plot the estimated density $\hat{f}(y)$ over the sample histogram

```
. gen fhat_y2 = exp((y==0)*ln(invlogit(_b[/theta1])+(1-invlogit(_b[/theta1]))*gammap(exp(_b[/theta2]),1/exp(_b[/theta3])
> ))+ ///
> (y>0)*ln((1-invlogit(_b[/theta1]))*(gammap(exp(_b[/theta2]),(y+1)/exp(_b[/theta3]))-gammap(exp(_b[/theta2]),y/exp(_b[/
> theta3])))))
. tw (hist y, width(1)) (line fhat_y fhat_y2 y, sort connect(J J)), name(p4, replace) legend(off)
. graph export p4.png, replace
(file p4.png written in PNG format)
```



Which model seems to fit better the data? Tabulate the observed and model-based predicted proportions.

```
. gen N = _N // This line and the following 2 are used to compute the observed (empirical) proportions
. bysort y: gen n = _N
. gen obs_p = n / N
. tabstat obs_p fhat_y fhat_y2, by(y) nototal format(%4.3f)
```

```
Summary statistics: mean
  by categories of: y
```

y	obs_p	fhat_y	fhat_y2
0	0.403	0.403	0.403
1	0.053	0.029	0.049
2	0.089	0.066	0.087
3	0.096	0.100	0.102
4	0.082	0.114	0.096
5	0.084	0.104	0.079
6	0.071	0.079	0.060
7	0.044	0.051	0.043
8	0.029	0.029	0.029
9	0.012	0.015	0.019
10	0.019	0.007	0.012
11	0.010	0.003	0.008
12	0.003	0.001	0.005
13	0.006	0.000	0.003

Extra

Let's refit the model in Exercise 3, but this time we use the optimization function `optimize()` (which is the function that `mlexp` calls behind the curtains). `optimize()` is part of Mata, Stata's matrix programming language.

```
. use https://raw.githubusercontent.com/anddis/fsm/master/data/lab3_1.dta, clear

. gen d = (age < 100)

.
. mata
------------------------------------------------- mata (type end to exit) ------------------------------------------------
: mata clear

:
: X = st_data(., ("age", "age100", "d" ))

:
: void model3(todo, beta, ll, S, H) {
> mu = beta[1]
> sigma = exp(beta[2])
> eta = invlogit(beta[3])
>
> external X
> age = X[., 1]
> age100 = X[., 2]
> d = X[., 3]
>
> G = exp((age100 :- mu) :/ sigma)
> g = exp((age100 :- mu) :/ sigma) :/ sigma
> f = exp(-G) :* g
> S = exp(-G)
>
> ll = colsum((age:<1) :* ln(eta) :+ (age:>=1) :* ln((1:-eta) :* ((f):^(d:==1) :* (S):^(d:==0))))
> }
note: argument todo unused
note: argument H unused

:
: S = optimize_init()

: optimize_init_evaluator(S, &model3())

: optimize_init_params(S, (100, log(10), logit(.5)))

: b = optimize(S)
Iteration 0:   f(p) = -168058.62  (not concave)
Iteration 1:   f(p) = -144679.84  (not concave)
Iteration 2:   f(p) = -134653.55  (not concave)
Iteration 3:   f(p) = -130907.79
Iteration 4:   f(p) =  -127911.4
Iteration 5:   f(p) = -127779.65
Iteration 6:   f(p) = -127776.38
Iteration 7:   f(p) = -127776.38

: se = sqrt(diagonal(invsym(-optimize_result_Hessian(S))))

:
: b´, se
                 1              2
```

```
  1 | 87.74611605    .0521504553
  2 | 2.207030536    .0045158228
  3 | -5.356118786   .0797434067

: end
```