

# Bayesian Estimation of the Spectral Density

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## 1 Motivation

To estimate the spectral density of a stationary time series using Bayesian methods using the asymptotic distributional properties of a periodogram. The motivation behind this work is the following model.

$$\begin{aligned} I_n(\omega_r) | f(\omega_r) &\sim \text{indep Exp}(2\pi f(\omega_r)) \\ f(\omega_r) &\sim \pi(\theta) \\ f(\omega_r) | I_n(\omega_r) &\propto 2\pi f(\omega_r) e^{-2\pi f(\omega_r)y} \pi(\theta) \end{aligned}$$

The periodogram values at  $\omega_r$  are only asymptotically distributed as independent exponentials, so to construct the Bayesian model it is essential that the asymptotic behavior holds. This research uses simulations to [validate](#) two results by Lahiri. [4] ([add if do Bayes, unlikely](#))

## 2 Background and Problem Statement

Two results drive the first part of our work.

**Result 1.** *Let  $f(\omega_r) \neq 0, 1 \leq r \leq k$  where  $f$  denotes the spectral density of a stationary time series  $\{X_t\}$ . Then when  $n \rightarrow \infty$  the joint distribution of the periodogram at  $\omega_r$ ,  $I_n(\omega_r)$ , tends to that of  $k$  mutually independent random variables distributed as  $\text{Exponential}(2\pi f(\omega_r))$  for  $0 < \omega_r < \pi$  [1].*

Result 1 describes the asymptotic distribution of a periodogram at fixed frequencies. In practice, this result has often been used with the Fourier frequencies in place of fixed frequencies [reference this](#)  $\{2\pi j/n : j = 1, \dots, n\}$ . However, it has been shown that the asymptotic behavior of the periodogram at the Fourier frequencies does not hold as  $n \rightarrow \infty$  ([dig into reference to flesh this out](#)). Lahiri [4] derived two results that describe behavior required for a set of frequencies to satisfy Result 1

**Result 2.** *In the absense of data tapering,*

- (a) *the periodogram values at asymptotically distant ordinates ( $I_n(\omega_r)$ ) are asymptotically independent. Asymptotically distant ordinates  $\{\omega_{l_n}\}, \{\omega_{k_n}\}$  satisfy  $|n(\omega_{l_n} - \omega_{k_n})| \rightarrow \infty$  as  $n \rightarrow \infty$ .*
- (b) *the periodogram values at asymptotically close ordinates which are asymptotically distant from the sequence  $\{0\}$  are asymptotically independent. Asymptotically close ordinates  $\{\omega_{l_n}\}, \{\omega_{k_n}\}$  satisfy  $|n(\omega_{l_n} - \omega_{k_n})| \rightarrow 2\pi l$  for some nonzero integer  $l$  as  $n \rightarrow \infty$  [4].*

Result 2 suggests that for a subset of sufficiently spaced and distant from 0 frequencies, the periodogram ordinates at these frequencies should be asymptotically independent. [We explore these two results to determine if](#), for large  $n$ ,

1. Result 1 will not hold at the Fourier frequencies, and
2. if (1) is true, a subset of Fourier frequencies that are sufficiently spaced and far from 0 can be constructed such that Result 1 holds.

### 3 Models

#### [WHAT ARE WE DOING?!?](#)

We used three separate models to explore the asymptotic behavior of the periodogram. The included models are *a)* IID Gaussian(0, 1); *b)* AR(1) with  $\phi = 0.5$ ; and *c)* AR(4) with  $\phi = [0.08, 0.33, 0.1, 0.45]$ .

#### **IID Gaussian**

The following result about the IID Gaussian model gives us knowledge of the exact, rather than asymptotic, behavior of the periodogram at any frequency.

**Result 3.** *For  $\{X_t\} \stackrel{IID}{\sim} N(0, \sigma^2)$ , the periodogram values  $\{I_n(\omega_j) : \omega_j \in \mathcal{F}_n, \omega_j \notin \{0, \pi\}\}$  are IID Exponential( $\sigma^2$ ) random variables [1].*

We use this model as a sanity check on our testing procedure. Should our procedure show failure with the IID Gaussian model, then this is an indication of an issue with the procedure, rather than the frequency spacings.

#### **ARMA(p,q)**

We used exclusively ARMA models as our experimental models because the spectral density of an ARMA model has a known and closed form. An ARMA model is a process  $\{X_t\}$  with the form  $\phi(B)X_t = \theta(B)Z_t$  where  $B$  is the backshift operator and  $\{Z_t\}$  is WN(0,  $\sigma^2$ ). Models of this form have spectral density

$$f_X(\omega) = \frac{\sigma^2}{2\pi} \frac{|\theta(e^{-i\omega})|^2}{|\phi(e^{-i\omega})|^2}, \omega \in [-\pi, \pi] \setminus \{0, \pi\}.$$

Without the exact knowledge of the spectral density function, we would not be able to test the distributions of the periodogram at each frequencies. [mention moving window?](#)

### 4 Procedures

[Explain two tests for independence and exponentials, pairwise tests even though not necessarily joint independence](#) We implement the following testing procedure and repeat  $s$  simulations of the tests to determine if we see similar rates of failures as the Type-I errors of our tests ( $\alpha$ ).

1. Simulate  $M$  draws from  $X_1, \dots, X_n$  where  $\{X_t\}$  is a stationary time series from one of three models.
2. Obtain  $M$  periodograms using the fourier frequencies from  $(0, \pi)$ ,  $\omega_j = \frac{2\pi j}{n} : j = 1, \dots, \lfloor n/2 \rfloor$ .

3. Obtain  $\lfloor \frac{n-1}{2} \rfloor$  values of the spectral density at each fourier frequency  $f(\omega_j)$ . These will be known by design.
4. Simulate  $M$  draws from the  $\lfloor \frac{n-1}{2} \rfloor$  different distributions  $\text{Exp}(2\pi f(\omega_r))$
5. Test the distribution of the  $M$  values of periodograms at each fourier frequency separately. Store the number of failed tests at each frequency.
6. Test the pairwise independence of each neighboring  $M$  values of periodograms at each fourier frequency. Store the number of failed tests at each frequency.
7. Inspect the results from steps 5 and 6 to determine if different spacings of frequencies are needed. If so, use sparcer frequencies and repeat from step 2 at the chosen frequencies rather than fourier frequencies.

## 4.1 Tests

### Spearman's Rank Correlation

**Kolmogorov-Smirnov Test of Distribution** The Kolmogorov-Smirnov (KS) test is a procedure that uses the maximum vertical distance between an empirical cumulative distribution function and a named cumulative distribution function as a measure of how much the two functions resemble each other. The only assumption of this test is that the data are a random sample. The KS test uses test statistic  $T = \sup_x |F^*(x) - S(x)|$  to test the null hypothesis  $H_0 : F(x) = F^*(x)$  for all  $x$ , where  $F^*(x)$  is the completely specified hypothesized distribution function,  $F(x)$  is the unknown distribution function of the data, and  $S(x)$  is the empirical distribution function of the data. For data of length  $n \leq 100$ , an exact p-values are available and for data of length  $n > 100$ , an asymptotic approximation is used [2]. The asymptotic distribution is called the Kolmogorov distribution, and is of the form  $P(T \leq \lambda) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2 \lambda^2}$  [3].

We used a two-sided KS test at the  $\alpha = 0.05$ -level to test whether the distributions of our periodogram values at each frequency  $\omega_j$  were exponentiall distributed with mean parameter  $2\pi f(\omega_j)$ , where  $f(\omega_j)$  is the spectral density evaluated at  $\omega_j$ .

## 5 Results

### 5.1 IID Gaussian

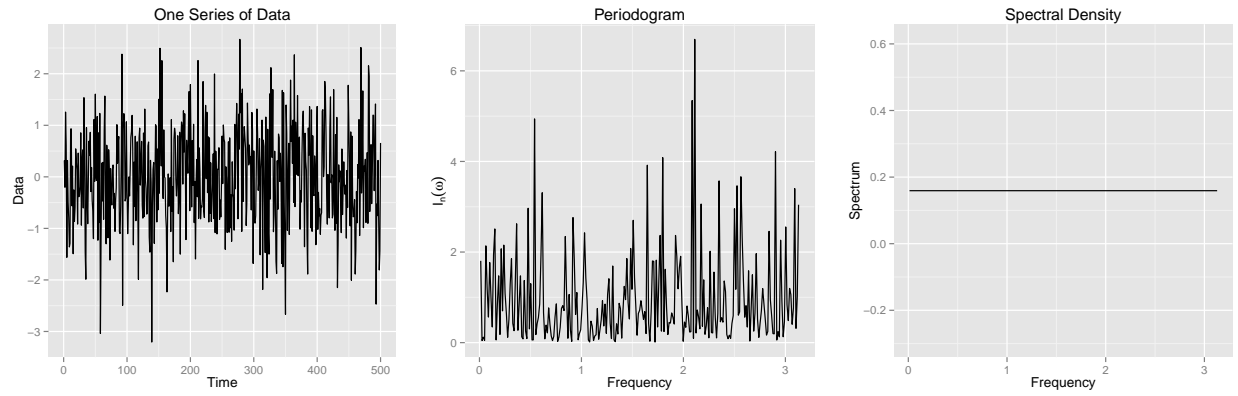


Figure 1: One draw, periodogram, and spectral density of a Gaussian IID Model.

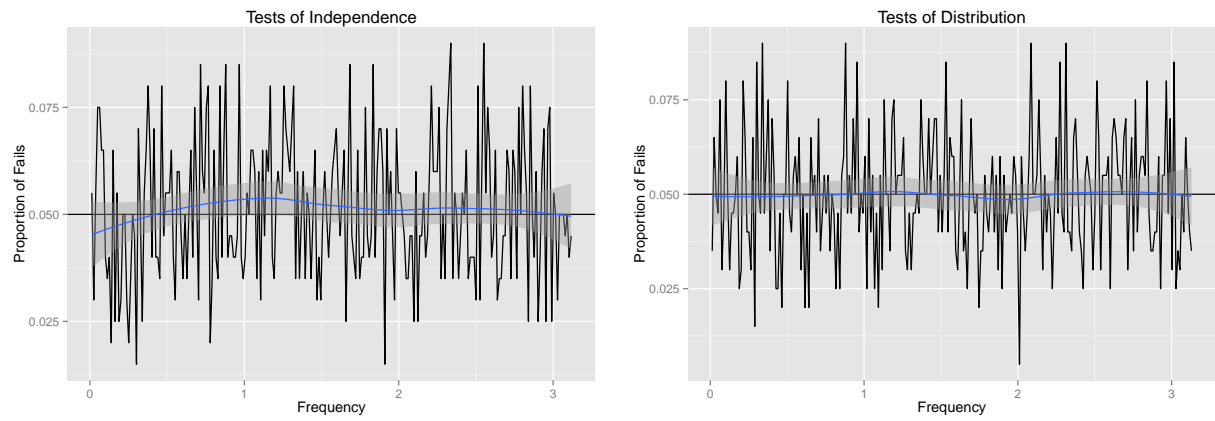


Figure 2: Tests of independence and distribution for a Gaussian IID Model.

## 5.2 AR(1)

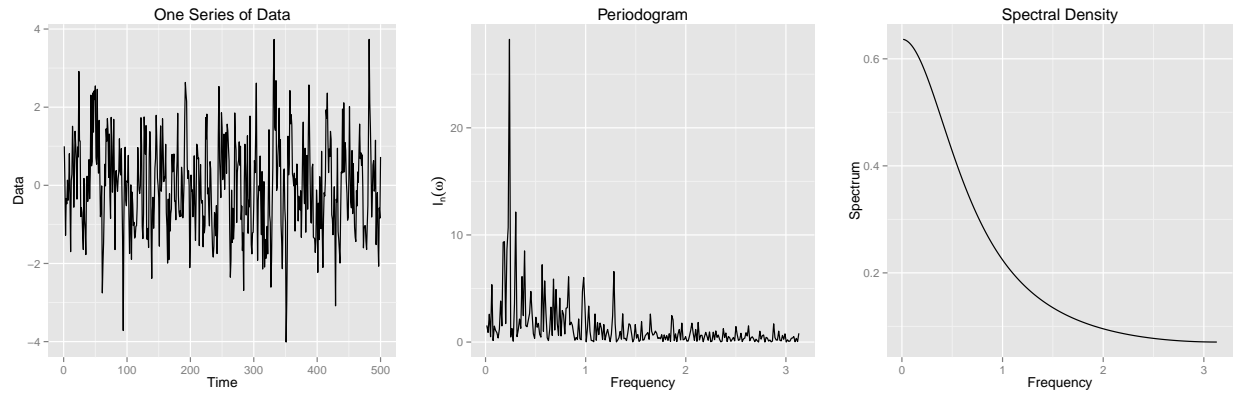


Figure 3: One draw, periodogram, and spectral density of an AR(1) model.

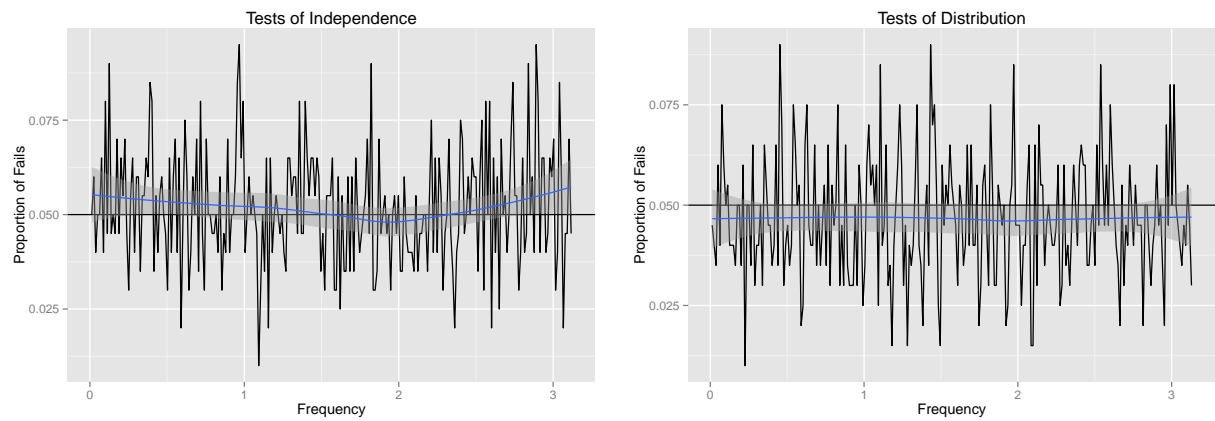


Figure 4: Tests of independence and distribution for an AR(1) model.

### 5.3 AR(4)

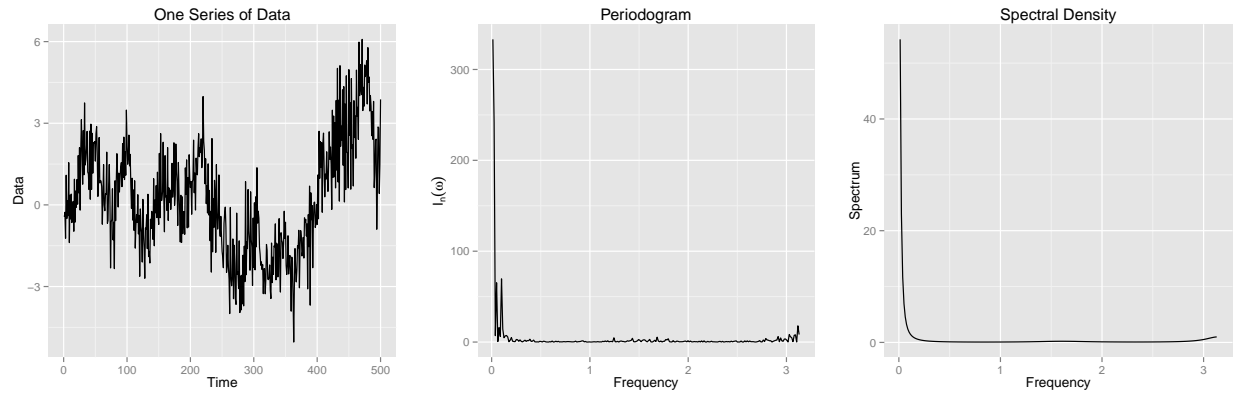


Figure 5: One draw, periodogram, and spectral density of an AR(4) model.

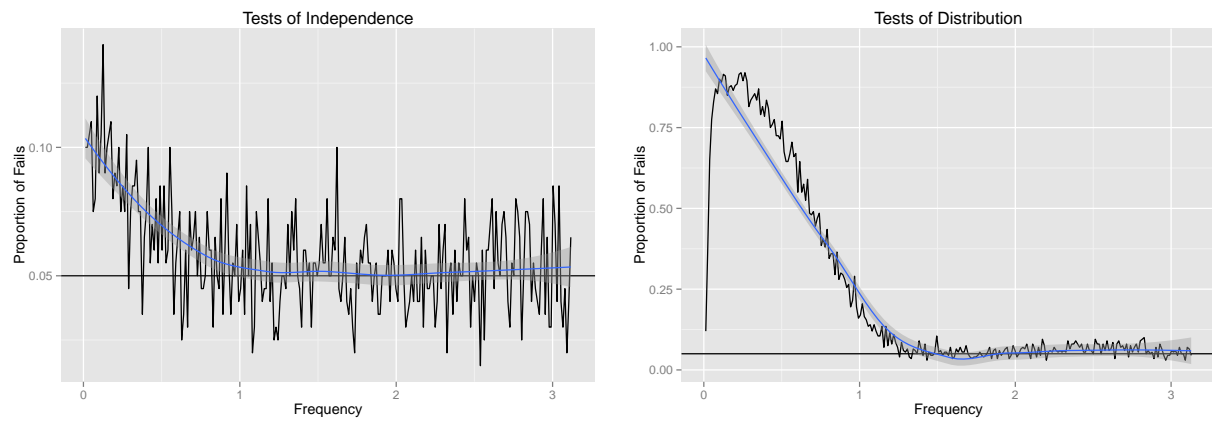


Figure 6: Tests of independence and distribution for an AR(4) model.

## 5.4 MA(1)

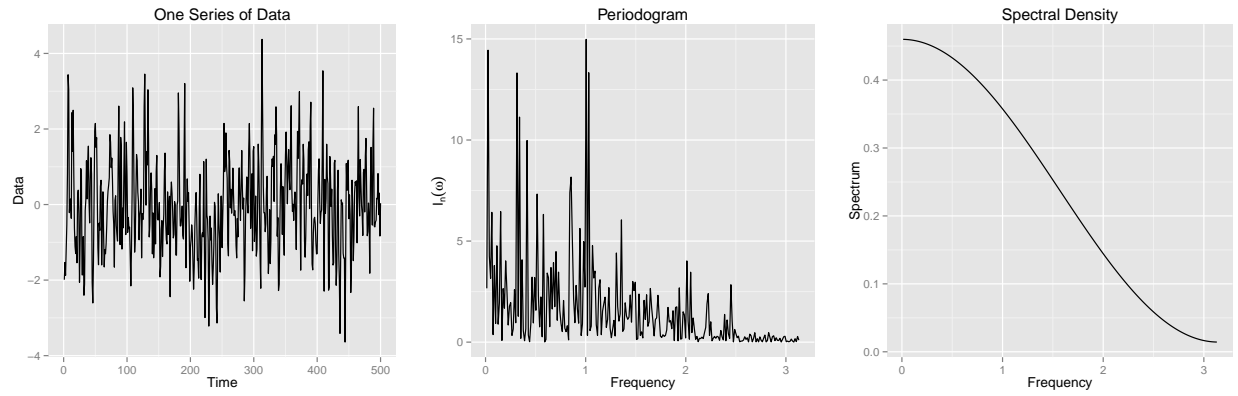


Figure 7: One draw, periodogram, and spectral density of an MA(1) model.

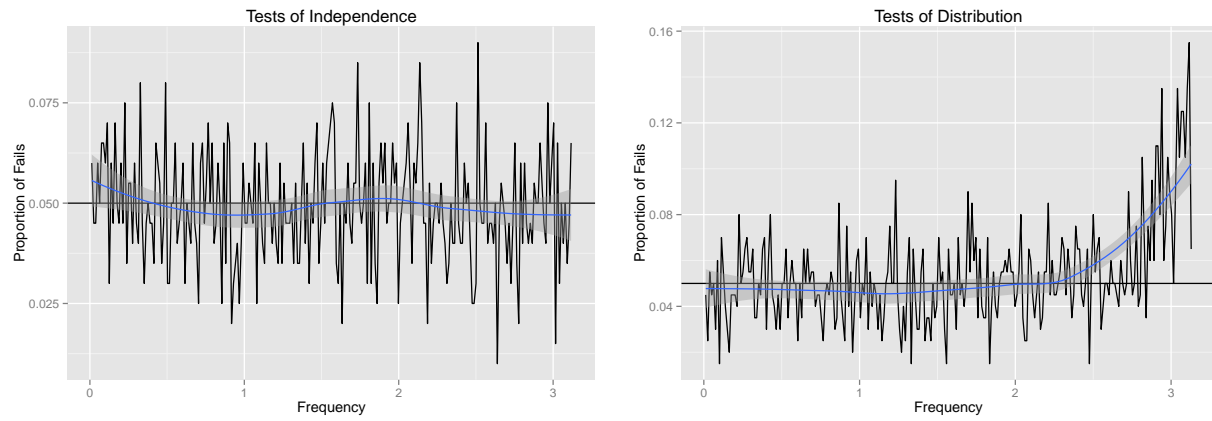


Figure 8: Tests of independence and distribution for an MA(1) model.

## 5.5 MA(2)

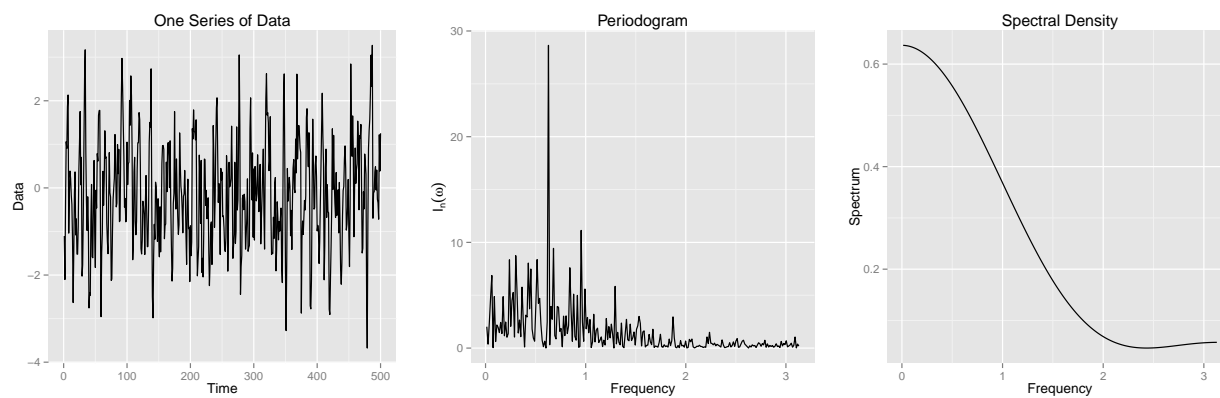


Figure 9: One draw, periodogram, and spectral density of an MA(2) model.

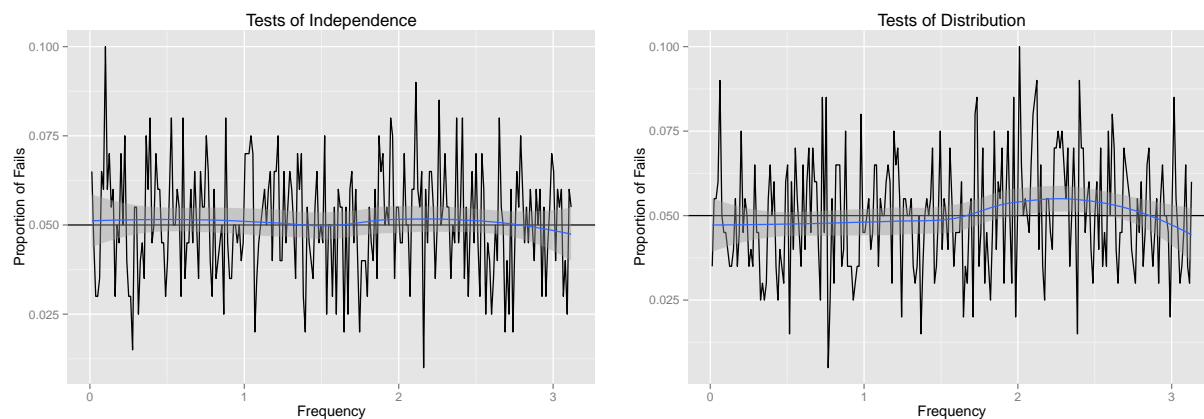


Figure 10: Tests of independence and distribution for an MA(2) model.

## 5.6 ARMA(4,1)



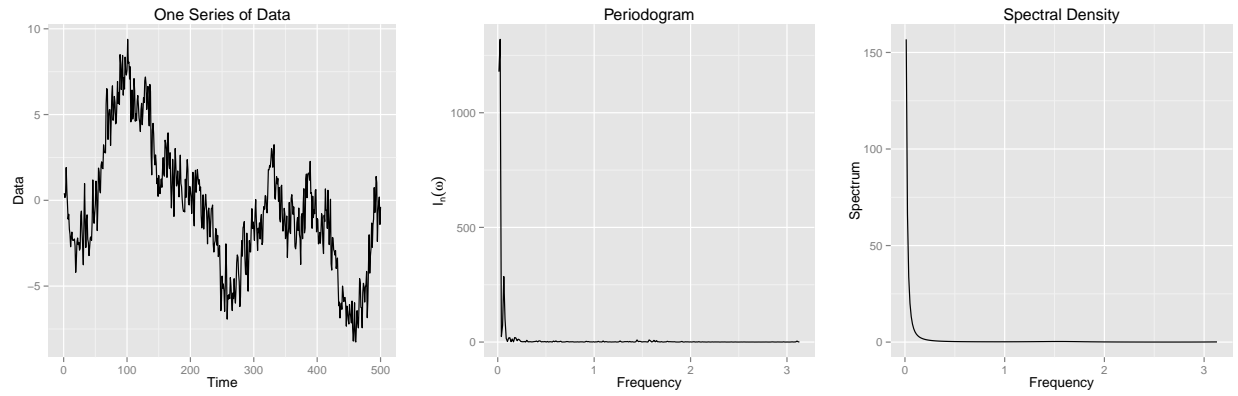


Figure 11: One draw, periodogram, and spectral density of an ARMA(4,1) model.

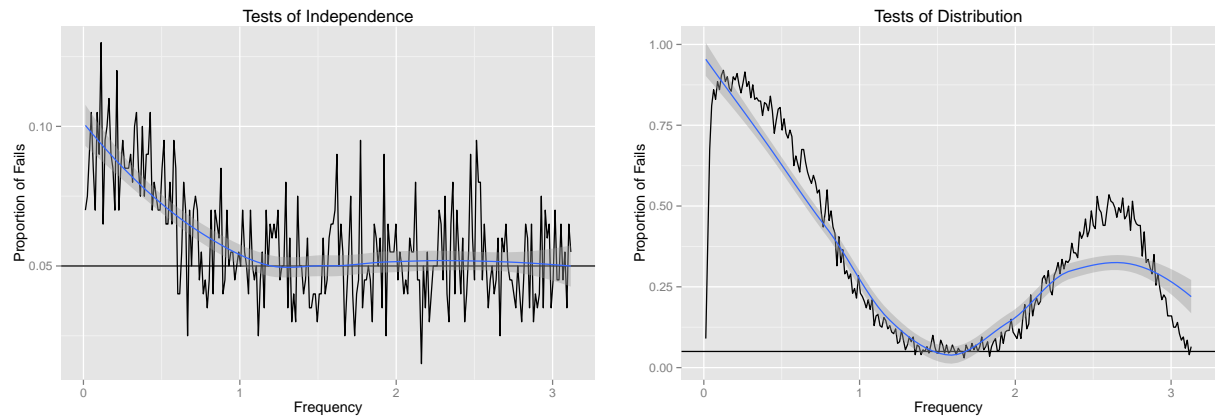


Figure 12: Tests of independence and distribution for an ARMA(4,1) model.

## 5.7 ARMA(4,2)

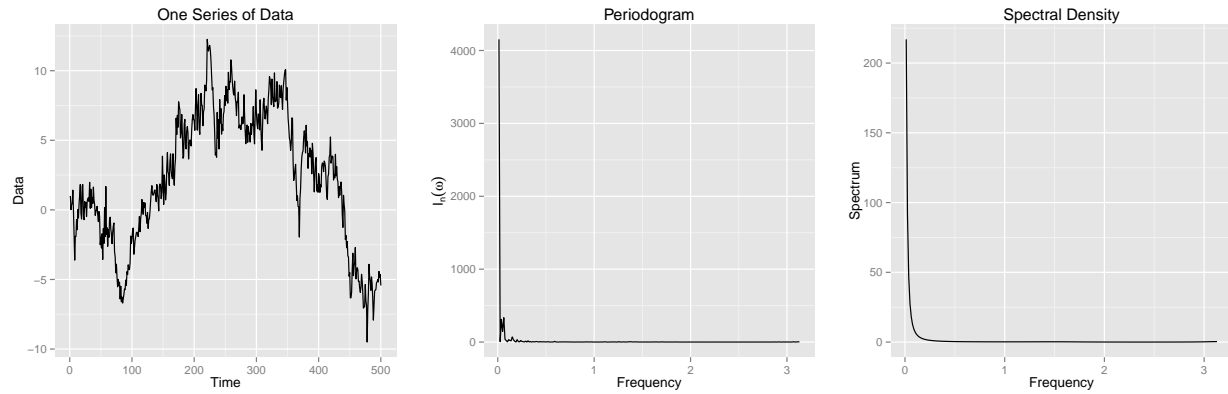


Figure 13: One draw, periodogram, and spectral density of an ARMA(4,2) model.

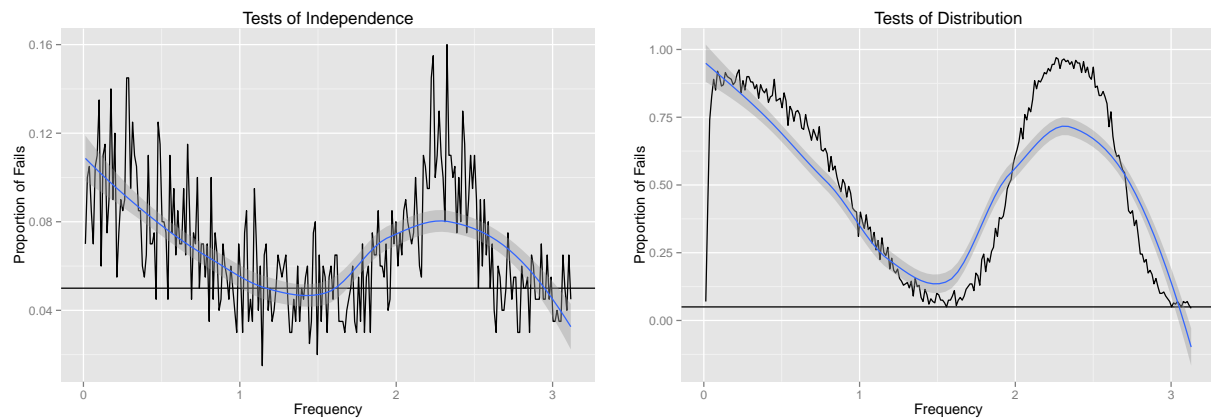


Figure 14: Tests of independence and distribution for an ARMA(4,2) model.

## 6 Discussion

### Issues

1. Multiple comparison
2. ARMA(4,2), MA(1), MA(2)
3. Exponential Fails
4. Usability
  - ARMA useless, Models w/more structure
  - spacing depends on model
  - can never really be used in practice
5. Computation time

## 6. Pairwise independence $\neq$ Joint independence

### References

- [1] Peter J Brockwell and Richard A Davis. *Introduction to time series and forecasting*. Taylor & Francis US, 2002.
- [2] William Jay Conover. “Practical nonparametric statistics”. In: (1998).
- [3] A Kolmogorov. “On the empirical determination of a distribution function”. In: *Breakthroughs in statistics*. Springer, 1992, pp. 106–113.
- [4] SN Lahiri. “A necessary and sufficient condition for asymptotic independence of discrete Fourier transforms under short-and long-range dependence”. In: *The Annals of Statistics* 31.2 (2003), pp. 613–641.