Bayesian Estimation of the Spectral Density

Gaussian IID Case

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Purpose

To check our testing procedure on a model with exact results. This will ensure that our procedure is reasonable and that when we see a periodogram at sparse frequencies does not behave as independent Exponential distribution, that we can trust this result.

Theoretical Result

Result. *For* $\{X_t\} \overset{IID}{\sim} N(0, \sigma^2)$*, the periodogram values* $\{I_n(\omega_j) : \omega_j \in \mathcal{F}_n, \omega_j \notin \{0, \pi\}\}$ *are IID Exponential(*σ^2*) random variables.*

Testing Procedure

1. Simulate M draws from $X_1, \dots, X_n$ where $\{X_t\} \overset{\text{IID}}{\sim} \text{N}(0, \sigma^2)$
2. Obtain M periodograms using the fourier frequencies from $(0, \pi)$, $\omega_j = \frac{2\pi j}{n} : j = 1, \dots, \lfloor n/2 \rfloor$
3. Simulate $M \times n$ draws from $\text{Exp}(\sigma^2)$
4. Multiply periodograms across frequencies to obtain M periodogram products.
5. Multiply exponential draws across n to obtain M draws from the product of exponential distributions (joint distribution)
6. Compare product data using a Kolmogorov-Smirnov test and examining a quantile-quantile plot

We anticipate that the KS test with fail to reject the null hypothesis, giving no indication that the samples are drawn from different distributions. Similarly, we expect that the quantile-quantile plot comparing the periodogram to exponentials should indicate a good fit.

Results

Using $M = 1000, n = 500, \sigma^2 = 1$, the K-S test statistic and p-value are:

```
##
##  Two-sample Kolmogorov-Smirnov test
##
## data:  perio.prod and exp.prod
## D = 0.03, p-value = 0.7591
## alternative hypothesis: two-sided
```

From these results we fail to reject H_0, which states that the two samples are from the same distribution. Thus there is no evidence that our periodogram values at the fourier frequencies are not independent Exponential(σ^2) random variables. This is consistent with our expectations. Let us also take a look at the quantile-quantile plot.

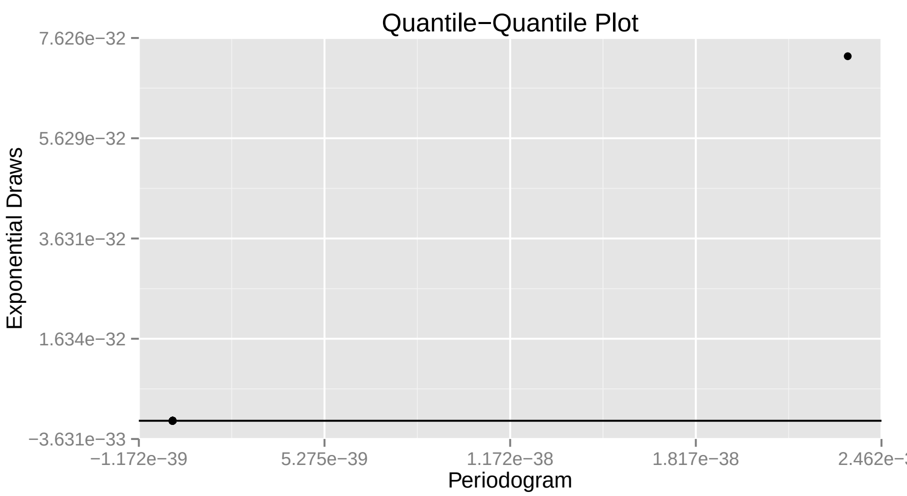

Figure 1: Quantile-quantile plot of the product of periodogram values at the Fourier frequencies versus the product of Exp(1) draws.

Clearly, this does not show a good fit between our exponential draws and the periodograms of simulated IID Gaussian data. All of our samples are actually very close to zero.

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Periodogram	1.5500E-95	5.7500E-69	4.0800E-63	2.9300E-41	2.5800E-57	2.3400E-38
Exponential	2.3300E-93	2.2000E-69	3.9900E-63	7.3100E-35	3.3100E-57	7.2600E-32

Table 1: Summaries of the products across frequencies of sample periodograms and samples from the Exponential(σ^2) distribution.

To further investigate the relationship, we look at the quantile-quantile plot for only the first half of quantiles.

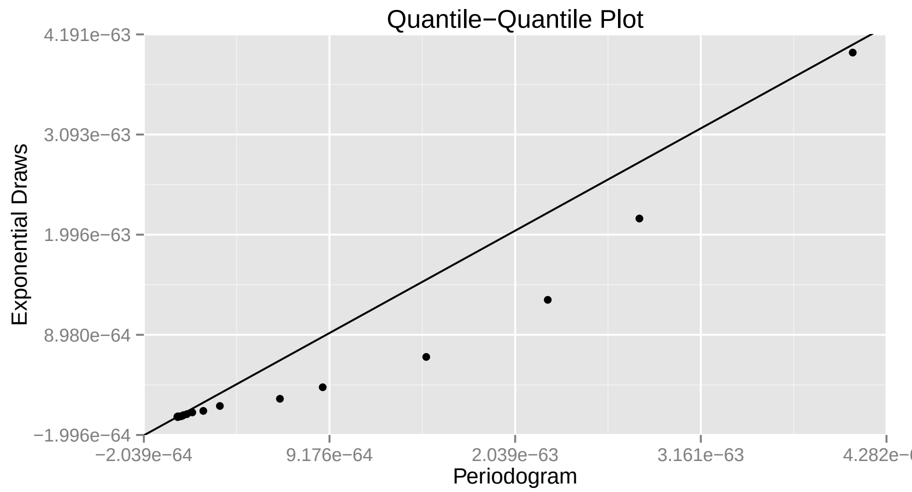

Figure 2: Quantile-quantile plot of the product of periodogram values at the Fourier frequencies versus the product of Exp(1) draws.

By zooming in on the first half of the points we can see there is a closer relationship between the quantiles, however not as linear as we would expect, given the exact result above.

The Problem

By taking the product of our periodograms (and exponential draws) over each frequency $\frac{n}{2}$ times, we are essentially amplifying the probability of obtaining values very close to zero. To illustrate this behavior, compare the Exp(1) density to the product of 50 Exp(1) densities.

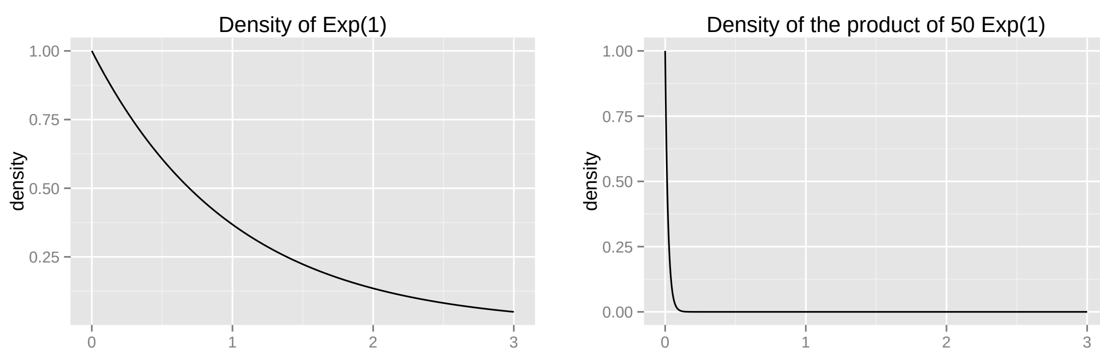

Figure 3: Comparison of the Exp(1) density to the product of 50 Exp(1) densities.

In our example above, we are actually taking the product of 249 Exp(1) distributions, leading to many values almose indistinguishable from zero (in the order of $10^{-}95$).

With this in mind, the question becomes **how robust is the Kolmogorov-Smirnov test to many values close to zero and is there a better way to test for independence as well as distribution?**