Extensive Assignment 1

STAT 601

Andee Kaplan & Maggie Johnson

February 28, 2014

Model

Let $\{Y_{ij} : i = 1, \ldots, n_j, j = 1, \ldots, m\}$ represent the quantity of green beans sold on day i at store j, let $\{Z_{ij} : i = 1, \ldots, n_j, j = 1, \ldots, m\}$ represent the unobservable construct of consumer interest on day i at store j, and let $\{x_{ij} : i = 1, \ldots, n_j, j = 1, \ldots, m\}$ be the price of green beans on day i at store j. There are $m = 191$ stores in the midwest region. Then, we impose the following model.

$$Z_{ij} \overset{\text{iid}}{\sim} \text{Bern}(p_{ij})$$
$$Y_{ij}|Z_{ij} = 1 \overset{\text{indep}}{\sim} \text{Pois}(\lambda_{ij})$$
$$Pr(Y_{ij} = 0|Z_{ij} = 0) = 1$$

This yields the following marginal distribution of Y_{ij} for $\lambda_{ij} > 0$ and $0 < p_{ij} < 1$:

$$\begin{aligned} f(y_{ij}|p_{ij}, \lambda_{ij}) &= [(1 - p_{ij}) + p_{ij} \exp(-\lambda_{ij})] \, \mathbb{I}\{y_{ij} = 0\} + \left[\frac{p_{ij}}{y_{ij}!} \lambda_{ij}^{y_{ij}} \exp(-\lambda_{ij})\right] \mathbb{I}\{y_{ij} > 0\} \\ &= [(1 - p_{ij}) + p_{ij} \exp(-\lambda_{ij})]^{\delta_{ij}} \left[\frac{p_{ij}}{y_{ij}!} \lambda_{ij}^{y_{ij}} \exp(-\lambda_{ij})\right]^{1-\delta_{ij}} \end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1 & y_{ij} = 0 \\ 0 & y_{ij} > 0 \end{cases}$$

Systematic Components

We would like to have both the parameters p_{ij} from the Bernoulli distribution and the parameters λ_{ij} from the Poisson portion of the model to be further modeled as a function of price (x_{ij}). So, we will use the constructs of GLM to model the expected values of Y_{ij} and Z_{ij} as inverse link functions of the simple regression equations.

$$\mathrm{E}(Z_{ij}) = p_{ij}$$
$$\mathrm{E}(Y_{ij}|Z_{ij}) = \lambda_{ij}$$

We will use a logit-link for p_{ij} and a log-link for λ_{ij}, which yields the following values.

$$\log\left(\frac{p_{ij}}{1-p_{ij}}\right) = \beta_0 + \beta_1 x_{ij}$$
$$\Rightarrow p_{ij} = \frac{\exp(\beta_0 + \beta_1 x_{ij})}{1+\exp(\beta_0 + \beta_1 x_{ij})}$$

$$\log(\lambda_{ij}) = \beta_2 + \beta_3 x_{ij}$$
$$\Rightarrow \lambda_{ij} = \exp(\beta_2 + \beta_3 x_{ij})$$

Likelihood

To obtain the store likelihood $L_j(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$, first we will write the joint density for store j.

$$\begin{aligned} L_j(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) &= f(y_{1j}, \ldots, y_{n_j,j}|\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) \\ &= \prod_{i=1}^{n_j} f(y_{ij}|\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) \qquad \text{(independence)} \\ &= \prod_{i=1}^{n_j} \left([(1-p_{ij}) + p_{ij}\exp(-\lambda_{ij})]^{\delta_{ij}} \left[\frac{p_{ij}}{y_{ij}!}\lambda_{ij}^{y_{ij}}\exp(-\lambda_{ij})\right]^{1-\delta_{ij}} \right) \end{aligned}$$

Where p_{ij} and λ_{ij} are functions of x_ij and $\beta_0, \ldots, \beta_3$. Then the log-likelihood function can be written as the following.

$$l_j(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) = \sum_{i=1}^{n_j} \left(\delta_{ij}\log\left[(1-p_{ij}) + p_{ij}\exp(-\lambda_{ij})\right] + (1-\delta_{ij})\left[\log(p_{ij}) - \log(y_{ij}!) + y_{ij}\log(\lambda_{ij}) - \lambda_{ij}\right]\right)$$

We can now find the Jacobian and the Hessian matrix to facilitate finding the MLEs using a Newton-Raphson method:

$$\begin{aligned} \frac{\partial l_j}{\partial \beta_k} &= \frac{\partial l_j}{\partial p_{ij}}\frac{\partial p_{ij}}{\partial \beta_k} + \frac{\partial l_j}{\partial \lambda_{ij}}\frac{\partial \lambda_{ij}}{\partial \beta_k} \\ \frac{\partial^2 l_j}{\partial \beta_k \partial \beta_l} &= \left[\frac{\partial^2 l_j}{\partial p_{ij}^2}\frac{\partial p_{ij}}{\partial \beta_l} + \frac{\partial^2 l_j}{\partial p_{ij}\partial \lambda_{ij}}\frac{\partial \lambda_{ij}}{\partial \beta_l}\right]\frac{\partial p_{ij}}{\partial \beta_k} + \frac{\partial^2 p_{ij}}{\partial \beta_k \partial \beta_l}\frac{\partial l_j}{\partial p_{ij}} + \\ &\qquad \left[\frac{\partial^2 l_j}{\partial \lambda_{ij}\partial p_{ij}}\frac{\partial p_{ij}}{\partial \beta_l} + \frac{\partial^2 l_j}{\partial \partial \lambda_{ij}^2}\frac{\partial \lambda_{ij}}{\partial \beta_l}\right]\frac{\partial \lambda_{ij}}{\partial \beta_k} + \frac{\partial^2 \lambda_{ij}}{\partial \beta_k \partial \beta_l}\frac{\partial l_j}{\partial \lambda_{ij}} \end{aligned}$$

for $k, l = 0, 1, 2, 3$.

Now let $T_1 = \exp \beta_0 + \beta_1 x_{ij}$ and $T_2 = \exp \beta_2 + \beta_3 x_{ij}$. Then, the partial derivatives can be computed as follows.

$$
\begin{aligned}
\frac{\partial l_i}{\partial p_{ij}} &= \sum_{i=1}^{n_j} \left\{ \delta_{ij} \frac{\exp(-\lambda_{ij}) - 1}{(1 - p_{ij}) + p_{ij} \exp(-\lambda_{ij})} + (1 - \delta_{ij}) \frac{1}{p_{ij}} \right\} \\
\frac{\partial l_i}{\partial \lambda_{ij}} &= \sum_{i=1}^{n_j} \left\{ \delta_{ij} \frac{-p_{ij} \exp(-\lambda_{ij})}{(1 - p_{ij}) + p_{ij} \exp(-\lambda_{ij})} + (1 - \delta_{ij}) \left[\frac{y_{ij}}{\lambda_{ij}} - 1 \right] \right\} \\
\frac{\partial p_{ij}}{\partial \beta_0} &= \frac{T_1}{(1 + T_1)^2} \\
\frac{\partial p_{ij}}{\partial \beta_1} &= x_{ij} \frac{T_1}{(1 + T_1)^2} \\
\frac{\partial \lambda_{ij}}{\partial \beta_2} &= T_2 \\
\frac{\partial \lambda_{ij}}{\partial \beta_3} &= x_{ij} T_2 \\
\frac{\partial^2 l_j}{\partial p_{ij}^2} &= \sum_{i=1}^{n_j} \left\{ -\delta_{ij} \left(\frac{\exp(-\lambda_{ij}) - 1}{(1 - p_{ij}) + p_{ij} \exp(-\lambda_{ij})} \right)^2 - (1 - \delta_{ij}) \frac{1}{p_{ij}^2} \right\} \\
\frac{\partial^2 l_j}{\partial p_{ij} \partial \lambda_{ij}} &= \sum_{i=1}^{n_j} \left\{ \delta_{ij} \left(\frac{-(\exp(-\lambda_{ij}) - 1) \exp(-\lambda_{ij})}{(1 - p_{ij}) + p_{ij} \exp(-\lambda_{ij})} - \frac{(\exp(-\lambda_{ij}) - 1) p_{ij} \exp(-\lambda_{ij})}{((1 - p_{ij}) + p_{ij} \exp(-\lambda_{ij}))^2} \right) \right\} \\
\frac{\partial^2 l_j}{\partial \lambda_{ij}^2} &= \sum_{i=1}^{n_j} \left\{ \delta_{ij} \left[\frac{p_{ij} \exp(-\lambda_{ij})}{(1 - p_{ij}) + p_{ij} \exp(-\lambda_{ij})} - \left(\frac{p_{ij} \exp(-\lambda_{ij})}{(1 - p_{ij}) + p_{ij} \exp(-\lambda_{ij})} \right)^2 \right] - (1 - \delta_{ij}) \left[\frac{y_{ij}}{\lambda_{ij}^2} \right] \right\} \\
\frac{\partial^2 p_{ij}}{\partial \beta_0 \partial \beta_0} &= \left(\frac{T_1}{1 + T_1} \right)^2 - 2 \left(\frac{T_1}{1 + T_1} \right)^3 \\
\frac{\partial^2 p_{ij}}{\partial \beta_0 \partial \beta_1} &= x_{ij} \left[\left(\frac{T_1}{1 + T_1} \right)^2 - 2 \left(\frac{T_1}{1 + T_1} \right)^3 \right] \\
\frac{\partial^2 p_{ij}}{\partial \beta_1 \partial \beta_1} &= x_{ij}^2 \left[\left(\frac{T_1}{1 + T_1} \right)^2 - 2 \left(\frac{T_1}{1 + T_1} \right)^3 \right] \\
\frac{\partial^2 \lambda_{ij}}{\partial \beta_2 \partial \beta_2} &= T_2 \\
\frac{\partial^2 \lambda_{ij}}{\partial \beta_2 \partial \beta_3} &= x_{ij} T_2 \\
\frac{\partial^2 \lambda_{ij}}{\partial \beta_3 \partial \beta_3} &= x_{ij}^2 T_2
\end{aligned}
$$

Where the remaining partial derivatives are zero.