A note on the instability and degeneracy of deep learning models

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Introduction

- A probability model exhibits *instability* if small changes in a data outcome result in large changes in probability
- Model *degeneracy* implies placing all probability on a small portion of the sample space

Goal: Quantify instability for general probability models defined on sequences of observations, where each sequence of length N has a finite number of possible outcomes

Notation

- $\boldsymbol{X} = (X_1, \dots, X_N)$ a set of discrete random variables with a finite sample space, $\mathcal{X}^N$
- For each N, $P_{\boldsymbol{\theta}_N}$ is a probability model on $\mathcal{X}^N$

FSFS models

Finitely Supported Finite Sequence (FSFS) model class

A series $P_{\boldsymbol{\theta}_N}$ of probability models, indexed by a generic sequence of parameters $\boldsymbol{\theta}_N$, to describe data of each length $N \geq 1$ with model support of $P_{\boldsymbol{\theta}_N}$ equaling the (finite) sample space $\mathcal{X}^N$.

- The size and structure of such parameters $\boldsymbol{\theta}_N$ are without restriction
- Natural cases include $\boldsymbol{\theta}_N \in \mathbb{R}^{q(N)}$ for some arbitrary integer-valued function $q(\cdot) \geq 1$

Discrete exponential family models

Exponential family model for $\boldsymbol{X}$ with pmf of the form

$$p_{N,\lambda}(\boldsymbol{x}) = \exp\left[\boldsymbol{\eta}^T(\boldsymbol{\lambda})\boldsymbol{g}_N(\boldsymbol{x}) - \psi(\boldsymbol{\lambda})\right], \quad \boldsymbol{x} \in \mathcal{X}^N,$$

for fixed positive dimensions of the parameter, $\boldsymbol{\lambda} \in \Lambda \subset \mathbb{R}^k$ and natural parameter $\boldsymbol{\eta} : \mathbb{R}^k \mapsto \mathbb{R}^L$ spaces, $\boldsymbol{g}_N : \mathcal{X}^N \mapsto \mathbb{R}^L$ a vector of sufficient statistics,

$$\psi(\boldsymbol{\lambda}) = \log \sum_{\boldsymbol{x} \in \mathcal{X}^N} \exp\left[\boldsymbol{\eta}^T(\boldsymbol{\lambda})\boldsymbol{g}_N(\boldsymbol{x})\right], \qquad \boldsymbol{\lambda} \in \Lambda,$$

the normalizing function, and $\Lambda = \{\boldsymbol{\lambda} \in \mathbb{R}^k : \psi(\boldsymbol{\lambda}) < \infty, k \leq q(N)\}$ is the parameter space.

Discrete exponential family models (cont'd)

- Such models arise with
  - Spatial data on a lattice (Besag 1974)
  - Network data (Wasserman and Faust 1994; Handcock 2003)
  - Binomial sampling with N iid Bernoulli random variables
- These models are special cases of the FSFS models
- $P_{\boldsymbol{\theta}_N}(\boldsymbol{x}) \equiv p_{N,\boldsymbol{\lambda}_N}(\boldsymbol{x})$ with $\boldsymbol{\theta}_N = \boldsymbol{\lambda}_N$ a sequence of elements of $\Lambda \subset \mathbb{R}^k$
- $P_{\boldsymbol{\theta}_N}(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \mathcal{X}^N$
- The dimension of the parameter $\boldsymbol{\theta}_N$ is the same for each N (k)
- Schweinberger (2011) considered *instability* in such exponential models

Restricted Boltzmann machines

Deep learning

Instability results

Implications

Thank you

Questions?

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References I

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