5.2 Continuous random variables

It is often convenient to think of a random variable as having a whole (continuous) interval for its set of possible values.

The devices used to describe continuous probability distributions differ from those that describe discrete probability distributions.

Examples of continuous random variables:

5.2.1 Probability density functions and cumulative distribution functions

A *probability density function (pdf)* is the continuous analogue of a discrete random variable's probability mass function (pmf).

Definition 5.12. A *probability density function (pdf)* for a continuous random variable X is a nonnegative function $f(x)$ with

$$\int_{-\infty}^{\infty} f(x) = 1$$

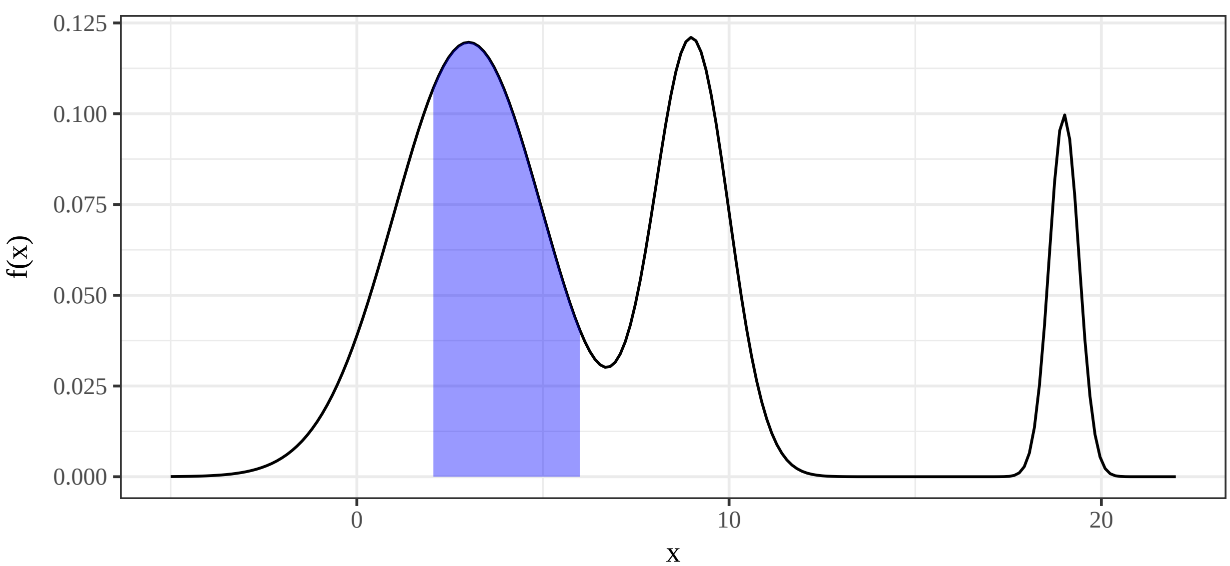
and such that for all $a \leq b$,

$$P[a \leq X \leq b] = \int_{a}^{b} f(x)dx.$$

1.

2.

3.



Example 5.17 (Compass needle)**.** Consider a de-magnetised compass needle mounted at its centre so that it can spin freely. It is spun clockwise and when it comes to rest the angle, θ, from the vertical, is measured. Let

$$Y = \text{the angle measured after each spin in radians}$$

What values can Y take?

What form makes sense for $f(y)$?

If this form is adopted, that what must the pdf be?

Using this pdf, calculate the following probabilities:

1. $P[Y < \frac{\pi}{2}]$

2. $P[\frac{\pi}{2} < Y < 2\pi]$

3. $P[\frac{\pi}{6} < Y < \frac{\pi}{4}]$

4. $P[Y = \frac{\pi}{6}]$

Definition 5.13. The *cumulative distribution function (cdf)* of a continuous random variable X is a function F such that

$$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(t)dt$$

$F(x)$ is obtained from $f(x)$ by integrtion, and applying the fundamental theorem of calculus yields

$$\frac{d}{dx}F(x) = f(x).$$

That is, $f(x)$ is obtained from $F(x)$ by differentiation.

As with discrete random variables, F has the following properties:

1.

2.

3.

Example 5.18 (Compass needle, cont'd)**.** Recall the compass needle example, with

$$f(x) = \begin{cases} \frac{1}{2\pi} & 0 \le y \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Find the cdf.

For $y < 0$

For $0 \le y \le 2\pi$

For $y > 2\pi$

Calculate the following using the cdf:

$F(1.5)$

$P[Y \leq \frac{4\pi}{5}]$

$P[Y > 5]$

$P[\frac{\pi}{3} < Y \leq \frac{\pi}{2}]$

5.2.2 Quantiles

Recall:

Definition 5.14. The *p-quantile of a random variable*, X, is the number $Q(p)$ such that $P[X \leq Q(p)] = p$.

In terms of the cumulative distribution function (for a continuous random variable),

Example 5.19 (Compass needle, cont'd)**.** Recall the compass needle example, with

$$f(x) = \begin{cases} \frac{1}{2\pi} & 0 \leq y \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$Q(.95)$:

You can also calculate quantiles directly from the cdf.

$$F(x) = \begin{cases} 0 & y < 0 \\ \frac{1}{2\pi} y & 0 \leq y \leq 2\pi \\ 1 & \text{otherwise} \end{cases}$$

$Q(.25)$:

$Q(.5)$

5.2.3 Means and variances for continuous distributions

It is possible to summarize continuous probability distributions using

1.

2.

3.

Definition 5.15. The *mean* or *expected value* of a continuous random variable X is

$$\mathrm{E}X = \int_{-\infty}^{\infty} x f(x) dx.$$

Example 5.20 (Compass needle, cont'd)**.** Calculate $\mathrm{E}Y$ where Y is the angle from vertical in radians that a spun needle lands on.

$$f(y) = \begin{cases} \frac{1}{2\pi} & 0 \leq y \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Example 5.21. Calculate EX where X follows the following distribution

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{3}e^{-x/3} & x \geq 0 \end{cases}$$

Definition 5.16. The *variance* of a continuous random variable X is

$$\mathrm{Var}X = \int_{-\infty}^{\infty} (x - \mathrm{E}X)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - (\mathrm{E}X)^2.$$

The *standard deviation* of X is $\sqrt{\mathrm{Var}X}$.

Example 5.22 (Library books)**.** Let X denote the amount of time for which a book on 2-hour hold reserve at a college library is checked out by a randomly selected student and suppose its density function is

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Calculate $\mathrm{E}X$ and $\mathrm{Var}X$.

Example 5.23 (Ecology)**.** An ecologist wishes to mark off a circular sampling region having radius 10m. However, the radius of the resulting region is actually a random variable R with pdf

$$f(r) = \begin{cases} \frac{3}{2}(10-r)^2 & 9 \leq r \leq 11 \\ 0 & \text{otherwise} \end{cases}$$

Calculate $\mathrm{E}R$ and $\mathrm{SD}(R)$.

Why does $\mathrm{E}X^2 = \int\limits_{-\infty}^{\infty} x^2 f(x)dx$?

Example 5.24 (Ecology, cont'd)**.** Calculate the expected *area* of the circular sampling region.

For a linear function, $g(x) = ax + b$, where a and b are constants,

$\mathrm{E}(ax + b)$

$\mathrm{Var}(ax + b)$

Example 5.25 (Ecology, cont'd)**.** Calculate the expected value and variance of the *diameter* of the circular sampling region.

Definition 5.17. *Standardization* is the process of transforming a random variable, X, into the signed number of standard deviations by which it is is above its mean value.

$$Z = \frac{X - \mathrm{E}X}{\mathrm{SD}(X)}$$

Z has mean 0

Z has variance (and standard deviation) 1

5.2.4 The Normal distribution

Just as there are a number of useful discrete distributions commonly applied to engineering problems, there are a number of standard continuous probability distributions.

Definition 5.18. The *normal* or *Gaussian*(μ, σ^2) distribution is a continuous probability distribution with probability density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \qquad \text{for all } x$$

for $\sigma > 0$.

A normal random variable is (often) a finite average of many repeated, independent, identical trials.

It is not obvious, but

1. $\int\limits_{-\infty}^{\infty} f(x)dx = \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx =$

2. $\mathrm{E}X = \int\limits_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx =$

3. $\mathrm{Var}X = \int\limits_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx =$

The Calc I methods of evaluating integrals via antidifferentiation will fail when it comes to normal densities. They do not have antiderivatives that are expressible in terms of elementary functions.

The use of tables for evaluating normal probabilities depends on the following relationship. If $X \sim \text{Normal}(\mu, \sigma^2)$,

$$P[a \leq X \leq b] = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx = \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = P\left[\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right]$$

where $Z \sim \text{Normal}(0, 1)$.

Definition 5.19. The normal distribution with $\mu = 0$ and $\sigma = 1$ is called the *standard normal distribution.*

So, we can find probabilities for all normal distributions by tabulating probabilities for only the standard normal distribution. We will use a table of the **standard normal cumulative probability function**.

$$\Phi(z) = F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2} dt.$$

Example 5.26 (Standard normal probabilities). $P[Z < 1.76]$

$P[.57 < Z < 1.32]$

We can also do it in reverse, find z such that $P[-z < Z < z] = .95$.

Example 5.27 (Baby food)**.** J. Fisher, in his article Computer Assisted Net Weight Control (*Quality Progress*, June 1983), discusses the filling of food containers with strained plums and tapioca by weight. The mean of the values portrayed is about 137.2g, the standard deviation is about 1.6g, and data look bell-shaped. Let

$$W = \text{the next fill weight.}$$

Let's find the probability that the next jar contains less food by mass than it's supposed to (declared weight = 135.05g).

Table B.3

Standard Normal Cumulative Probabilities

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985

Example 5.28 (More normal probabilities)**.** Using the standard normal table, calculate the following:

$P(X \leq 3), X \sim \text{Normal}(2, 64)$

$P(X > 7), X \sim \text{Normal}(6, 9)$

$P(|X - 1| > 0.5), X \sim \text{Normal}(2, 4)$

We can find standard normal quantiles by using the standard normal table in reverse.

Example 5.29 (Baby food, cont'd)**.** For the jar weights $X \sim \text{Normal}(137.2, 1.62^2)$, find $Q(0.1)$.

Table B.3
Standard Normal Cumulative Probabilities

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985

Example 5.30 (Normal quantiles)**.** Find:

$Q(0.95)$ of $X \sim \text{Normal}(9, 3)$.

c such that $P(|X - 2| > c) = 0.01$, $X \sim \text{Normal}(2, 4)$

5.2.5 Other special continuous distributions

Table B.3
Standard Normal Cumulative Probabilities

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2297	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Table B.3

Standard Normal Cumulative Probabilities (*continued*)

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9773	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9983	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

This table was generated using MINITAB.