6 Introduction to formal statistical inference

Formal statistical inference uses probability theory to quantify the reliability of data-based conclusions. We want information on a population. We can use:

1. Point estimates:

2. Interval estimates:

6.1 Large-sample confidence intervals for a mean

Many important engineering applications of statistics fit the following mold. Values for parameters of a data-generating process are unknown. Based on data, the goal is

1.
2.

Definition 6.1. A *confidence interval* for a parameter (or function of one or more parameters) is a data-based interval of numbers thought likely to contain the parameter (or function of one or more parameters) possessing a stated probability-based confidence or reliability.

A confidence interval is a **random interval**, an interval on the real line with a random variable at one or both of the endpoints.

Example 6.1 (Instrumental drift)**.** Let Z be a measure of instrumental drift of a random voltmeter that comes out of a certain factory. Say $Z \sim N(0, 1)$. Define a random interval:

$$(Z - 2, Z + 2)$$

What is the probability that -1 is inside the interval?

Example 6.2 (More practice)**.** Calculate:

1. $P(2 \text{ in } (X-1, X+1))$, $X \sim N(2,4)$

2. $P(6.6 \text{ in } (X-2, X+1))$, $X \sim N(7,2)$

Example 6.3 (Abstract random intervals)**.** Let's say $X_1, X_2, \ldots, X_n$ are iid with $n \geq 25$, mean μ, variance σ^2. We can find a random interval that provides a lower bound for μ with $1 - \alpha$ probability:

Calculate:

1. $P(\mu \in (-\infty, \overline{X} + z_{1-\alpha}\frac{\sigma}{\sqrt{n}})), X \sim N(\mu, \sigma^2)$

2. $P(\mu \in (\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}})), X \sim N(\mu, \sigma^2)$

6.1.1 A Large-n confidence interval for μ involving σ

A $1-\alpha$ **confidence interval** for an unknown parameter is the realization of a random interval that contains that parameter with probability $1-\alpha$.

For random variables $X_1, X_2, \ldots, X_n$ iid with $\mathrm{E}(X_1) = \mu$, $\mathrm{Var}(X_1) = \sigma^2$, a $1-\alpha$ confidence interval for μ is

$$(\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}})$$

which is a **realization** from the random interval

$$(\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}).$$

- Two-sided $1-\alpha$ confidence interval for μ

- One-sided $1-\alpha$ confidence interval for μ with a upper confidence bound

- One-sided $1-\alpha$ confidence interval for μ with a lower confidence bound

Example 6.4 (Fill weight of jars)**.** Suppose a manufacturer fills jars of food using a stable fillin gprocess with a known standard deviation of $\sigma = 1.6$g. We take a sample of $n = 47$ jars and measure the sample mean weight $\overline{x} = 138.2$g. A two-sided 90% confidence interval ($\alpha = 0.1$) for the true mean weight μ is:

Interpretation:

What if we just want to be sure that the true mean fill weight is high enough?

Example 6.5 (Hard disk failures)**.** F. Willett, in the article "The Case of the Derailed Disk Drives?" (*Mechanical Engineering*, 1988), discusses a study done to isolate the cause of link code A failure in a model of Winchester harddisk drive. For each disk, the investigator measured the breakaway torque (in. oz.) required to loosen the drive's interrupter flag on the stepper motor shaft. Breakaway torques for 26 disk drives were recorded, with a sample mean of 11.5 in. oz. Suppose you know the true standard deviation of the breakaway torques is 5.1 in. oz. Calculate and interpret:

1. A two-sided 90% confidence interval for the true mean breakaway torque of the relevant type of Winchester drive.

2. An analogous two-sided 95% confidence interval.

Example 6.6 (Width of a CI)**.** If you want to estimate the breakaway torque with a 2-sided, 95% confidence interval with ± 2.0 in. oz. of precision, what sample size would you need?

6.1.2 A generally applicable large-n confidence interval for μ

Although the equations for a $1-\alpha$ confidence interval is mathematically correct, it is severely limited in its usefulness because

If $n \geq 25$ and σ is *unknown*, $Z = \frac{\overline{X}-\mu}{s/\sqrt{n}}$, where

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}.$$

is still **approximately standard normally distributed**. So, you can replace σ in the confidence interval formula with the sample standard deviation, s.

- Two-sided $1-\alpha$ confidence interval for μ

- One-sided $1-\alpha$ confidence interval for μ with a upper confidence bound

- One-sided $1-\alpha$ confidence interval for μ with a lower confidence bound

Example 6.7. Suppose you are a manufacturer of construction equipment. You make 0.0125 inch wire rope and need to determine how much weight it can hold before breaking so that you can label it clearly. Here are breaking strengths, in kg, for 41 sample wires:

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 [1] 100.37  96.31  72.57  88.02 105.89 107.80  75.84  92.73  67.47  94.87
[11] 122.04 115.12  95.24 119.75 114.83 101.79  80.90  96.10 118.51 109.66
[21]  88.07  56.29  86.50  57.62  74.70  92.53  86.25  82.56  97.96  94.92
[31]  62.00  93.00  98.44 119.37 103.70  72.40  71.29 107.24  64.82  93.51
[41]  86.97
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The sample mean breaking strength is 91.85 kg and the sample standard deviation is 17.6 kg. Using the appropriate 95% confidence interval, try to determine whether the breaking strengths meet the requirement of at least 85 kg.

6.2 Small-sample confidence intervals for a mean

6.2.1 The Student t distribution

6.2.2 Small-sample confidence intervals, σ unknown

6.3 Large-sample significance tests for a mean

6.3.1 Hypothesis testing using the CI

6.4 Inference for matched pairs and two-sample data

6.4.1 Matched pairs

6.4.2 Two-sample data

6.4.3 Small samples

6.5 Prediction intervals