The words “eigenvalues” and “eigenvector” are derived from the German word “eigen” which means “proper” or “characteristic”. Eigenvalues and eigenvectors have many applications in several engineering disciplines ranging from the analysis of structures in civil engineering to circuit analysis, signal and image processing, and control in electrical engineering.

Find a basis for the row space of

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

The reduced row echelon form of A is

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

basis for the row space of A is

$$\{[1 \quad 0 \quad 1 \quad 0 \quad -1], [0 \quad 1 \quad 2 \quad 0 \quad 3], [0 \quad 0 \quad 0 \quad 1 \quad 4]\}$$

Find a basis for the column space of the matrix A.

What is the fastest way to find this basis? Use the columns of A that correspond to the columns of R *containing the leading 1s*. A basis for col(A) is

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Find a basis for the null space of matrix A

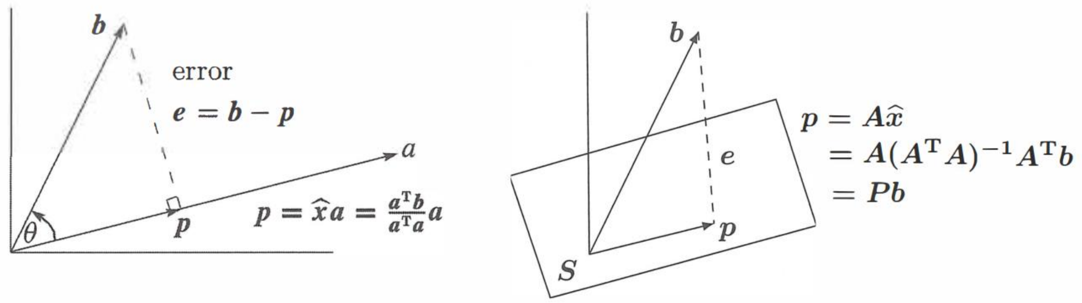

Figure 4.6: The projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto a line and onto $\boldsymbol{S}$ = column space of A.

The final augmented matrix is

$$[R\,|\,\mathbf{0}] = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Setting $x_3 = s$ and $x_5 = t$, we obtain

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s + t \\ -2s - 3t \\ s \\ -4t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} = s\mathbf{u} + t\mathbf{v}$$

Find bases of range(A^T) and null(A^T), where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 1 & 4 \\ 2 & 1 & 1 & -1 & -3 \end{bmatrix}.$$

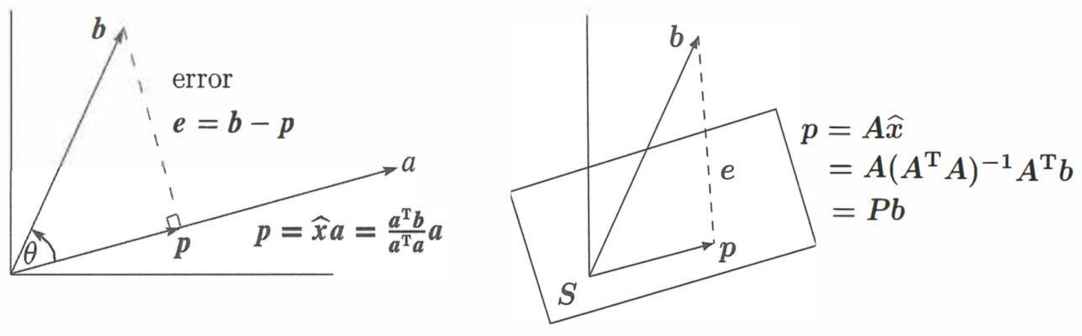

Figure 4.6: The projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto a line and onto $\boldsymbol{S}$ = column space of A.

We start by finding the reduced row echelon form of A^T:

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 4 & -3 \end{bmatrix} \xrightarrow[R_5+R_1]{R_3-R_1} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & 3 & -1 \end{bmatrix}$$

$$\xrightarrow[R_5-2R_2]{\substack{R_1-R_2 \\ R_3+R_2}} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -3 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow[R_5-3R_3]{R_1+R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

basis of range(A^T). Explicitly, this basis is

$$\{(1,0,1,0,-1),(1,1,0,0,1),(-1,0,-1,1,4)\}.$$

$$(x_1,x_2,x_3,x_4) = x_4(0,-1,1,1),$$

so $\{(0,-1,1,1)\}$ is a basis of null(A^T).

Apply the Gram-Schmidt Process to construct an orthonormal basis for the subspace $W = \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ of $\mathbb{R}^4$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Solution First we note that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set, so it forms a basis for W. We begin by setting $\mathbf{v}_1 = \mathbf{x}_1$. Next, we compute the component of $\mathbf{x}_2$ orthogonal to $W_1 = \text{span}(\mathbf{v}_1)$:

$$\begin{aligned}\mathbf{v}_2 = \text{perp}_{W_1}(\mathbf{x}_2) &= \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\ &= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \left(\tfrac{2}{4}\right)\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\end{aligned}$$

Find the orthogonal complement of the subspace S of R^4 spanned by the two column vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the matrix A.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{array}{cc} \mathbf{v}_1 & \mathbf{v}_2 \end{array} \end{array}$$

A vector $\mathbf{u} \in R^4$ is in the orthogonal complement of S when its dot product with each of the columns of A, $\mathbf{v}_1$ and $\mathbf{v}_2$, is zero. So, the orthogonal complement of S consists of all the vectors $\mathbf{u}$ such that $A^T\mathbf{u} = \mathbf{0}$.

$$A^T\mathbf{u} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

That is, the orthogonal complement of S is the nullspace of the matrix A^T:

$$S^\perp = N(A^T).$$

Find the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$N(A)$ = nullspace of A $\qquad$ $N(A^T)$ = nullspace of A^T

$R(A)$ = column space of A $\qquad$ $R(A^T)$ = column space of A^T

$$R(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad R(A^T) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$N(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad N(A^T) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Fundamental Subspaces of a Matrix

If A is an $m \times n$ matrix, then

1. $R(A)$ and $N(A^T)$ are orthogonal subspaces of R^m.
2. $R(A^T)$ and $N(A)$ are orthogonal subspaces of R^n.
3. $R(A) \oplus N(A^T) = R^m$.
4. $R(A^T) \oplus N(A) = R^n$.

Orthogonal Subspaces In Exercises 5–8, determine whether the subspaces are orthogonal.

5. $S_1 = \text{span}\left\{\begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ $\quad S_2 = \text{span}\left\{\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}\right\}$

6. $S_1 = \text{span}\left\{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$ $\quad S_2 = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

7. $S_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ $\quad S_2 = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix}\right\}$

Projection Onto a Subspace In Exercises 17–20, find the projection of the vector $\mathbf{v}$ onto the subspace S.

17. $S = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

18. $S = \text{span}\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Fundamental Subspaces In Exercises 21–24, find bases for the four fundamental subspaces of the matrix A.

21. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

22. $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

24. $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Finding the Least Squares Solution In Exercises 25–28, find the least squares solution of the system $A\mathbf{x} = \mathbf{b}$.

25. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$

26. $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

Finding the Orthogonal Complement and Direct Sum In Exercises 9–14, (a) find the orthogonal complement $S^\perp$, and (b) find the direct sum $S \oplus S^\perp$.

9. $S = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$

10. $S = \text{span}\left\{\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\right\}$

11. $S = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right\}$

12. $S = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}\right\}$

1. Find the QR-factorization of each matrix.

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Orthogonal and Orthonormal Sets In Exercises 1–12, (a) determine whether the set of vectors in R^n is orthogonal, (b) if the set is orthogonal, then determine whether it is also orthonormal, and (c) determine whether the set is a basis for R^n.

1. $\{(2, -4), (2, 1)\}$

2. $\{(-3, 5), (4, 0)\}$

3. $\left\{\left(\frac{3}{5}, \frac{4}{5}\right), \left(-\frac{4}{5}, \frac{3}{5}\right)\right\}$

4. $\left\{(2, 1), \left(\frac{1}{3}, -\frac{2}{3}\right)\right\}$

5. $\{(4, -1, 1), (-1, 0, 4), (-4, -17, -1)\}$

6. $\{(2, -4, 2), (0, 2, 4), (-10, -4, 2)\}$

7. $\left\{\left(\frac{\sqrt{2}}{3}, 0, -\frac{\sqrt{2}}{6}\right), \left(0, \frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right), \left(\frac{\sqrt{5}}{5}, 0, \frac{1}{2}\right)\right\}$

8. $\left\{\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right), \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)\right\}$

9. $\{(2, 5, -3), (4, 2, 6)\}$

10. $\{(-6, 3, 2, 1), (2, 0, 6, 0)\}$

Applying the Gram-Schmidt Process In Exercises 25–34, apply the Gram-Schmidt orthonormalization process to transform the given basis for R^n into an orthonormal basis. Use the vectors in the order in which they are given.

25. $B = \{(3, 4), (1, 0)\}$

26. $B = \{(-1, 2), (1, 0)\}$

27. $B = \{(0, 1), (2, 5)\}$

28. $B = \{(4, -3), (3, 2)\}$

29. $B = \{(2, 1, -2), (1, 2, 2), (2, -2, 1)\}$

30. $B = \{(1, 0, 0), (1, 1, 1), (1, 1, -1)\}$

31. $B = \{(4, -3, 0), (1, 2, 0), (0, 0, 4)\}$

32. $B = \{(0, 1, 2), (2, 0, 0), (1, 1, 1)\}$

Finding a Coordinate Matrix **In Exercises 19–24, find the coordinate matrix of w relative to the orthonormal basis B in R^n.**

19. $\mathbf{w} = (1, 2)$, $B = \left\{\left(-\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}\right), \left(\frac{3\sqrt{13}}{13}, \frac{2\sqrt{13}}{13}\right)\right\}$

20. $\mathbf{w} = (4, -3)$, $B = \left\{\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right), \left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{3}\right)\right\}$

21. $\mathbf{w} = (2, -2, 1)$,

$B = \left\{\left(\frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10}\right), (0, 1, 0), \left(-\frac{3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10}\right)\right\}$

Consider the set of polynomials $S = \{1 \; t \; t^2\}$ defined over the interval of $-1 \leq t \leq 1$. Using the Gram–Schmidt orthogonalization process, obtain an orthonormal set.

Consider the set $S = \{x_1 \; x_2 \; x_3 \; x_4\} \in R^3$ where:

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, x_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, x_4 = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

Use the Gram–Schmidt orthogonalization process to obtain an orthonormal set.

Find the QR factorization of the matrix below:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 4 \\ 2 & 0 & 3 \\ -5 & -3 & 1 \end{bmatrix}$$

Solve the following set of linear equations using QR factorization.

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 14 \end{bmatrix}$$

Solve the following set of linear equations using QR factorization:

$$x_1 + 2x_2 + 3x_3 = 14$$

$$4x_1 + 5x_2 + 6x_3 = 32$$

$$7x_1 - 3x_2 - 2x_3 = -5$$