The material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

Conservation laws

Consider a property L that is measureable within a finite volume Ω with boundary $\partial\Omega$. The rate of change equals the amount that is created/consumed inside the volume or what flows in/out through the boundary. This can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} L \mathrm{d}V = -\int_{\partial\Omega} L\mathbf{v} \cdot \mathbf{n} \mathrm{d}A - \int_{\Omega} Q \mathrm{d}V,$$

where $\mathbf{n}$ is the normal vector to the boundary pointing outwards, $\mathbf{v}$ is the fluid velocity and Q represents the sources or sinks inside the fluid. If we apply the divergence theorem, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} L \mathrm{d}V = -\int_{\Omega} \nabla \cdot (L\mathbf{v}) \mathrm{d}V - \int_{\Omega} Q \mathrm{d}V,$$

or simply

$$\int_{\Omega} \left(\frac{\partial L}{\partial t} + \nabla \cdot (L\mathbf{v}) + Q \right) \mathrm{d}V = 0.$$

But since the volume Ω is arbitrary chosen, the integrand must be zero

$$\frac{\partial L}{\partial t} + \nabla \cdot (L\mathbf{v}) + Q = 0.$$

Conservation of momentum

By using $L = \rho\mathbf{v}$, we get the conservation given as

$$\frac{\partial \rho\mathbf{v}}{\partial t} + \nabla \cdot (\rho\mathbf{v}\mathbf{v}) + \mathbf{Q} = 0.$$

The Q term, the source or sink, is simply the body force, i.e. external forces that we denote by $\mathbf{b}$. The term with $\mathbf{vv}$ is a dyad, i.e. a second rank tensor. This then becomes

$$\mathbf{v}\frac{\partial \rho}{\partial t} + \rho\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\mathbf{v} \cdot \nabla\rho + \rho\mathbf{v}\nabla \cdot \mathbf{v} + \rho\mathbf{v} \cdot \nabla\mathbf{v} = \mathbf{b}.$$