We look at heat conduction in a body with density ρ, heat capacity c and heat conduction coefficient α. The equation for the temperature distribution $T(x, y, z, t)$ can be written as

$$\rho c T_t = \nabla \cdot (\alpha(x, y, z)\nabla T).$$

The body is a cylinder of length L with isolated cylindrical surface such that $-\alpha\partial_n T = 0$ here. The ends are kept at constant temperatures T_0 and T_1. The left half of the cylinder is made of a material with heat capacity c_0 and conduction coefficient α_0, while the right half has values c_1 and α_1. At first, these are two separate cylinders, but at $t = 0$ the cylinders are brought together.

1 Show that the simplification T=T(x,t) is possible in the described problem, where x is a coordinate along the cylinder (just insert T(x,t) in the original problem and see that it fulfills all equations). Set up the simplified PDE with proper boundary and initial conditions.

We assume that the temperature is only dependent of x, so that $T(x, y, z, t) = T(x, t)$. We insert into the diffusion equation

$$\rho c \frac{\partial T(x,t)}{\partial t} = \nabla \cdot (\alpha(x, y, z)\nabla T) = \nabla\alpha \cdot \nabla T + \alpha\nabla^2 T$$

Since T is only dependent of x, this reduces to

$$\rho c \frac{\partial T}{\partial t} = \nabla\alpha \cdot \nabla T + \alpha\nabla^2 T = \frac{\partial \alpha}{\partial x}\frac{\partial T}{\partial x} + \alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\alpha\frac{\partial T}{\partial x}\right)$$

with

$$\begin{aligned}
\alpha(x) &= \begin{cases} \alpha_0 & x < L/2 \\ \alpha_1 & x \geq L/2 \end{cases} \\
c(x) &= \begin{cases} c_0 & x < L/2 \\ c_1 & x \geq L/2 \end{cases} \\
T(x, 0) &= \begin{cases} T_0 & x < L/2 \\ T_1 & x \geq L/2 \end{cases} \\
T(0) &= T_0 \\
T(L) &= T_1
\end{aligned}$$

2 The 1D PDE problem is discretized by the Forward Euler, Backward Euler, and Crank-Nicolson schemes. Derive the discrete equations for one of these schemes.

We discretize in the usual way

$$T(x, t) \approx T(i\Delta x, n\Delta t) \equiv T_i^n$$

2.1 Forward Euler

$$\left[\rho c D_t^+ T = D_x \alpha [D_x T]\right]_i^n$$

where we have used the centered difference scheme in the spatial part of the equation. Written out with indices, this becomes

$$\rho c \frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{1}{\Delta x^2}\Big[\alpha_{i+1/2}\big(T_{i+1}^n - T_i^n\big) - \alpha_{i-1/2}\big(T_i^n - T_{i-1}^n\big)\Big]$$

which can be solved directly for T_i^{n+1}

$$T_i^{n+1} = C\Big[\alpha_{i+1/2}\big(T_{i+1}^n - T_i^n\big) - \alpha_{i-1/2}\big(T_i^n - T_{i-1}^n\big)\Big] + T_i^n$$

where $C = \frac{\Delta t}{\rho c \Delta x^2}$.

3 Assume for simplicity that $c_0 = c_1$ and that $\alpha_0 = \alpha_1$. With a discontinuous initial conditions, numerical artifacts may appear in the solutions with the three methods. Illustrate such artifacts. A suitable program to play around with is demo_osc.py.

4 Present the ideas and results of an analysis that can explain the artifacts in the previous subproblem.

When α is a constant, the equation is reduced to

$$u_t = C u_x x$$