1 Explain briefly how this equation arises from basic principles in physics and what the individual terms model.

We look at a sphere with radius r, volume V and density ρ falling with gravity g in a viscous fluid with dynamic viscosity μ, density ρ_f. The forces are divided into three parts

$$\begin{aligned} F_g &= -mg = -V\rho g \\ F_d &= -6\pi r\mu v \\ F_f &= V\rho_f g \end{aligned}$$

where $F_G = F_g + F_f = gV(\rho_f - \rho)$ is the total force due to gravity including the uplift from archimedes principle. Using Newtons 2nd law, we have

$$\begin{aligned} V\rho\dot{v} &= -V\rho g - 6\pi r\mu v + V\rho_f g \\ &= gV(\rho_f - \rho) - 6\pi r\mu \cdot v \\ \dot{v} &= \frac{g(\rho_f - \rho)}{\rho} - \frac{6\pi r\mu}{V\rho} v = A + Bv, \end{aligned} \tag{1}$$

where $A = g(\rho_f - \rho)/\rho$ and $B = -6\pi r\mu/V\rho$. This is a first order differential equation for the velocity v. Note that the friction term is proportional to the radius and the viscosity of the fluid. This term is called Stokes' law and can be derived by solving the Navier-Stokes equations assuming Stokes flow. Stokes flow has low Reynold numbers, and this is also a requirement for (1) to hold. The reynold number is defined as

$$\text{Re} = \frac{\rho d|v|}{\mu}, \tag{2}$$

where d is a characteristic length of the object, i.e. the diameter. The thermal velocity is given when the acceleration is zero

$$v_{max} = -\frac{A}{B} = \frac{Vg(\rho_f - \rho)}{6\pi r\mu},$$

whereas the full analytical solution to the differential equation is

$$v(t) = c_1 \exp(Bt) - \frac{A}{B}.$$

If we insert the initial condition $v(0) = 0$, we have

$$v(t) = \frac{Vg(\rho_f - \rho)}{6\pi r\mu}\left[1 - \exp\left(-\frac{6\pi r\mu}{V\rho}t\right)\right].$$

2 Derive a Forward Euler, Backward Euler, and a Crank-Nicolson scheme for the equation. Mention other possible schemes too.

In order to solve this system numerically, we have to discretize the time dimension so that

$$t = n\Delta t$$

and our numerical solutions obeys

$$v(t) \approx v(n\Delta t) \equiv v^n$$

The only way to evolve the system in time is by connecting v_n to v^{n+1} by using a scheme for the time derivative.

2.1 Operator notation

We use the operator notation to describe the different schemes, where the differential equation

$$\frac{\partial u}{\partial t} = f(u)$$

is discretized as

$$[D_t^+ v = f(v)]^n \qquad \text{Forward Euler} \tag{3}$$
$$[D_t^- v = f(v)]^n \qquad \text{Backward Euler} \tag{4}$$
$$[D_t v = f(v)]^{n+1/2} \qquad \text{Crank-Nicolson} \tag{5}$$

where we'll investigate these in detail in the following sections.

2.2 Forward Euler

The FE scheme is written with operator notation

$$[D_t^+ v = A + Bv]^n = \frac{v^{n+1} - v^n}{\Delta t} = A + Bv^n$$

which can be solved directly to give an explicit scheme for v^{n+1}

$$v^{n+1} = \Delta t(A + Bv^n) + v^n = v^n(1 + \Delta tB) + \Delta tA \tag{6}$$

2.3 Backward Euler

If we instead evaluate the derivative in timestep $n + 1$, we get the Backward Euler scheme

$$[D_t^- v = A + Bv]^n = \frac{v^n - v^{n-1}}{\Delta t} = A + Bv^n$$

which also gives an explicit scheme

$$\begin{aligned} v^n(1 - \Delta tB) &= \Delta tA + v^{n-1} \\ v^n &= \frac{\Delta tA + v^{n-1}}{1 - \Delta tB} \end{aligned}$$

2.4 Crank-Nicolson

In the Crank-Nicolson scheme, we expand the function around $n + 1/2$, so that

$$\begin{aligned} [D_t v = f(v)]^{n+1/2} &= \frac{v^{n+1/2+1/2} - v^{n+1/2-1/2}}{2\Delta t/2} = \frac{v^{n+1} - v^n}{\Delta t} \\ &= f(v^{n+1/2}) = A + Bv^{n+1/2}, \end{aligned}$$

where we have to find a way to evalute $v^{n+1/2}$. A usual approach is to use the arithmetic mean $v^{n+1/2} = 1/2(v^{n+1} + v^n)$ which again will give us an explicit scheme for v^{n+1}

$$\begin{aligned} \frac{v^{n+1} - v^n}{\Delta t} &= A + \frac{B}{2}(v^{n+1} + v^n) \\ v^{n+1} &= \frac{A\Delta t + v^n\left(1 + \frac{B\Delta t}{2}\right)}{\left(1 - \frac{B\Delta t}{2}\right)} \end{aligned}$$

2.5 θ-rule

All these can be summarized into one scheme with a parameter θ determining which scheme we end up with. We define $f(v^{n+\theta}) = f(\theta v^{n+1} + (1-\theta)v^n)$

$$\frac{v^{n+1} - v^n}{\Delta t} = A + B\left(\theta v^{n+1} + (1-\theta)v^n\right)$$

which we can solve for u^{n+1}

$$v^{n+1} = \frac{A\Delta t + v^n\left[1 + B\Delta t(1-\theta)\right]}{1 - \theta B \Delta t} \tag{7}$$

which will give Forward Euler, Crank-Nicolson and Backward Euler for $\theta\{0, \frac{1}{2}, 1\}$ respectivly.

2.6 Runge-Kutta

3 Illustrate what kind of numerical artifacts that may appear when using the Forward Euler, Backward Euler, and a Crank-Nicolson schemes. Explain the reason for the artifacts (motivated by a mathematical analysis of the schemes).

3.1 Oscillations and divergence

If we start with the initial velocity v_0 and use (7), we can find v_1, v_2 etc

$$\begin{aligned}
v^1 &= \frac{A\Delta t + v^0\left[1 + B\Delta t(1-\theta)\right]}{1 - \theta B \Delta t} = v^0 C + D \\
v_2 &= v_1 C + D = (v_0 C + D)C + D = v_0 C^2 + DC + D \\
v_3 &= v_2 C + D = (v_1 C + D)C + D = ((v_0 C + D)C + D)C + D \\
&= v_0 C^3 + DC^2 + DC + D.
\end{aligned}$$

where

$$\begin{aligned}
C &= \frac{1 + B\Delta t(1-\theta)}{1 - \theta B \Delta t} \\
D &= \frac{A\Delta t}{1 - \theta B \Delta t}.
\end{aligned}$$

We recognize a pattern and can write the n'th term as

$$v_n = v_0 C^n + D\sum_{k=0}^{n-1} C^k = v_0 C^n + D\frac{1 - C^n}{1 - C}$$

where we have used the sum of a geometric series. In order to have a convergent series, we require $|C| < 1$, which gives a stability criterion for Δt. If we also want non-oscillating solutions (which are non-physical), we require $0 < C < 1$

$$\begin{aligned}
0 &< C \\
0 &< \frac{1 + B\Delta t(1-\theta)}{1 - \theta B \Delta t}.
\end{aligned}$$

Since B is negative, the denominator will always be greater than 1, so this inequality reduces to

$$\begin{aligned}
1 + B\Delta t(1-\theta) &\geq 0 \\
\Delta t &\leq -\frac{1}{B(1-\theta)}
\end{aligned} \tag{8}$$

Note that (8) always holds for Backward Euler, i.e. $\theta = 1$. The other inequality gives

$$\begin{aligned}
C &\leq 1 \\
\frac{1 + B\Delta t(1-\theta)}{1 - \theta B\Delta t} &\leq 1 \\
1 + B\Delta t - \theta B\Delta t &\leq 1 - \theta B\Delta t \\
1 + B\Delta t &\leq 1 \\
B\Delta t &\leq 0,
\end{aligned}$$

which also holds because $B < 0$. The divergence problem appears only when $C < -1$, which is taken care of by the first requirement.

4 Which one of the three schemes will you recommend for solving this equation with a) large time steps and b) small time steps?