Exercise 10

In this exercise we will look at a finite element approximation of the function $f(x) = \sin(x)$. We will work in the domain $\Omega = [0, \pi]$ and use two P1 elements, each if size $\pi/2$. The local P1 elements are given by the Lagrange polynomials

$$\tilde{\phi}_0(X) = \frac{1}{2}(1 - X)$$
$$\tilde{\phi}_1(X) = \frac{1}{2}(1 + X)$$

The affine transform is given by

$$x = x_m + \frac{1}{2}hX$$

where x_m is the midpoint in the interval and $h = \pi/2$. We have the midpoints for the elements $x_m^{e=1} = \pi/4$ and $x_m^{e=2} = 6\pi/4$. The jacobi determinants are the same, $\det J = h/2 = \pi/4$. Being general in mind, keeping the h factor, the coefficient matrix is given as

$$\begin{aligned}
\tilde{A}^(e)_{0,0} &= \int_{-1}^{1} \mathrm{d}X \det J \tilde{\phi}_0(X)\tilde{\phi}_0(X) \\
&= \frac{h}{2}\int_{-1}^{1} \mathrm{d}X \frac{1}{2}(1 - X)\frac{1}{2}(1 - X) \\
&= \frac{h}{8}\int_{-1}^{1} \mathrm{d}X(1 - X)^2 = \frac{h}{24}\Big[(1 - X)^3\Big]_{-1}^{1} = \frac{h}{3}.
\end{aligned}$$

Similarly we get that

$$\tilde{A}^(e)_{1,0} = A^(e)_{0,1} = \frac{h}{6}$$
$$\tilde{A}^(e)_{1,1} = A^(e)_{0,0} = \frac{h}{3},$$

so it finally looks like

$$M' = \begin{pmatrix} \frac{h}{3} & \frac{h}{6} \\ \frac{h}{6} & \frac{h}{3} \end{pmatrix}.$$

The right hand side, the vector b, is given as

$$\begin{aligned}
\tilde{b}_0^{(e)} &= \int_{-1}^{1} \mathrm{d}X f(x(X))\tilde{\phi}_0(X)\frac{h}{2} \\
&= \frac{h}{4}\int_{-1}^{1} \mathrm{d}X \sin(x_m^e + \frac{1}{2}hX)(1 - X)
\end{aligned}$$