The 1d wave equation looks like

$$u_{tt} = c^2 u_{xx}$$

a)

We use a finite difference scheme in the time dimension

$$\frac{u^{n-1} - 2u^n + u^{n+1}}{\Delta t^2} = c^2 u_{xx}$$

This can be solved for the next time step

$$u^{n+1} = \Delta t^2 c^2 u_{xx} - u^{n-1} + 2u^n$$

c)

We rearrange the last equation

$$u^{n+1} - \Delta t^2 c^2 u_{xx} + u^{n-1} - 2u^n = \mathcal{L}(u) = R$$

We want R to be orthogonal to the vector space $\mathcal{V}$, so we multiply by a basis function φ_i and integrate

$$\begin{aligned}
(R, \varphi_i) &= \int_0^L \mathrm{d}x R\varphi_i \\
&= \int_0^L \mathrm{d}x \Big[u^{n+1} - 2u^n + u^{n-1} - c^2 \Delta t^2 u'' \Big] \varphi_i \\
&= \int_0^L \mathrm{d}x u^{n+1}\varphi_i - 2\int_0^L \mathrm{d}x u^n \varphi_i + \int_0^L \mathrm{d}x u^{n-1}\varphi_i - c^2\Delta t^2 \int_0^L \mathrm{d}x u''\varphi_i,
\end{aligned}$$

where we use u'' instead of u_{xx}. We look at the second derivative term and apply integration by parts

$$\int_0^L \mathrm{d}x u''\varphi_i = u'\varphi_i\Big|_0^L - \int_0^L \mathrm{d}x u'\varphi_i' = -\int_0^L \mathrm{d}x u'\varphi_i' = -(u', \varphi_i'),$$

because the boundary terms are zero. We can then write the residual as

$$(R, \varphi_i) = (u^{n+1}, \varphi_i) - 2(u^n, \varphi_i) + (u^{n-1}, \varphi_i) + C^2(u', \varphi_i') = 0$$

where we have defined $C^2 = \Delta t^2 c^2$, using that the residual should be orthogonal to all basis vectors and evaluating each u at timestep n. By inserting $u = \sum_j c_j \varphi_j$, we get

$$(\sum_j c_j^{n+1}\varphi_j, \varphi_i) = 2(\sum_j c_j^n \varphi_j, \varphi_i) - (\sum_j c_j^{n-1}\varphi_j, \varphi_i) - C^2(\sum_j c_j^n \varphi_j', \varphi_i').$$

Since the P1 elements have the property that $u(x_i) = \sum_j c_j \varphi_j(x_i) = c_j \delta_{ij}$, we have

$$c_j^{n+1}(\varphi_j, \varphi_i) = 2c_j^n(\varphi_j, \varphi_i) - c_j^{n-1}(\varphi_j, \varphi_i) + C^2 c_j^n(\varphi_j', \varphi_i').$$

We recognize this as equation i in the matrix multiplication

$$Mc^{n+1} = 2Mc^n - Mc^{n-1} + C^2 Kc^n,$$

where $M_{ij} = (\varphi_i, \varphi_j)$ and $K_{ij} = (\varphi_i', \varphi_j')$. With the P1 elements, we then have the 5×5 M matrix given as

$$M = \frac{h}{6}\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

and the 5×5 K matrix looks like

$$K = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

$$\Big[D_t D_t (u + \frac{1}{6}\Delta x^2 D_x D_x u) = c^2 D_x D_x u\Big]_i^n \tag{1}$$

Stability analysis

We expand u in a discrete Fourier basis so that

$$u_p^n = \exp\Big[i(kp\Delta x - \tilde{\omega} n \Delta t)\Big] \tag{2}$$

Writing equation (1) on discretized form

$$\begin{aligned} 6C^2\Big[u_{i-1}^n - 2u_i^n + u_{i+1}^n\Big] &= 6(u_i^{n-1} - 2u_i^n + u_i^{n+1}) + (u_{i-1}^{n-1} - 2u_{i-1}^n + u_{i-1}^{n+1}) \\ &\quad - 2(u_i^{n-1} - 2u_i^n + u_i^{n+1}) + (u_{i+1}^{n-1} - 2u_{i+1}^n + u_{i+1}^{n+1}) \\ &= 4(u_i^{n-1} - 2u_i^n + u_i^{n+1}) + (u_{i-1}^{n-1} - 2u_{i-1}^n + u_{i-1}^{n+1}) \\ &\quad + (u_{i+1}^{n-1} - 2u_{i+1}^n + u_{i+1}^{n+1}) \end{aligned}$$

and inserting (2) into the LHS

$$\begin{aligned} L.H.S. &= 6C^2 e^{-i\tilde{\omega} n\Delta t}\Big[e^{ik(p-1)\Delta x} - 2e^{ikp\Delta x} + e^{ik(p+1)\Delta x}\Big] \\ &= 6C^2 e^{i(kp\Delta x - \tilde{\omega} n \Delta t)}\Big[e^{-ik\Delta x} - 2 + e^{ik\Delta x}\Big] \\ &= -24C^2 e^{i(kp\Delta x - \tilde{\omega} n\Delta t)} \sin^2\Big(\frac{k\Delta x}{2}\Big) \end{aligned}$$

and into RHS

$$\begin{aligned} R.H.S &= -16 e^{i(kp\Delta x - \tilde{\omega} n \Delta t)} \sin^2\Big(\frac{\tilde{\omega}\Delta t}{2}\Big) - 4e^{i(k(p-1)\Delta x - \tilde{\omega} n\Delta t)} \sin^2\Big(\frac{\tilde{\omega}\Delta t}{2}\Big) \\ &\quad - 4e^{i(k(p+1)\Delta x - \tilde{\omega} n \Delta t)} \sin^2\Big(\frac{\tilde{\omega}\Delta t}{2}\Big) \\ &= -4e^{i(kp\Delta x - \tilde{\omega} n\Delta t)} \sin^2\Big(\frac{\tilde{\omega}\Delta t}{2}\Big)\Big[e^{-ik\Delta x} + 4 + e^{ik\Delta x}\Big] \\ &= -8e^{i(kp\Delta x - \tilde{\omega} n \Delta t)} \sin^2\Big(\frac{\tilde{\omega}\Delta t}{2}\Big)\Big[\cos\Big(k\Delta x\Big) + 2\Big]. \end{aligned}$$

Combining both sides gives

$$\sin^2\Big(\frac{\tilde{\omega}\Delta t}{2}\Big) = \frac{3C^2}{\cos(k\Delta x) + 2}\sin^2\Big(\frac{k\Delta x}{2}\Big),$$

and by square rooting, we get

$$\sin\left(\frac{\tilde{\omega}\Delta t}{2}\right) = \frac{\sqrt{3}C}{\left[\cos(k\Delta x) + 2\right]^{\frac{1}{2}}} \sin\left(\frac{k\Delta x}{2}\right).$$

The sine on the right hand side has its maximum values ± 1 when $k\Delta x = (n+1)\pi$. The cosine in the denominator is then ± 1. The worst case scenario is if $\cos(k\Delta x) = -1$, where we have the stability criterion $C\sqrt{3} \leq 1$ giving

$$C \leq \frac{1}{\sqrt{3}},$$

because the sine on the left hand side must evaluate to less than 1. The solution -1 will not appear since we actually have a sine squared.