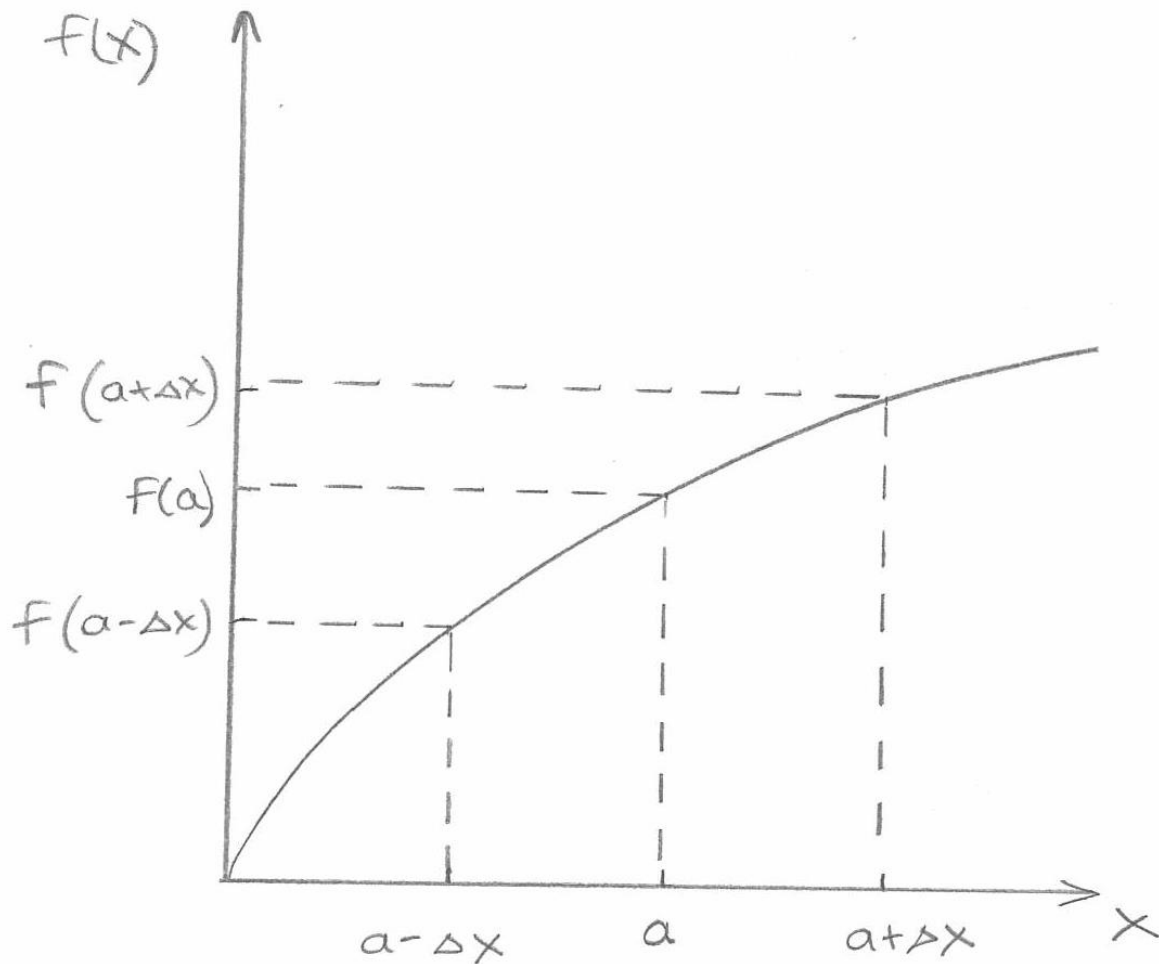


## Section 2: A Review of Finite Difference Approximations and Iterative Solvers

### Lecture 1: Finite Difference Methods



### Taylor-Series Expansions

$$f(a + \Delta x) = f(a) + \left. \frac{df}{dx} \right|_a \Delta x + \left. \frac{d^2 f}{dx^2} \right|_a \frac{\Delta x^2}{2!} + \left. \frac{d^3 f}{dx^3} \right|_a \frac{\Delta x^3}{3!} + O(\Delta x^4)$$

$$f(a - \Delta x) = f(a) - \left. \frac{df}{dx} \right|_a \Delta x + \left. \frac{d^2 f}{dx^2} \right|_a \frac{\Delta x^2}{2!} - \left. \frac{d^3 f}{dx^3} \right|_a \frac{\Delta x^3}{3!} + O(\Delta x^4)$$

What if we want the derivative of  $f$  evaluated at  $a$ ?

Subtract the backward expansion from the forward expansion:

$$f(a + \Delta x) - f(a - \Delta x) = 2 \left. \frac{df}{dx} \right|_a \Delta x + O(\Delta x^3)$$

Solve for the derivative:

$$\left. \frac{df}{dx} \right|_a = \frac{f(a + \Delta x) - f(a - \Delta x)}{2\Delta x} + O(\Delta x^2)$$

This is known as a second order accurate central difference approximation to the derivative at  $x=a$ .

First order forward and backward derivatives may be determined right from the Taylor series expansions as:

$$\left. \frac{df}{dx} \right|_a = \frac{f(a + \Delta x) - f(a)}{\Delta x} + O(\Delta x)$$

$$\left. \frac{df}{dx} \right|_a = \frac{f(a) - f(a - \Delta x)}{\Delta x} + O(\Delta x)$$

We could also develop an expression for the second derivative by adding the two Taylor series expansions. This results in:

$$\left. \frac{d^2 f}{dx^2} \right|_a = \frac{f(a + \Delta x) - 2f(a) + f(a - \Delta x)}{\Delta x^2} + O(\Delta x^2)$$

The “big  $O$ ” notation denotes a truncation error. What does it mean and how does it relate to CFD?

### Example

Consider the function  $f(x) = x^2$

Then  $f'(0) = 0$ .

If we evaluate  $f'(0)$  using our 2<sup>nd</sup> order accurate central finite difference approximation, with  $\Delta x = 1$ , we find

$$f'(0) = \frac{1 - 1}{2} = 0$$

which is the exact solution.

### Why do we get the exact solution?

If we evaluate  $f'(0)$  using the first order forward difference we find

$$f'(0) = \frac{1 - 0}{1} = 1$$

If we let  $\Delta x = 1/2$ , then

$$f'(0) = \frac{1/4 - 0}{1/2} = 1/2$$

Similarly, if we let  $\Delta x = 1/4$ ,  $f'(0) = 1/4$ .

What do we learn from this?

Second order method results in exact derivative.

**First order methods show error, which is decreased by a factor of 2 each time the mesh spacing is decreased by a factor of 2.**

Further discussion...

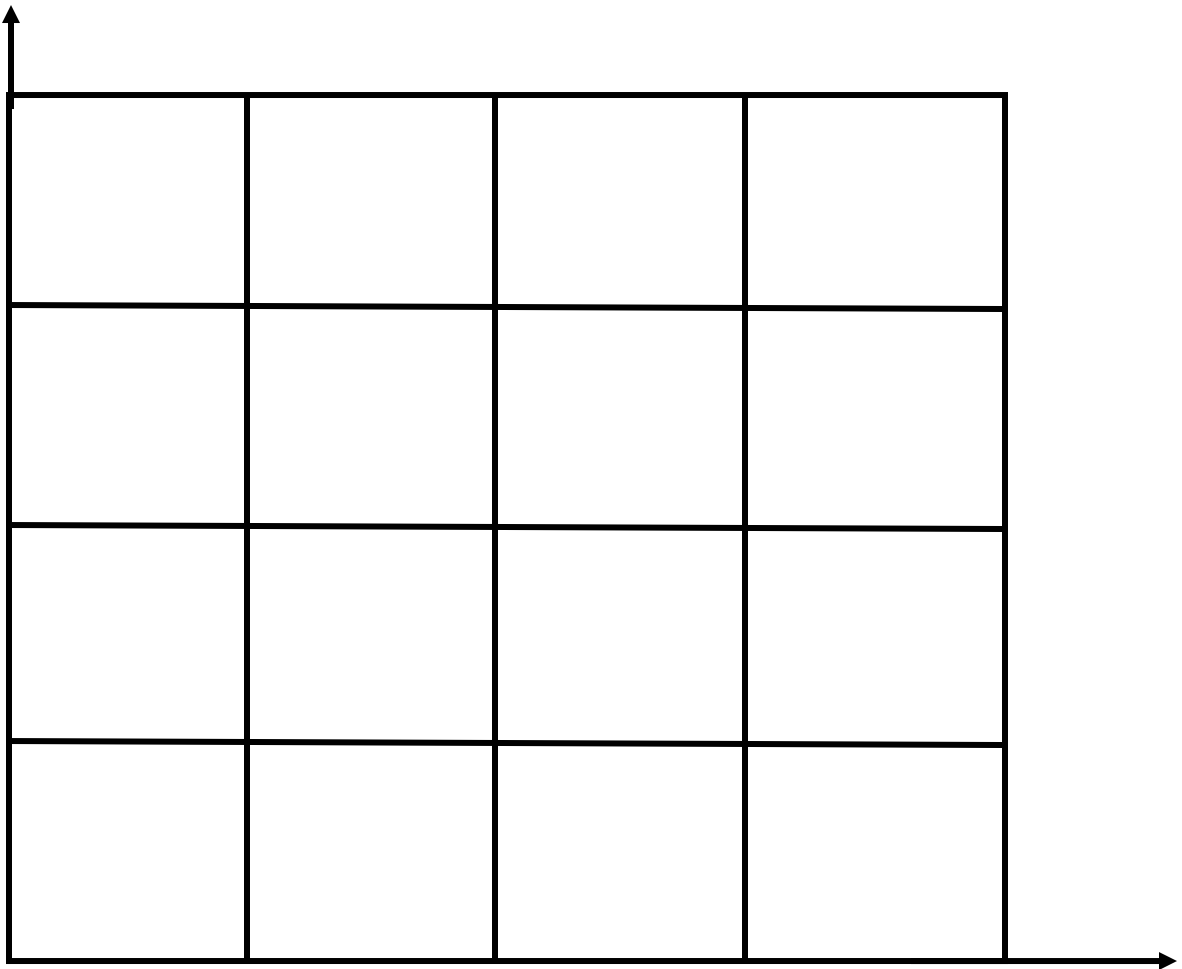
## Lecture 2: Iterative Solvers

### Point-by-point solvers

Jacobi, Gauss-Seidel, Successive Over Relation (SOR)

By Example:

Consider the following “finite difference” mesh:



We wish to determine the temperature at each of the interior points, given a fixed temperature at all the points along the boundaries.

The governing differential equation to solve is given as:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

It turns out that the temperature at any interior point is given approximately by the average of the temperatures at the 4 surrounding points.

$$T(i, j) = \frac{1}{4} (T(i + 1, j) + T(i - 1, j) + T(i, j + 1) + T(i, j - 1))$$

To solve for the temperature distribution, we “write” this equation at each interior node. This leads to two approaches: Direct Solvers (i.e., Gauss Elimination), and Iterative Solvers. We will use the iterative solver approach in this class.

Let’s place the above finite difference equation within a DO loops in the pseudo-code below:

```
DO OUTER_ITERATIONS=1,"Big Number"
```

```
DO I=1,3
```

```
DO J=1,3
```

$$T(i, j) = \frac{1}{4} (T(i + 1, j) + T(i - 1, j) + T(i, j + 1) + T(i, j - 1))$$

```
END DO
```

```
END DO
```

```
END DO
```

At this point we have a choice regarding how to treat the temperatures on the right side. That is, do we use the most recently computed values, or those from the previous outer iteration?

Depending on the choice, we get either a Jacobi method, or a Gauss-Seidel method.

To converge in fewer iterations, or to handle nonlinear problems, we can use an over/under relaxation technique:

$$T(i,j) = T(i,j) + \frac{\Omega}{4} (T(i+1,j) + T(i-1,j) + T(i,j+1) + T(i,j-1) - 4T(i,j))$$

Here,  $0 < \Omega < 2$  is an over/under relaxation factor.

This SOR technique is what we will use in the course. It's simple to implement and reasonably fast.

There are of course faster solvers, but that is not the emphasis of this course.

We are more interested in the formulation of the discretized equations, not solving in as fast a manner as possible.