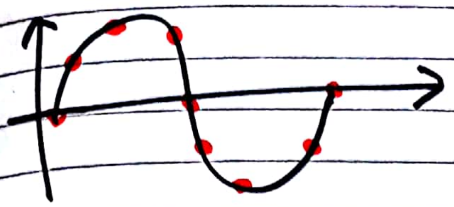
Numerical Method in Physical Simulation

(Prof. Heiner Igel, LMU, Coursera)

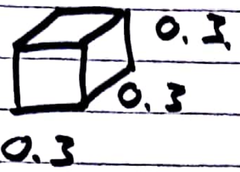
WEEK 1

Date

$c = 3\ km/s$; $c = \lambda f$; $f = 1\ Hz$



10 grid points per λ



0.3 grid volume.
0.3 grid spacing.

Simulating wave propagation in the entire earth:

radius of earth $= 6371\ km$

volume $= \frac{4}{3}\pi r_E^3$

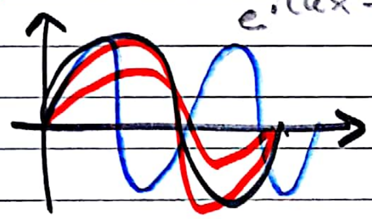
So there are $\frac{V_E}{(0.3)^3} = 4\times10^{13}$ grids ~ 8 bytes $\times 4\times10^{13}$

$= 320$ terabytes

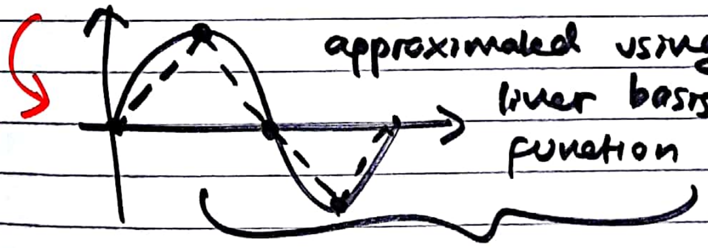
* Meshes
* Pseudospectral method
* Finite-element methods.

$u.v \rightarrow u'v + uv'$

$e^{i(kx-\omega t)}-$



approximated using linear basis function



techniques of discretizing wavelike problem

Acoustic wave equation: $\boxed{\partial_t^2 p(x,t) = c(x)^2 \Delta p(x,t)}$

if $c =$ constant, simple solution.

substitute

laplacian of pressure in 1D

1D : $\partial_x^2 p(x,t)$

Exercise:

$p(x,t) = p_0 e^{i(kx-\omega t)}$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} p_0 e^{i(kx-\omega t)}\right) = c(x)^2 \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} p_0 e^{i(kx-\omega t)}\right).$$

$$\frac{\partial}{\partial t}\left(-\omega e^{i(kx-\omega t)}\right) = c \frac{\partial}{\partial x}\Big($$

$$u.v. = u'v + uv'$$

Date

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} p_o e^{\overbrace{i(kx-\omega t)}^{}}\right) = c^2 \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} p_o e^{i(kx-\omega t)}\right).$$

$ikx - i\omega t$

$$\frac{\partial}{\partial t}\Big[\underbrace{e^{i(kx-\omega t)}}_{A} + \overbrace{p_o \cdot -i\omega\, e^{i(kx-\omega t)}}^{B}\Big] = c^2 \frac{\partial}{\partial x}\Big[\underbrace{e^{i(kx-\omega t)}}_{C} + \underbrace{p_o - ik\, e^{i(kx-\omega t)}}_{D}\Big]$$

$$A : \frac{\partial}{\partial t}\left[e^{i(kx-\omega t)}\right] = -i\omega e^{i(kx-\omega t)}$$

$$B : \frac{\partial}{\partial t}\left[p_o \cdot -i\omega e^{i(kx-\omega t)}\right] = e^{i(kx-\omega t)} - p_o i\omega \cdot -i\omega e^{i(kx-\omega t)}$$

$$= e^{i(kx-\omega t)} + p_o i^2 \omega^2 e^{i(kx-\omega t)}$$

$$i^2 = 1$$

$$C : \frac{\partial}{\partial x}\left[e^{i(kx-\omega t)}\right] = -ik e^{i(kx-\omega t)}$$

$$D : \frac{\partial}{\partial x}\left[p_o \cdot -ik e^{i(kx-\omega t)}\right] = e^{i(kx-\omega t)} - p_o ik \cdot -ik e^{i(kx-\omega t)}$$

$$= e^{i(kx-\omega t)} + p_o i^2 k^2 e^{i(kx-\omega t)}$$

$$i^2 = 1$$

$$A + B = c^2 (C + D)$$

$$-i\omega e^{i(kx-\omega t)} + e^{i(kx-\omega t)} + p_o \omega^2 e^{i(kx-\omega t)}$$

$$= c^2\left[-ik e^{i(kx-\omega t)} + e^{i(kx-\omega t)} + p_o k^2 e^{i(kx-\omega t)}\right]$$

$$\cancel{e^{i(kx-\omega t)}}\left[-i\omega + 1 + p_o\omega^2\right] = c^2 \cancel{e^{i(kx-\omega t)}}\left[-ik + 1 + p_o k^2\right]$$

$$-i\omega + 1 + p_o\omega^2 = c^2\left[-ik + 1 + p_o k^2\right]$$

Date

Result must be the dispersion relation:

$$\boxed{c = \frac{\omega}{k} = \frac{2\pi f}{k} = \frac{2\pi / T}{2\pi/\lambda} = \lambda f.}$$

* Another form of the equation $\partial_t^2 p(x,t) = c^2 \partial_x^2 p(x,t)$

$$\boxed{\partial_t p + c\,\partial_x p = 0} \Rightarrow \boxed{p_t + c p_x = 0}$$

$$A p_{xx} + B p_{xt} + C p_{tt} + D p_x + E p_t + F p = 0$$

↓ Fourier transform

$$A x^2 + B x t + C t^2 + D x + E t + F = 0$$

↓

discriminant types:

$B^2 - 4AC = 0 \rightarrow$ parabolic

$B^2 - 4AC < 0 \rightarrow$ eliptic

$B^2 - 4AC > 0 \rightarrow$ hyperbolic.

Apply to:

$$1\,\partial_t^2 p(x,t) = c^2 \partial_x^2 p(x,t)$$

↓

$p_{tt} - c^2 p_{xx} = 0$ ← $C p_{tt}$ (coeff of p_{tt}) is 1.

$B^2 - 4AC = 0$

$0 - 4(-c^2).1 = \boxed{4c^2 > 0}$ → thus is HYPERBOLIC

$A p_{xx}$ (coefficient of p_{xx} is c^2

$B p_{xt}$ (coefficient of p_{xt}) is 0.

* Introduction to Parallel Simulations

Architecture :

SISD	MISD
serial computer	cryptographic decoding.
SIMD	MIMD
GPU cluster, CM2	supercomputers PC cluster.

SISD : Single Instruction Single Data

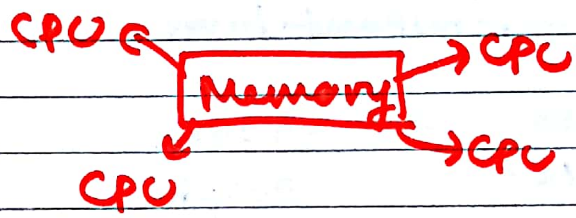
MISD : Multiple Instruction Single Data

SIMD : Single Instruction Multiple Data

$\boxed{\text{MIMD}}$: Multiple Instruction Multiple Data

type of memory

* Shared Memory. → several CPUs have access to the same memory space.



* Distributed memory.

← CPU ---- Network ---- CPU →
memory memory.

* Hybrid Distributed Shared Memory.

Programming with Python → using : Multiple – Passing Interface (MPI) library.

Demonstrate that your algorithm is efficiently scaling !

* Wave Equation:

$$\partial_t^2 p(x,t) = c(x)^2 \partial_x^2 p(x,t) + s(x,t)$$

Analytical solutions :

* if $c(x) = c_0$ and $s(x,t) = 0$
$p(x, t=0) = p_0$
$\partial_t p(x, t=0) = 0$

* if $s(x,t) = \delta(x-x_0)\delta(t-t_0)$
↳ impulse function

* Green's function characterises the impulse function

$$\partial_t^2 \boxed{G(x, t ; x_0, t_0)} - c^2 \Delta \boxed{G(x, t, x_0, t_0)}$$
$$= \delta(x - x_0)\, \delta(t - t_0)$$

* Elastic wave equation :

u_y displacement (particle motion perpendicular to wave propagation)

$$\boxed{\underbrace{\rho}_{\text{density}} \partial_t^2 u_y = \partial_x (\underbrace{\mu}_{\text{shear modulus}} \partial_x u_y) + f_y}$$

Homogenous medium :

① ρ and μ are independent of space

$$\boxed{\partial_t^2 u_y = \frac{\mu}{\rho} \partial_x^2 u_y + f_y} \; ; \; c = \sqrt{\frac{\mu}{\rho}}$$

More complicated for heterogeneous medium

* Plane wave equation

$$\boxed{p(x,t) = p_0 \sin(kx - \omega t)} \; ; \; \vec{k} \, , \; \vec{x}$$

position vector

wavenumber vector → pointing in the direction of wave propagation

$$p(x,t) = p_0 e^{i(kx - \omega t)}$$

* Wave equation is symmetric in time ! → reciprocity

* Time reversal → reverse acoustics.

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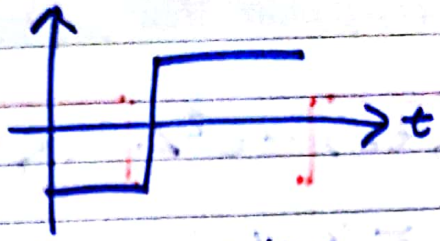
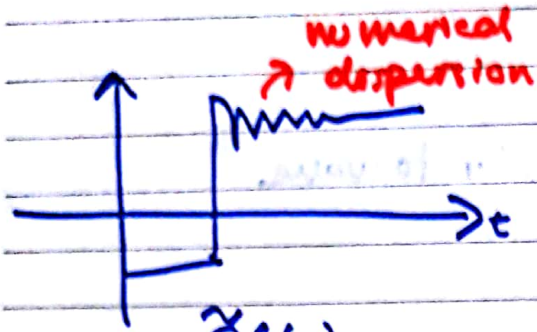
source function $s(t)$

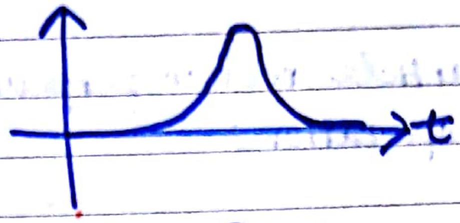


Green function
$G(t)$
heaviside



$\tilde{G}(t)$

$G(t)$ injected with delta function



$G(t) \otimes s(t)$
convolution

Quiz Week 1

① ~~1~~, 3, ~~4~~

② 2, ~~3~~

③ $c = 1490$ m/s

$f = 20$ kHz $= 20 \times 10^3$ Hz

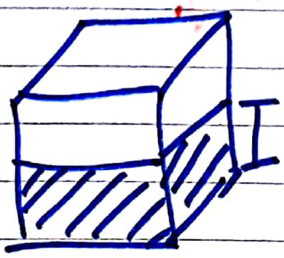
$\lambda = \dfrac{c}{f} = \dfrac{1490 \text{ m/s}}{20 \times 10^3 \text{ Hz}} = 0.0745$ m

grid spacing $= \dfrac{0.0745}{20}$

V block $= 100 \times 100 \times 10$

grid numbers $= \dfrac{100 \times 100 \times 10}{\left(\dfrac{0.0745}{20}\right)^3} = 1.93 \times 10^{12}$ grids (3)

④



$\frac{1}{2}$ height of box is sand

$c = 3$ km/s $= 3000$ m/s

$f = 20 \times 10^3$

$\lambda = 0.15$ m

grids spacing $= \dfrac{0.15 \text{ m}}{20}$

grid numbers $= \dfrac{100 \times 100 \times 5}{\left(\dfrac{0.15}{20}\right)^3}$

Date

$\frac{1}{2}$ of the box is water

$c = 1490$ m/s

grid spacing $= \dfrac{1490/20\times10^3}{20}$.

grid numbers $= \dfrac{100\times100\times5}{\left(\dfrac{0.0745}{20}\right)^3}$

total grid numbers multiplied by 8 bytes

$= 8.68\times10^{12}$ bytes

$= 8.68$ TB (1)

⑤ 4

⑥ coeff. A : ~~fxx~~ Pxx

B : Pxt

C : Ptt

$$\underset{C}{\partial_t}\phi = \alpha\,\underset{A}{\partial_x^2}\phi \Rightarrow \partial_t\phi - \alpha\,\partial_x^2\phi = 0$$

$$\underbrace{B^2 - 4AC}_{} = 0 - 4(-\alpha)(1) = 4\alpha > 0.$$

wave eq. is ~~HYPERBOLIC~~ PARABOLIC 2nd order