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SHORT COMMUNICATIONS

NUMERICAL SOLUTION OF HEAT CONDUCTION PROBLEMS BY PARABOLIC TIME–SPACE ELEMENT

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SUMMARY

In this paper, heat conduction problems are solved by a quasi-variational approach. A parabolic time–space element based on the above formulation is developed, and the stability of the above scheme is established. The results indicate that the scheme is suitable for various auxiliary conditions and that it is unconditionally stable.

INTRODUCTION

Many heat conduction problems have been analytically solved by Carslaw and Jaeger;[1] however, their solutions are only feasible for simple boundary conditions and geometrical shapes. Several numerical methods, such as finite difference and finite element methods, had been used to solve such problems. The finite element solution for the steady-state field problem was first presented by Zienkiewicz and Cheung,[2] while Wilson and Nickell[3] used Gurtin's variational principle[4] to solve the transient field problems. Zienkiewicz[5] proposed a Galerkin's formulation in place of the traditional finite difference formulation for the time-marching scheme. Other investigators[6–9] had also carried out studies for different time-marching schemes.

Real time–space finite elements were used by Argyris *et al.*[10,11] to analyse structural problems. Bruch[12] and his co-workers[13,14] presented the solution for one-dimensional problems, while Bruch and Zyvoloski[15] solved two-dimensional heat conduction problems by prismatic element with 8 nodes based on Galerkin's formulation. In both cases, the elements are only linear in the time domain.

In this paper, parabolic time–space elements (Figure 1) are used to solve heat conduction problems by quasi-variational formulation. Unlike the traditional three-point recurrence scheme,[5] which marched from prescribed values at two initial time-levels (t_n and t_{n-1}), the present scheme marched directly from the initial value at t_{n-1} and yield values for t_n and t_{n+1} simultaneously. In fact, the present approach can be generalized to other isoparametric elements and the use of any number of time stations within each time step Δt. Such a technique was used for one-dimensional problem be Cheung and Tham[16,17] and has been proved to be of advantage over least square and collocation least square formulation. The stability criterion for the above scheme was established and stability studies were made to demonstrate its validity.

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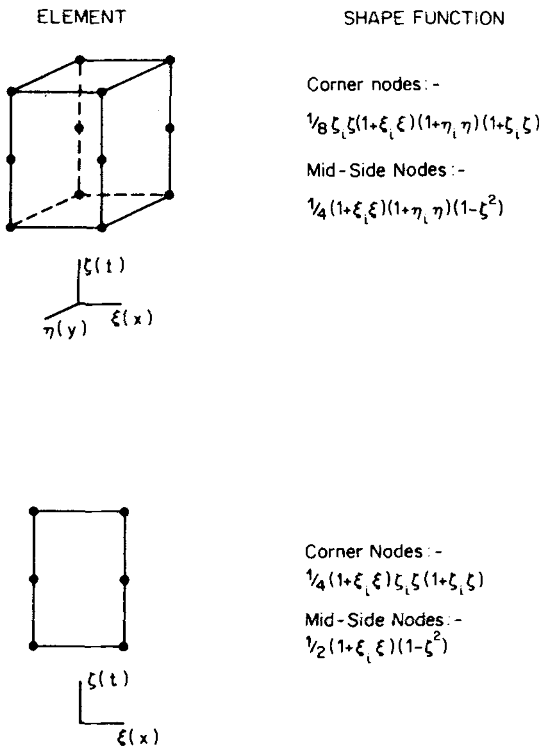

Figure 1. Parabolic time–space elements: (a) two-dimensional element; (b) one-dimensional element

PROBLEM

The general two-dimensional heat conduction equation is described by

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial\Phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial\Phi}{\partial y}\right)+Q-C\frac{\partial\Phi}{\partial t}=0 \tag{1}$$

and subject to the boundary conditions

$$\Phi=\Phi_B \quad \text{on } S_1\times[0,\infty) \tag{2a}$$

$$k_x\frac{\partial\Phi}{\partial x}l_x+k_y\frac{\partial\Phi}{\partial y}l_y+q+\alpha\Phi=0 \qquad \text{on } S_2\times[0,\infty) \tag{2b}$$

and the initial condition

$$\Phi(x,y,0)=\Phi_0(x,y) \tag{2c}$$

where $S=S_1+S_2$ is the boundary of the domain Ω ($R\times(0,\infty)$) (Figure 2), and l_x and l_y are the direction cosines of the outward normals to the boundary surface. The other quantities can be identified as Φ for the temperature, Q for the external heat input per unit volume, C for specific heat, and q and α for the boundary heat input per unit area and heat transfer coefficient, respectively. For the one-dimensional case, one of the spatial variables simply drops out.

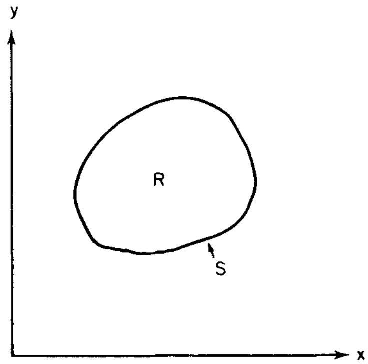

Figure 2. Spatial domain

THEORY

Using the quasi-variational approach, the functional χ^e for each element is written as

$$\chi^e = \int_{\Omega^e} \left(\frac{1}{2}\left(k_x\left(\frac{\partial \Phi}{\partial x}\right)^2 + k_y\left(\frac{\partial \Phi}{\partial y}\right)^2 \right) - \left(Q - \tfrac{1}{2}C\frac{\partial \Phi}{\partial t}\right)\Phi \right) \mathrm{d}\Omega + \int_{S_2^e} (q\Phi + \tfrac{1}{2}\alpha\Phi^2)\,\mathrm{d}s \qquad (3)$$

χ^e is minimized with respect to the nodal parameters by differentiating equation (3) such that

$$\frac{\partial \chi^e}{\partial \Phi_i} = \int_{\Omega^e} \left(k_x \frac{\partial \Phi}{\partial x}\frac{\partial}{\partial \Phi_i}\left(\frac{\partial \Phi}{\partial x}\right) + k_y \frac{\partial \Phi}{\partial y}\frac{\partial}{\partial \Phi_i}\left(\frac{\partial \Phi}{\partial y}\right) - \left(Q - \tfrac{1}{2}C\frac{\partial \Phi}{\partial t}\right)\frac{\partial \Phi}{\partial \Phi_i}\right) \mathrm{d}\Omega$$

$$+ \int_{S_2^e} \left(q\frac{\partial \Phi}{\partial \Phi_i} + \alpha\Phi\frac{\partial \Phi}{\partial \Phi_i}\right) \mathrm{d}s = 0 \qquad (4)$$

or

$$\int_{\Omega^e} \left(k_x \left\{\frac{\partial N}{\partial x}\right\}^{\mathrm{T}} \left\{\frac{\partial N}{\partial x}\right\} + k_y \left\{\frac{\partial N}{\partial y}\right\}^{\mathrm{T}} \left\{\frac{\partial N}{\partial y}\right\} + C\{N\}^{\mathrm{T}}\left\{\frac{\partial N}{\partial t}\right\}\right)\{\Phi\}^e\,\mathrm{d}\Omega$$

$$- \int_{\Omega^e} Q\{N\}^{\mathrm{T}}\,\mathrm{d}\Omega + \int_{S_2^e} (q\{N\}^{\mathrm{T}} + \alpha\{N\}^{\mathrm{T}}\{N\}\{\Phi\}^e)\,\mathrm{d}s = 0 \qquad (4a)$$

Note that the shape function matrix $\{N\}$ is now a function of both space (x and y) and time (t).

The above equation (equation 4a) can be written in the matrix form as

$$[H_1]^e\{\Phi\}^e + [H_2]^e\{\Phi\}^e - \{F_1\}^e + \{F_2\}^e = 0 \qquad (5)$$

in which

$$[H_1]^e = \int_{\Omega^e} \left(k_x \left\{\frac{\partial N}{\partial x}\right\}^{\mathrm{T}} \left\{\frac{\partial N}{\partial x}\right\} + k_y \left\{\frac{\partial N}{\partial y}\right\}^{\mathrm{T}} \left\{\frac{\partial N}{\partial y}\right\} + C\{N\}^{\mathrm{T}}\left\{\frac{\partial N}{\partial t}\right\}\right) \mathrm{d}\Omega \qquad (6)$$

$$[H_2]^e = \int_{S_2^e} \alpha\{N\}^{\mathrm{T}}\{N\}\,\mathrm{d}s \qquad (7)$$

$$\{F_1\}^e = \int_{\Omega^e} -Q\{N\}^{\mathrm{T}}\,\mathrm{d}\Omega \qquad (8)$$

$$\{F_2\}^e = \int_{S^e} q\{N\}^{\mathrm{T}}\,\mathrm{d}s \qquad (9)$$

Minimizing the functional over the whole domain with respect to the nodal parameters is simply summing of equation (5) for all elements, that is

$$[H]\{\Phi\}+\{F\}=0 \tag{10}$$

for which

$$[H]=\Sigma([H_1]^e+[H_2]^e)$$
$$\{F\}=\Sigma(\{F_1\}^e+\{F_2\}^e)$$

If the boundary condition along S_1 and the initial conditions are applied, equation (10) can be solved for the unknowns at time Δt. At each new time step, the results of the previous time step are taken as the initial conditions and the procedure repeated.

STABILITY ANALYSIS

The stability of the scheme is established according to the method given by Zienkiewicz.[5] It will be shown in Appendix I that the parabolic time–space element presented herein is equivalent to the coupling of the linear acceleration and the Galerkin's scheme. In general, the characteristic equation for the tree-points recurrence scheme[5] is

$$(\gamma+\beta\rho_i)(\alpha_i)_{n+1}+[(1-2\gamma)+(\tfrac{1}{2}-2\beta+\gamma)\rho_i](\alpha_i)_n+[-(1-\gamma)+(\tfrac{1}{2}+\beta-\gamma)\rho_i](\alpha_i)_{n-1}=0$$

where γ, β are parameters depending on the scheme, and

$\rho_i=\Delta t\omega_i$

$\omega_i=$ eigenvalues of the natural frequency of the system.

$(\alpha_i)_{n+1}$, $(\alpha_i)_n$, $(\alpha_i)_{n-1}=$ modal participation variables for natural frequency ω_i at t_{n+1}, t_n, t_{n-1}, respectively.

The parameters γ, β for the above two schemes are given in Table I, and by substituting in the appropriate parameters into equation (11), it becomes

(i) For the Galerkin scheme:

$$(\tfrac{3}{2}+\tfrac{4}{5}\rho_i)(\alpha_i)_{n+1}+(-2+\tfrac{2}{5}\rho_i)(\alpha_i)_n+(\tfrac{1}{2}-\tfrac{1}{5}\rho_i)(\alpha_i)_{n-1}=0 \tag{11a}$$

(ii) For the linear acceleration scheme:

$$(\tfrac{1}{2}+\tfrac{1}{10}\rho_i)(\alpha_i)_{n+1}+(\tfrac{4}{5}\rho_i)(\alpha_i)_n+(-\tfrac{1}{2}+\tfrac{1}{10}\rho_i)(\alpha_i)_{n-1}=0 \tag{11b}$$

Eliminating $(\alpha_i)_n$ from the above two equations, the relation between $(\alpha_i)_{n+1}$ and $(\alpha_i)_{n-1}$ can be established as

$$(1+\tfrac{6}{5}\rho_i+\tfrac{15}{25}\rho_i^2)(\alpha_i)_{n+1}=(1-\tfrac{4}{5}\rho_i+\tfrac{5}{25}\rho_i^2)(\alpha_i)_{n-1} \tag{12}$$

Table I. The parameters γ and β for the recurrence fumulae

Scheme	γ	β
Linear acceleration	1/2	1/10
Galerkin	3/2	4/5

If $(\alpha_i)_{n+1} = \lambda_i^{n+1}(\alpha_i)_{n-1}$, then equation (12) can be written as

$$(1+\tfrac{6}{5}\rho_i+\tfrac{15}{25}\rho_i^2)\lambda_i^{n+1} = (1-\tfrac{4}{5}\rho_i+\tfrac{5}{25}\rho_i^2) \tag{12a}$$

For the above scheme to be stable, $|\lambda_i^{n+1}|<1$, that is

$$1+\tfrac{6}{5}\rho_i+\tfrac{15}{25}\rho_i^2 > 1-\tfrac{4}{5}\rho_i+\tfrac{5}{25}\rho_i^2$$

or

$$\rho_i(\tfrac{2}{5}\rho_i+2)>0 \tag{13}$$

Furthermore, by eliminating $(\alpha_i)_{n+1}$, a similar relationship can be established:

$$1+\tfrac{6}{5}\rho_i+\tfrac{15}{25}\rho_i^2 > 1+\tfrac{4}{20}\rho_i-\tfrac{5}{50}\rho_i^2$$

or

$$\rho_i(\tfrac{7}{10}\rho_i+1)>0 \tag{14}$$

Combining the above two criteria (equations 13 and 14), it is easy to conclude that the system is unconditionally stable.

NUMERICAL EXAMPLES

Heat conduction problems governed by equation (1) but subject to different auxiliary conditions were solved by the above method.

Example 1

A one-dimensional problem is solved to test the above stability criteria. The problem is subjected to the following auxiliary conditions:

$$\Phi(0, t) = \Phi(L_x, t) = 0$$

and

$$\Phi(x, 0) = 1{\cdot}0$$

where L_x is the length in the x-direction. The closed form solution is

$$\Phi(x, t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)}\sin\frac{(2n+1)x}{L_x}\exp\left(\frac{-k_x(2n+1)^2\pi^2}{L_x^2}t\right) \tag{15}$$

In Table II, the results for different $\Delta t/\Delta x^2$ are given, respectively. The results indicates that the time–space element is stable for all time steps.

Table II. Temperature distribution at $T = 8{\cdot}0$ by time–space element for Example 1 ($k_x = 1{\cdot}0$; $l_x = 10{\cdot}0$)

		$\Delta t/\Delta x^2$					
x	Analytical	0·1	0·2	0·5	0·8	1·0	2·0
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
1·0	0·1789	0·1762	0·1762	0·1762	0·1763	0·1763	0·1836
2·0	0·3401	0·3350	0·3350	0·3350	0·3351	0·3351	0·3304
3·0	0·4678	0·4609	0·4609	0·4609	0·4610	0·4612	0·4622
4·0	0·5496	0·5416	0·5416	0·5417	0·5418	0·5418	0·5430
5·0	0·5778	0·5694	0·5694	0·5695	0·5696	0·5697	0·5702

Example 2

If the conductivities k are dependent on the temperature, equation (1) becomes nonlinear. In this study, a nonlinear example based on the following equation is considered:

$$\frac{\partial}{\partial x}\left(\Psi_1\Phi\frac{\partial\Phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Psi_2\Phi\frac{\partial\Phi}{\partial y}\right)+Q=C\frac{\partial\Phi}{\partial t}$$

For simplicity, Ψ_1, Ψ_2 and also C are taken as unity, while Q is taken as zero and the auxiliary conditions are given by

$$\Phi(0, y, t)=\Phi(x, 0, t)=\Phi(L_x, y, t)=\Phi(x, L_y, t)=1{\cdot}0$$
$$\Phi(x, y, 0)=0{\cdot}1$$

The parabolic time–space element is used in conjunction with an iterative process[13] to solve the above problem in a unit square domain. The results at $T=1{\cdot}0$ are given in Table III and they are compared with a semi-analytical solution.[18] No simple analytical solution is available for such a case.

Table III. Temperature distribution at $t=0{\cdot}1$ hr, using finite element method with $\Delta T=0{\cdot}05$ hr (Example 2)

	Finite element method		
1·000	1·000	1·000	1·000
1·000	0·941	0·902	0·871
1·000	0·902	0·812	0·744
1·000	0·871	0·744	0·647
1·000	0·871	0·744	0·647
1·000	0·902	0·812	0·744
1·000	0·941	0·902	0·871
1·000	1·000	1·000	1·000
	Semi-analytical method[14]		
1·000	1·000	1·000	1·000
0·814	0·903	0·949	1·000
0·742	0·807	0·903	1·000
0·642	0·742	0·874	1·000
0·642	0·742	0·874	1·000
0·742	0·807	0·903	1·000
0·814	0·903	0·949	1·000
1·000	1·000	1·000	1·000

CONCLUSION

The parabolic time–space element has been proved to be suitable for time-marching problems. It has been demonstrated that such a direct time–space formulation yields stable solutions and is an attractive alternative from the traditional time-marching schemes based on finite difference or Galerkin's formulation.

APPENDIX I

In this appendix, the recurrence formula is derived for the one-dimensional case. The 'stiffness' matrix for the parabolic time–space element is

$$\left[\begin{array}{cc|c|c|cc}
\frac{4}{15\Delta x^2}-\frac{4}{9\Delta t} & \frac{-4}{15\Delta x^2}-\frac{2}{9\Delta t} & \frac{32}{15\Delta x^2} & \frac{-32}{15\Delta x^2} & \frac{4}{15\Delta x^2}+\frac{4}{9\Delta t} & \frac{-4}{15\Delta x^2}+\frac{2}{9\Delta t} \\
\frac{-4}{15\Delta x^2}-\frac{2}{9\Delta t} & \frac{4}{15\Delta x^2}-\frac{4}{9\Delta t} & \frac{-32}{15\Delta x^2} & \frac{32}{15\Delta x^2} & \frac{-4}{15\Delta x^2}+\frac{2}{9\Delta t} & \frac{4}{15\Delta x^2}+\frac{4}{9\Delta t} \\
\hline
\frac{-2}{15\Delta x^2}+\frac{1}{9\Delta t} & \frac{2}{15\Delta x^2}+\frac{1}{18\Delta t} & \frac{4}{15\Delta x^2}-\frac{4}{9\Delta t} & \frac{-4}{15\Delta x^2}-\frac{2}{9\Delta t} & \frac{8}{15\Delta x^2}+\frac{1}{3\Delta t} & \frac{-8}{15\Delta x^2}+\frac{1}{6\Delta t} \\
\frac{2}{15\Delta x^2}+\frac{1}{18\Delta t} & \frac{-2}{15\Delta x^2}+\frac{1}{9\Delta t} & \frac{-4}{15\Delta x^2}-\frac{2}{9\Delta t} & \frac{4}{15\Delta x^2}-\frac{4}{9\Delta t} & \frac{-8}{15\Delta x^2}+\frac{1}{6\Delta t} & \frac{8}{15\Delta x^2}+\frac{1}{3\Delta t}
\end{array}\right] \tag{16}$$

The first two rows are eliminated by applying the initial conditions, and equation (10) can be written as (the 'loading' term is taken as zero in the stability analysis)

$$\left[\begin{array}{c|c}
\frac{4}{15\Delta x^2}-\frac{4}{9\Delta t} & \frac{-4}{15\Delta x^2}-\frac{2}{9\Delta t} \\
\hline
\frac{-4}{15\Delta x^2}-\frac{2}{9\Delta t} & \frac{4}{15\Delta x^2}-\frac{4}{9\Delta t}
\end{array}\right]
\begin{bmatrix}\Phi^1 \\ \Phi^2\end{bmatrix}_{n-1}
+\left[\begin{array}{c|c}
\frac{32}{15\Delta x^2} & \frac{-32}{15\Delta x^2} \\
\hline
\frac{-32}{15\Delta x^2} & \frac{32}{15\Delta x^2}
\end{array}\right]
\begin{bmatrix}\Phi^1 \\ \Phi^2\end{bmatrix}_{n}
+\left[\begin{array}{c|c}
\frac{4}{15\Delta x^2}+\frac{4}{9\Delta t} & \frac{-4}{15\Delta x^2}+\frac{2}{9\Delta t} \\
\hline
\frac{-4}{15\Delta x^2}+\frac{2}{9\Delta t} & \frac{4}{15\Delta x^2}+\frac{4}{9\Delta t}
\end{array}\right]
\begin{bmatrix}\Phi^1 \\ \Phi^2\end{bmatrix}_{n+1}=0 \tag{17}$$

and

$$\left[\begin{array}{c|c}
\frac{-2}{15\Delta x^2}+\frac{1}{9\Delta t} & \frac{2}{15\Delta x^2}+\frac{1}{18\Delta t} \\
\hline
\frac{2}{15\Delta x^2}+\frac{1}{18\Delta t} & \frac{-2}{15\Delta x^2}+\frac{1}{9\Delta t}
\end{array}\right]
\begin{bmatrix}\Phi^1 \\ \Phi^2\end{bmatrix}_{n-1}
+\left[\begin{array}{c|c}
\frac{4}{15\Delta x^2}-\frac{4}{9\Delta t} & \frac{-4}{15\Delta x^2}-\frac{2}{9\Delta t} \\
\hline
\frac{-4}{15\Delta x^2}-\frac{2}{9\Delta t} & \frac{4}{15\Delta x^2}-\frac{4}{9\Delta t}
\end{array}\right]
\begin{bmatrix}\Phi^1 \\ \Phi^2\end{bmatrix}_{n}
+\left[\begin{array}{c|c}
\frac{8}{15\Delta x^2}+\frac{1}{3\Delta t} & \frac{-8}{15\Delta x^2}+\frac{1}{6\Delta t} \\
\hline
\frac{-8}{15\Delta x^2}+\frac{1}{6\Delta t} & \frac{8}{15\Delta x^2}+\frac{1}{3\Delta t}
\end{array}\right]
\begin{bmatrix}\Phi^1 \\ \Phi^2\end{bmatrix}_{n+1}=0 \tag{18}$$

The eigenvalues for each of the square matrices can be easily determined and the corresponding eigenvectors obtained. It can be shown that the eigenvectors for all square matrices are $(1,-1)^{\mathrm{T}}$ and $(1,1)^{\mathrm{T}}$. By premultiplying equation (17) and equation (18) by $(1,-1)$, the equations will be decomposed into modal parameter as the modal decomposition method in the structural mechanics, and they can be written as

$$\left(\frac{8}{15\Delta x^2}+\frac{2}{9\Delta t}\right)(\alpha_i)_{n+1}+\frac{64}{15\Delta x^2}(\alpha_i)_n+\left(\frac{8}{15\Delta x^2}-\frac{2}{9\Delta t}\right)(\alpha_i)_{n-1}=0 \qquad (17a)$$

$$\left(\frac{16}{15\Delta x^2}+\frac{1}{6\Delta t}\right)(\alpha_i)_{n+1}+\left(\frac{8}{15\Delta x^2}-\frac{2}{9\Delta t}\right)(\alpha_i)_n+\left(\frac{-4}{15\Delta x^2}+\frac{1}{18\Delta t}\right)(\alpha_i)_{n-1}=0 \qquad (18a)$$

After same modification, and noting that for the one-dimensional case $\rho_i = 12/\Delta x^2$, the recurrence equations (equations 11a and 11b) can be easily obtained. The other modal parameters will be no interest in the stability analysis as it will not involve the time step size in the recurrence equations.

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