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Quasi-implicit and two-level explicit finite-difference procedures for solving the one-dimensional advection equation

Mehdi Dehghan

Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Avenue, Tehran, Iran

Abstract

A variety of explicit and implicit algorithms has been studied dealing with the solution of the one-dimensional advection equation. These schemes are based on the weighted finite difference approximations. This solution approach proves to be very effective. This procedure is carried out in many disciplines including hydrology, oceanography and meteorology. The main idea behind the finite-difference methods for obtaining the solution of a given partial differential equation is to approximate the derivatives appearing in the equation by a set of values of the function at a selected number of points. The most usual way to generate these approximations is through the use of Taylor series. The basis of analysis of the finite-difference equations considered here is the modified equivalent partial differential equation approach. It is worth noting that from the truncation error of the modified equivalent equation, it is possible to eliminate the dominant error terms associated with the finite-difference equations that contain weights, thus leading to more accurate techniques. Quasi-implicit and two-level explicit finite-difference techniques are considered in this article to approximately solve the one-dimensional advection problem. Stability of the techniques are investigated using the von Neumann stability analysis. The results of a numerical experiment are presented, and the accuracy and central processor unit (CPU) time needed are discussed

E-mail address: mdehghan@cic.aku.ac.ir

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and compared. Numerical tests show that the new third-order or fourth-order methods are more accurate than either of the second-order techniques.

Keywords: Two-level explicit techniques; Quasi-implicit schemes; Advection processes; CFL condition; Numerical differentiation; Implicit finite-difference formulae; Modified equivalent partial differential equations; Stability; The order of accuracy; Courant number

1. Introduction

This paper is devoted to study the one-dimensional advection equation

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = 0, \quad 0 < x < 1, \ 0 < t \leq T, \tag{1}$$

with initial condition

$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant 1 \tag{2}$$

and boundary conditions

$$u(0,t) = g_0(t), \quad 0 < t \leqslant T, \tag{3}$$

$$f(0) = g_0(0), \tag{4}$$

where f, and g_0 are known functions, while the function u is unknown. Note that $\beta > 0$ is considered to be a positive constant quantifying the advection process. Note that in solving initial-boundary value problems such as (1) by finite difference schemes, we must use the boundary conditions required by the partial differential equation in order to determine the finite difference solution. Many schemes also require additional boundary conditions called, numerical boundary conditions, to determine the solution uniquely. Particularly, numerical boundary condition should be some form of extrapolation that determines the solution on the boundary in terms of the solution in the interior [20]. It is worth pointing out that we assume that f and g_0 have the necessary conditions for the existence and uniqueness of the solution of our one-dimensional advection equation.

Eq. (1) is a linearized one-dimensional version of the partial differential equations which describe advection of quantities such as mass, heat, energy, vorticity, etc. [1,2,4–6,8,9,17]. Mass, heat and momentum transport in moving fluids may be simulated on digital computers by numerically solving the governing partial differential equations. This procedure is carried out in many disciplines including hydrology, oceanography and meteorology [10,17,19,22].

Because the velocity field is generally computed on a fixed grid, it is necessary to accurately model the advective contribution to the transport process on such a grid. Unfortunately, like other numerical techniques finite-difference methods seldom model accurately advective terms in the governing Eulerian equations, errors in speed of propagation of component waves being common.

Consequently, over the last 30 years, much effort has been put into developing stable and accurate numerical solutions of (1). In the present research various numerical schemes will be developed and compared for solving this equation [3,4,7,12,13,15,16,20].

This paper is divided into seven sections:

The main idea behind the finite-difference techniques is given in Section 2. The two-level finite-difference methods for the solution of (1)–(4) are described in Section 3. Two different quasi-implicit techniques are introduced in Section 4. Some fourth-order explicit finite-difference formulae are given in Section 3. Four different implicit procedures are discussed in Section 5. The accuracy and efficiency of the methods developed are also presented in Sections 3–5. For each technique the modified equivalent equation is investigated. Thus using the truncation error of the corresponding modified equivalent equation, it is possible to eliminate the dominant error terms associated with the finite-difference equations that contain free parameters. The resulted high-order techniques are given in Sections 3 and 5. The results of a numerical experiment are presented in Section 6. Section 7 is devoted to a brief conclusion. Finally some references are introduced at the end.

2. The finite-difference schemes

The main idea behind the finite-difference methods for obtaining the solution of a given partial differential equation is to approximate the derivatives appearing in the equation by a set of values of the function at a selected number of points. The most usual way to generate these approximations is through the use of Taylor series. The numerical techniques developed here are based on the modified equivalent partial differential equation as described by Warming and Hyett [21] which allows the simple determination of the theoretical order of accuracy, thus allowing methods to be compared with one another.

The solution domain of the problem is covered by a mesh of grid-lines:

$$x_i = i\Delta x, \quad i = 0, 1, 2, \ldots, M, \tag{5}$$

$$t_n = n\Delta t, \quad n = 0, 1, 2, \ldots, N, \tag{6}$$

parallel to the space and time coordinate axes, respectively. Approximations u_i^n to $u(i\Delta x, n\Delta t)$ are calculated at the point of intersection of these lines, namely, $(i\Delta x, n\Delta t)$ which is referred to as the (i,n) grid-point. The constant spatial and temporal grid-spacing are $\Delta x = \frac{1}{M}$ and $\Delta t = \frac{T}{N}$, respectively.

3. The two-level explicit methods

Consider the following approximations of the derivatives in the advection Eq. (1) which incorporate weights θ, ϕ and γ in the following form:

$$\left.\frac{\partial u}{\partial t}\right|_i^n + \beta \left.\frac{\partial u}{\partial x}\right|_i^n = 0, \tag{7}$$

$$\left.\frac{\partial u}{\partial t}\right|_i^n \simeq \frac{(u_i^{n+1} - u_i^n)}{\Delta t}, \tag{8}$$

$$\begin{aligned}\left.\frac{\partial u}{\partial x}\right|_i^n \simeq{}& \theta \frac{(u_i^n - u_{i-1}^n)}{\Delta x} + \phi \frac{(3u_i^n - 4u_{i-1}^n + u_{i-2}^n)}{2\Delta x} \\ &+ \gamma \frac{(-u_{i+2}^n + 8u_{i+1}^n - 8u_{i-1}^n + u_{i-2}^n)}{12\Delta x} + (1 - \phi - \theta - \gamma)\frac{(u_{i+1}^n - u_{i-1}^n)}{2\Delta x}.\end{aligned} \tag{9}$$

This gives the following weighted explicit finite-difference formula:

$$\begin{aligned}u_i^{n+1} ={}& \frac{-c}{12}(6\phi + \gamma)u_{i-2}^n + \frac{c}{6}(3 + 3\theta + 9\phi + \gamma)u_{i-1}^n \\ &+ \frac{-c}{6}(3 - 3\theta - 3\phi + \gamma)u_{i+1}^n + \frac{c\gamma}{12}u_{i+2}^n + \frac{1}{2}(2 - 3c\phi - 2c\theta)u_i^n,\end{aligned} \tag{10}$$

for $0 \leqslant n \leqslant N - 1$, where

$$c = \beta \frac{(\Delta t)}{(\Delta x)}. \tag{11}$$

This is usually referred as the Courant number.

It can be seen that this technique incorporates numerical diffusion with a coefficient of $\frac{\beta \Delta x}{2}(\theta - c)$. Hence $\theta \geqslant c$, is a necessary (not sufficient) condition for stability.

The modified equivalent partial differential equation of this method is in the following form [21]:

$$\begin{aligned}&\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} + \frac{\beta(\Delta x)}{2}(c - \theta)\frac{\partial^2 u}{\partial x^2} + \frac{\beta(\Delta x)^2}{3!}(1 + 2c^2 - \gamma - 3\phi - 3c\theta)\frac{\partial^3 u}{\partial x^3} \\ &\quad + \frac{\beta(\Delta x)^3}{4!}\left[4c + 6c^3 - 4c\gamma + 6\phi(1 - 2c) - \theta(1 + 12c^2) + 3c\theta^2\right]\frac{\partial^4 u}{\partial x^4} \\ &\quad + \frac{\beta(\Delta x)^4}{5!}\left[-15\theta(1 + 4c^2) + 10\theta c\gamma + 30c\theta\phi + 30c^2\theta^2\right]\frac{\partial^5 u}{\partial x^5} + \mathrm{O}\{(\Delta x)^5\} \\ &= 0.\end{aligned} \tag{12}$$

Choice of certain values for the weights θ, ϕ and γ produces some of the finite-difference formulae already developed in the literature.

3.1. The FTCS-type scheme

Putting $\phi = \theta = \gamma = 0$ in (10) gives the following FTCS-type finite-difference formula for solving the advection equation:

$$u_i^{n+1} = \frac{c}{2}(u_{i-1}^n - u_{i+1}^n) + u_i^n, \tag{13}$$

for $i = 1, 2, \ldots, M - 1$.

It can be easily seen that this scheme is unstable. This means that this method is of no practical use.

The modified equivalent partial differential equation of this method is in the following form:

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} + \frac{c\beta(\Delta x)}{2}\frac{\partial^2 u}{\partial x^2} + \mathrm{O}\{(\Delta x)^2\} = 0. \tag{14}$$

3.2. The Lax–Friedrichs technique

By replacing the term u_i^n in (13) with the average of the values u_{i-1}^n and u_{i+1}^n, the Lax's finite-difference formula is obtained.

$$u_i^{n+1} = \frac{1}{2}(1 + c)u_{i-1}^n + \frac{1}{2}(1 - c)u_{i+1}^n. \tag{15}$$

It can be seen that this scheme is stable for

$$0 < c \leqslant 1. \tag{16}$$

The modified equivalent partial differential equation which corresponds to the finite-difference formula (15) consistent with the pure advection equation can be written in the following form:

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} - \frac{\beta(\Delta x)(1 - c^2)}{2c}\frac{\partial^2 u}{\partial x^2} + \mathrm{O}\{(\Delta x)^2\} = 0. \tag{17}$$

Note that Eq. (17) shows that this finite-difference scheme is free of numerical diffusion. This technique was first developed by Lax and Richtmyer [13].

3.3. The upwind explicit formula

Setting $\theta = 1$, and $\phi = \gamma = 0$ in (10) gives the following upwind-type finite-difference explicit technique:

$$u_i^{n+1} = cu_{i-1}^n + (1-c)u_i^n, \tag{18}$$

which is stable for

$$0 < c \leqslant 1. \tag{19}$$

This condition is usually referred to as the Courant–Friedrichs–Lewy (CFL) condition. It requires that $\beta\Delta t \leqslant \Delta x$, which means that the fluid advecting the property u should not travel more than one grid spacing in the x direction in one time step.

This scheme was developed by Courant, Isaacson and Rees in order to overcome instability of the FTCS-type technique [3].

This method has been applied under various names. Oceanographers and hydrologists call this upstream differencing. Meteorologists refer to upwind differencing (stabilizing effect of weather). Mathematicians call this finite-difference formula with positive coefficients.

If $c = 1$, the upwind formula (18) becomes

$$u_i^{n+1} = u_{i-1}^n. \tag{20}$$

Therefore, application of (18) gives the exact solution of (1). That is using a Courant number of one in the upwind finite-difference formula gives the exact solution to the advection equation.

The modified equivalent partial differential equation of this method is in the following form:

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} - \frac{\beta(\Delta x)(1-c)}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\beta(\Delta x)^2}{6}(1-c)(1-2c)\frac{\partial^3 u}{\partial x^3} + \mathrm{O}\{(\Delta x)^3\} = 0. \tag{21}$$

Hence using the upwind formula introduces artificial (or numerical) diffusion with a coefficient of $\mathrm{O}(\Delta x)$ due to the introduction of the non-physical coefficient $\frac{\beta\Delta x(1-c)}{2}$ of $\frac{\partial^2 u}{\partial x^2}$.

Note that this coefficient is positive as $0 < c \leqslant 1$.

Hence this method incorporated numerical diffusion which suppressed peaks in the true solution.

3.4. The FTBS type method

Choice of $\theta = \gamma = 0$, and $\phi = 1$ in (10) gives the following forward time backward space finite-difference explicit formula:

$$u_i^{n+1} = \left(1 - \frac{3c}{2}\right)u_i^n + 2cu_{i-1}^n - \frac{c}{2}u_{i-2}^n. \tag{22}$$

Unfortunately, this choice produces an unstable method.

The modified equivalent partial differential equation of this procedure is in the following form:

$$\frac{\partial u}{\partial t}+\beta\frac{\partial u}{\partial x}-\frac{\beta(\Delta x)c}{2}\frac{\partial^2 u}{\partial x^2}+\frac{\beta(\Delta x)^2}{6}2(c^2-1)\frac{\partial^3 u}{\partial x^3}+\mathrm{O}\{(\Delta x)^3\}=0. \tag{23}$$

3.5. The Leith's scheme

Setting $\theta = c$, $\phi = \gamma = 0$ in (10) gives the following upwind-type finite-difference explicit technique:

$$u_i^{n+1}=\frac{1}{2}(c+c^2)u_{i-1}^n+(1-c^2)u_i^n+\frac{1}{2}(-c+c^2)u_{i+1}^n, \tag{24}$$

which is stable for

$$0<c\leqslant 1. \tag{25}$$

The numerical techniques developed here are based on the modified equivalent partial differential equation as described by Warming and Hyett [21]. This approach allows the simple determination of the theoretical order of accuracy, thus allowing methods to be compared with one another. Also from the truncation error of the modified equivalent equation, it is possible to eliminate the dominant error terms associated with the finite-difference equations that contain free parameters (weights), thus leading to more accurate methods.

The accuracy of the finite-difference schemes in this paper are assessed by comparing the size of the truncation errors in their modified equivalent partial differential equation (MEPDE), which may be written in the general form

$$\frac{\partial u}{\partial t}+\beta\frac{\partial u}{\partial x}+\sum_{q=2}^{\infty}\frac{\beta(\Delta x)^{q-1}}{q!}d_q\frac{\partial^q u}{\partial x^q}=0. \tag{26}$$

The coefficients d_q are $(q-1)$th order accurate in the spatial grid size and hence tend to zero as the grid spacing tends to zero, leaving the one-dimensional advection equation in the limit.

It is worth noting that the existence of the MEPDE requires the grid function u to be continuously differentiable in the domain $[0,1]\times[0,T]$. Therefore, results obtained from an analysis of the size of the error in (26) don't apply when discontinuities occur in the initial-boundary conditions.

Clearly, in order to improve the accuracy of finite difference procedures used to compute the approximations to $u(x,t)$ successive error terms in the MEPDE should be eliminated by setting $d_q = 0$ for as many $q = 2,3,4,\dots$ as possible.

The modified equivalent partial differential equation of this method is in the following form:

$$\frac{\partial u}{\partial t}+\beta\frac{\partial u}{\partial x}+\frac{\beta(\Delta x)^2(1-c^2)}{6}\frac{\partial^3 u}{\partial x^3}+\frac{\beta(\Delta x)^3}{8}c(1-c^2)\frac{\partial^4 u}{\partial x^4}+\mathrm{O}\{(\Delta x)^4\}=0. \tag{27}$$

Therefore, the Leith's procedure is second-order accurate.

3.6. The second-order upwind formula

Putting $\theta = c$, $\phi = 1 - c$ and $\gamma = 0$ in (10) gives the following upwind-type finite-difference explicit technique:

$$u_i^{n+1}=\frac{1}{2}(-c+c^2)u_{i-1}^n+(2c-c^2)u_i^n+\frac{1}{2}(2-3c+c^2)u_{i+1}^n, \tag{28}$$

which is von Neumann stable in the following range:

$$0<c\leqslant 2. \tag{29}$$

The modified equivalent partial differential equation of this method is in the following form:

$$\frac{\partial u}{\partial t}+\beta\frac{\partial u}{\partial x}-\frac{\beta(\Delta x)^2(1-c)(2-c)}{6}\frac{\partial^3 u}{\partial x^3}+\frac{\beta(\Delta x)^3}{8}(2-c)(1-c^2)\frac{\partial^4 u}{\partial x^4}+\mathrm{O}((\Delta x)^4)=0. \tag{30}$$

Note that this scheme is second-order accurate.

3.7. The Fromm's Technique

Choice of $\theta = c,\ \phi = \frac{1-c}{2}$ and $\gamma = 0$, in (10) yields the following finite-difference scheme which is known as the Fromm's formula:

$$u_i^{n+1}=\frac{c(c-1)}{4}u_{i-2}^n+\frac{c(5-c)}{4}u_{i-1}^n+\frac{(1-c)(c+4)}{4}u_i^n+\frac{c(c-1)}{4}u_{i+1}^n, \tag{31}$$

which is stable in the range (25).

Fromm's method has the following modified equivalent partial differential equation:

$$\frac{\partial u}{\partial t}+\beta\frac{\partial u}{\partial x}-\frac{\beta(\Delta x)^2(1-c)(1-2c)}{12}\frac{\partial^3 u}{\partial x^3}+\frac{\beta(\Delta x)^3}{8}(1-c)(1-c+c^2)\frac{\partial^4 u}{\partial x^4}+\mathrm{O}\{(\Delta x)^4\}=0. \tag{32}$$

It is worth nothing that this formula can be obtained using the arithmetic mean of the Leith's second-order scheme (24) and the upwind second-order method (28).

3.8. The third-order upwind procedure

Putting $\theta = c$, $\phi = \frac{(1-c^2)}{3}$ and $\gamma = 0$, in (10) yields the following third-order finite-difference formula:

$$u_i^{n+1} = \frac{c(c^2-1)}{6}u_{i-2}^n + \frac{c(1+c)(2-c)}{2}u_{i-1}^n + \frac{(c^2-1)(c-2)}{2}u_i^n + \frac{c(1-c)(c-2)}{6}u_{i+1}^n, \tag{33}$$

which is stable for (25).

The modified equivalent partial differential equation of this method is in the following form [17]:

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} + \frac{\beta(\Delta x)^3(1-c^2)(2-c)}{24}\frac{\partial^4 u}{\partial x^4} - \frac{\beta(\Delta x)^4}{60}(1-c^2)(2-c)(1-2c)\frac{\partial^5 u}{\partial x^5} + \mathrm{O}\{(\Delta x)^5\} = 0. \tag{34}$$

So this technique is third-order accurate.

It has been stated that this third-order upwinding formula is the rational basis for the development of clean and robust algorithms for computational fluid mechanics [20].

3.9. The Rusanov's third-order scheme

Setting $\theta = c$, $\phi = \frac{2(1-c^4)}{15c}$ and $\gamma = \frac{(c^2-1)(1-2c)(2-c)}{5c}$, gives the following third-order formula which is known as the Rusanov's third-order method [18]:

$$u_i^{n+1} = \frac{1}{60}(c^2-1)(1+2c)(2+c)u_{i-2}^n + \frac{1}{30}(4-c^2)(1+c)(1+4c)u_{i-1}^n + \frac{1}{5}(c^2-1)(c^2-4)u_i^n + \frac{1}{30}(c^2-4)(c-1)(1-4c)u_{i+1}^n + \frac{1}{60}(c^2-1)(2c-1)(c-2)u_{i+2}^n, \tag{35}$$

which is stable for (25).

This scheme has the following modified equivalent partial differential equation:

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} + \frac{\beta(\Delta x)^3}{120c}(c^2-1)(c^2-4)\frac{\partial^4 u}{\partial x^4} + \frac{\beta(\Delta x)^5}{720c}(1-c^2)(4-c^2)(1-2c^2) \times \frac{\partial^6 u}{\partial x^6} + \mathrm{O}\{(\Delta x)^6\} = 0. \tag{36}$$

The modified equivalent partial differential equation shows that this technique is third-order accurate.

3.10. The Rusanov's fourth-order formula

Setting $\theta = c$, $\phi = \frac{c(1-c^2)}{6}$ and $\gamma = \frac{(2-c)(1-c^2)}{2}$, in (10) yields the following fourth-order finite-difference procedure:

$$u_i^{n+1} = \frac{1}{24}c(c^2-1)(2+c)u_{i-2}^n + \frac{1}{6}c(1+c)(4-c^2)u_{i-1}^n + \frac{1}{4}(1-c^2) \times (4-c^2)u_i^n + \frac{1}{6}c(c-1)(4-c^2)u_{i+1}^n + \frac{1}{24}c(1-c^2)(2-c)u_{i+2}^n, \tag{37}$$

which is stable in the range (25).

This technique is called Rusanov's formula.

The modified equivalent partial differential equation of this method is in the following form [17]:

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} - \frac{\beta(\Delta x)^4(1-c^2)(4-c^2)}{120}\frac{\partial^5 u}{\partial x^5} - \frac{\beta(\Delta x)^5}{144}c(1-c^2)(4-c^2) \times \frac{\partial^6 u}{\partial x^6} + \mathrm{O}\{(\Delta x)^6\} = 0. \tag{38}$$

So this technique is fourth-order accurate.

3.11. Carpenter's method

Choice of $\theta = c$, $\phi = \frac{c(1-c^2)}{3}$ and $\gamma = (1-c^2)(1-c)$ gives the Carpenter's procedure which is stable in the range $0 < c \leqslant 1$.

$$u_i^{n+1} = \frac{c}{12}(c^3+c^2-c-1)u_{i-2}^n + \frac{c}{6}(-2c^3-c^2+5c+4)u_{i-1}^n + \frac{1}{2}(c^4-3c^2+2)u_i^n - \frac{c}{6}(2c^3-c^2-4c+4)u_{i+1}^n + \frac{c}{12}(c^3-c^2-c+1)u_{i+2}^n. \tag{39}$$

This scheme has the following modified equivalent partial differential equation:

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} - \frac{\beta(\Delta x)^4}{4!}c\left[2c^5-4c^4+4c^3-8c^2+3\right]\frac{\partial^5 u}{\partial x^5} + \mathrm{O}\{(\Delta x)^5\} = 0. \tag{40}$$

The modified equivalent analysis shows that this scheme is fourth-order accurate.

3.12. Gadd's technique

Setting $\theta = c$, $\phi = \frac{3c(1-c^2)}{2}$ and $\gamma = \frac{3(1-c^2)(1-c)}{2}$ in (10) gives the Gadd's technique which is stable for $0 < c \leqslant 1$.

$$\begin{aligned}u_i^{n+1} &= \frac{c}{8}(5c^3 + c^2 - 5c - 1)u_{i-2}^n + \frac{c}{2}(-8c^3 - c^2 + 10c + 3)u_{i-1}^n \\ &\quad + \frac{1}{2}(9c^3 - 2c^2 - 9c + 4)u_i^n + \frac{c}{2}(-4c^3 + c^2 + 6c - 1)u_{i+1}^n \\ &\quad + \frac{c}{8}(c^3 - c^2 - c + 1)u_{i+2}^n. \end{aligned} \tag{41}$$

The modified equivalent partial differential equation of this method is in the following form [17]:

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} + \frac{\beta(\Delta x)^4}{8}c(-2c^4 + c^3 - 2c^2 + c - 1)\frac{\partial^5 u}{\partial x^5} + \mathrm{O}\{(\Delta x)^5\} = 0. \tag{42}$$

So this technique is fourth-order accurate.

3.13. Crowley's scheme

Putting $\theta = c$, $\phi = \frac{-c}{2}$ and $\gamma = \frac{3c(2-c)}{4}$ in (10) yields the Crowley's method, which is stable for $0 < c \leqslant 1$.

$$\begin{aligned}u_i^{n+1} &= \frac{c^2}{16}(c + 2)u_{i-2}^n + \frac{c}{8}(4 - c^2)u_{i-1}^n + \frac{1}{4}(4 - c^2)u_i^n + \frac{c}{8}(c^2 - 4)u_{i+1}^n \\ &\quad + \frac{c^2}{16}(2 - c)u_{i+2}^n. \end{aligned} \tag{43}$$

This scheme has the following modified equivalent partial differential equation which shows the fourth-order accuracy of the Crowley's technique.

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} - \frac{\beta(\Delta x)^4 c}{12}(9c^4 - 24c^2 + 4c^3 - 6)\frac{\partial^5 u}{\partial x^5} + \mathrm{O}\{(\Delta x)^5\} = 0. \tag{44}$$

4. The quasi-implicit finite-difference procedures

The explicit finite-difference techniques introduced in Section 3 are simple to implement on a computer, but have very restricted ranges of Courant numbers for which they are stable. For example, the FTCS type formula is never stable and the upwind method is only stable if the Courant number satisfies the CFL condition, namely $0 < c \leqslant 1$.

In this section we are aiming to introduce some implicit schemes which involve two adjacent grid points at the new time level. Because they are implicit in nature, they may be more stable than the explicit techniques, in the same way as the implicit schemes for approximately solving the diffusion equation were found to be generally more stable than the explicit methods [11,14,23–26].

Due to the nature of their computational molecules implicit techniques of the type described can be marched in the positive x-direction across the latest time line. This means that they use less central processor time in their implementation than implicit schemes involving three or more grid points at the latest time level in their computational stencil, as these require the solution of a set of linear algebraic equations at each new time level.

In this section two numerical quasi-implicit techniques are developed for obtaining approximate solutions to the initial boundary value problem for the one-dimensional advection Eq. (1).

4.1. The box method

This technique uses the following approximations:

$$\left.\frac{\partial u}{\partial t}\right|_i^n \simeq \frac{1}{2}\left[\frac{u_{i-1}^{n+1}-u_{i-1}^n}{\Delta t}+\frac{u_i^{n+1}-u_i^n}{\Delta t}\right], \tag{45}$$

$$\left.\frac{\partial u}{\partial x}\right|_i^n \simeq \frac{1}{2}\left[\frac{u_i^n-u_{i-1}^n}{\Delta x}+\frac{u_i^{n+1}-u_{i-1}^{n+1}}{\Delta x}\right]. \tag{46}$$

The above approximations leads to the following difference equation:

$$(1-c)u_{i-1}^{n+1}+(1+c)u_i^{n+1}=(1+c)u_{i-1}^n+(1-c)u_i^n. \tag{47}$$

This scheme is unconditionally von Neumann stable.

Even though the box scheme is implicit, the values of u_i^{n+1} can be computed in an explicit manner using the following formula:

$$u_i^{n+1}=u_{i-1}^n+\left(\frac{1-c}{1+c}\right)(u_i^n-u_{i-1}^{n+1}). \tag{48}$$

Commencing with $i=1$, the value of u_i^{n+1} is computed from u_1^n and the known boundary values u_0^n and u_0^{n+1} at $x=1$. Then with $i=2$ values of u_2^{n+1}, is computed in terms of the known values of u_1^n, u_2^n and u_1^{n+1} and so on until (48) with $i=M$ gives u_M^{n+1}.

In this way the quasi-implicit box scheme can be used to march across the $(n + 1)$th time line in the positive x-direction.

The modified equivalent partial differential equation of this method is in the following form [17]:

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} - \frac{\beta(\Delta x)^2(1-c^2)}{12}\frac{\partial^3 u}{\partial x^3} - \frac{\beta(\Delta x)^4}{240}(1-c^2)(3c^2-2)\frac{\partial^5 u}{\partial x^5} + \mathrm{O}\{(\Delta x)^6\} = 0. \tag{49}$$

Hence this method is second-order accurate.

4.2. Roberts and Weiss scheme

This technique uses the following approximations for solving (1):

$$\left.\frac{\partial u}{\partial t}\right|_i^n \simeq \frac{1}{2}\left[\frac{u_i^{n+1} - u_i^n}{\Delta t}\right], \tag{50}$$

$$\left.\frac{\partial u}{\partial x}\right|_i^n \simeq \frac{1}{2}\left[\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + \frac{u_{i+1}^n - u_i^n}{\Delta x}\right]. \tag{51}$$

Using the above approximations we will have:

$$-cu_{i-1}^{n+1} + (2+c)u_i^{n+1} = (2+c)u_i^n - cu_{i+1}^n. \tag{52}$$

A von Neumann stability analysis shows that this quasi-implicit scheme is unconditionally stable.

The values of u_i^{n+1} can be computed in an explicit manner in the following form:

$$u_i^{n+1} = u_i^n + \frac{c}{2+c}(u_{i-1}^{n+1} - u_{i+1}^n). \tag{53}$$

This procedure has the following modified equivalent partial differential equation:

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} + \frac{\beta(\Delta x)^2(1+c)(2+c)}{12}\frac{\partial^3 u}{\partial x^3} + \frac{\beta(\Delta x)^4}{240}(1+c)(2+c) \times (1+6c+3c^2)\frac{\partial^5 u}{\partial x^5} + \mathrm{O}((\Delta x)^6) = 0. \tag{54}$$

Since no term of the form $\frac{\partial^2 u}{\partial x^2}$ occurs in the modified equation, there is no numerical diffusion introduced by this method.

5. The implicit finite-difference techniques

Mass, heat and momentum transport in moving fluids may be simulated on digital computers by numerically solving the governing partial differential equations. This procedure is carried out in many disciplines including hydrology, oceanography and meteorology.

Two-level explicit finite-difference methods to approximately solve the one-dimensional advection problem were considered in Section 3. The quasi-implicit finite-difference methods were discussed in Section 4. In this section some implicit finite-difference procedures will be described.

If more than two values are not known at the new time level in the finite-difference formula being used an additional boundary condition is required in order to solve the finite-difference formula which approximate the advection equation.

Note that the box scheme and Roberts and Wiess method discussed in Section 4, are implicit in the sense that they involve more than one unknown value at the new time level. However, both may be implemented in an explicit fashion by marching across the grid from the known boundary values at $x = 0$ to the boundary at $x = 1$. For this reason we call them as the quasi-implicit techniques.

5.1. The BTCS type formula

By applying the backward time approximation for the time derivative and the centred space approximation to Eq. (1) we will get the following finite-difference formula:

$$-cu_{i-1}^{n+1} + 2u_i^{n+1} + cu_{i+1}^{n+1} = 2u_i^n. \tag{55}$$

The von Neumann stability analysis shows that this technique is unconditionally stable.

If there are three unknown values at the new time level and the implicit finite-difference formula is diagonally dominant, the very efficient Thomas algorithm may be used in the solution of the resulting tridiagonal system of algebraic equations if values of u at the left and right boundary are known for $0 < t \leqslant T$.

Note that the resulting system is strictly diagonally dominant only if $0 < c < 1$.

The modified equivalent partial differential equation of this method is in the following form:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\beta(\Delta x)c}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\beta(\Delta x)^2(1+2c^2)}{6}\frac{\partial^3 u}{\partial x^3} + \mathrm{O}\{(\Delta x)^3\} = 0. \tag{56}$$

The modified equivalent equation analysis indicates that this like the standard FTCS scheme is only first-order accurate.

5.2. The Crank–Nicolson type scheme

If we replace the spatial derivative with the average of their values at the n and $n+1$ time levels and then substitute centred-difference forms for all derivatives, we get the Crank–Nicolson type scheme.

$$\left.\frac{\partial u}{\partial t}\right|_i^{n+\frac{1}{2}} + \beta\left.\frac{\partial u}{\partial x}\right|_i^{n+\frac{1}{2}} = 0, \tag{57}$$

$$\left.\frac{\partial u}{\partial t}\right|_i^{n+\frac{1}{2}} \simeq \frac{1}{2}\,\frac{(u_i^{n+1} - u_i^n)}{\Delta t}, \tag{58}$$

$$\left.\frac{\partial u}{\partial x}\right|_i^{n+\frac{1}{2}} \simeq \frac{1}{4}\left(\frac{(u_{i+1}^n - u_{i-1}^n)}{\Delta x}\right) + \frac{1}{4}\left(\frac{(u_{i+1}^{n+1} - u_{i-1}^{n+1})}{\Delta x}\right). \tag{59}$$

The time derivative is approximated in the following way.

The Crank–Nicolson type implicit finite-difference method uses the following formula for the solution of (1).

$$-cu_{i-1}^{n+1} + 4u_i^{n+1} + cu_{i+1}^{n+1} = cu_{i-1}^n + 4u_i^n - cu_{i+1}^n, \tag{60}$$

for $i = 1,2,\ldots,M-1$.

This technique is unconditionally stable.

The modified equivalent partial differential equation of this method is in the following form:

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} + \frac{\beta(2+c^2)(\Delta x)^2}{12}\frac{\partial^3 u}{\partial x^3} + \frac{\beta(\Delta x)^4(2+10c^2+3c^4)}{240}\frac{\partial^5 u}{\partial x^5} + \mathrm{O}\{(\Delta x)^6\} = 0. \tag{61}$$

This equation indicates that the Crank–Nicolson type scheme is second-order accurate and incorporates no numerical diffusion.

Note that like other implicit schemes this technique cannot be used to advantage on vector or parallel computers.

5.3. The second-order implicit method

Replacing the time derivative by a weighted average at the points $(x_{i-1}, t_{n+\frac{1}{2}})$ $(x_i, t_{n+\frac{1}{2}})$ and $(x_{i+1}, t_{n+\frac{1}{2}})$ gives:

$$\left.\frac{\partial u}{\partial t}\right|_i^{n+\frac{1}{2}} \simeq \frac{1}{6}\left(\left.\frac{\partial u}{\partial t}\right|_{i-1}^{n+\frac{1}{2}} + \left.\frac{\partial u}{\partial t}\right|_{i+1}^{n+\frac{1}{2}}\right) + \frac{2}{3}\left.\frac{\partial u}{\partial t}\right|_i^{n+\frac{1}{2}}. \tag{62}$$

Now we use the centred-difference forms for all time derivatives.

The space derivative is approximated as in (59).

So the second-order implicit finite-difference method uses the following formula for the solution of (1).

$$(2-3c)u_{i-1}^{n+1} + 8u_i^{n+1} + (2+3c)u_{i+1}^{n+1} = (2+3c)u_{i-1}^n + 8u_i^n + (2-3c)u_{i+1}^n, \tag{63}$$

for $i = 1, 2, \ldots, M-1$.

This technique is unconditionally stable.

The modified equivalent partial differential equation of this method is in the following form:

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} + \frac{\beta c^2(\Delta x)^2}{12}\frac{\partial^3 u}{\partial x^3} + \frac{\beta(\Delta x)^4(9c^4-4)}{720}\frac{\partial^5 u}{\partial x^5} + \mathrm{O}\{(\Delta x)^6\} = 0. \tag{64}$$

Eq. (64) indicates that this implicit scheme is second-order accurate and incorporates no numerical diffusion.

Note that this method is solvable only if $0 < c \leqslant \frac{4}{3}$.

It is worth noting that this technique is called the linear finite-element Crank–Nicolson implicit scheme (or the improved Crank–Nicolson method) by some authors.

5.4. The new implicit procedure

We approximate the time derivative in the following way:

$$\left.\frac{\partial u}{\partial t}\right|_i^{n+\frac{1}{2}} \simeq \frac{2+c^2}{12}\frac{(u_{i-1}^{n+1}-u_{i-1}^n)}{\Delta t} + \frac{4-c^2}{12}\frac{(u_i^{n+1}-u_i^n)}{\Delta t} + \frac{2+c^2}{12}\frac{(u_{i+1}^{n+1}-u_{i+1}^n)}{\Delta t}. \tag{65}$$

The space derivative is approximated in the following way:

$$\left.\frac{\partial u}{\partial x}\right|_i^{n+\frac{1}{2}} \simeq \frac{1}{4}\left(\frac{(u_{i+1}^n-u_{i-1}^n)}{\Delta x} + \frac{(u_{i+1}^{n+1}-u_{i-1}^{n+1})}{\Delta x}\right). \tag{66}$$

Hence the new implicit finite-difference technique uses the following formula for the solution of (1).

$$\begin{aligned}&(2-3c+c^2)u_{i-1}^{n+1} + 2(4-c^2)u_i^{n+1} + (2+3c+c^2)u_{i+1}^{n+1}\\ &\quad = (2+3c+c^2)u_{i-1}^n + 2(4-c^2)u_i^n + (2-3c+c^2)u_{i+1}^n,\end{aligned} \tag{67}$$

for $i = 1, 2, \ldots, M-1$.

This procedure is unconditionally stable in the von Neumann sense.

Values of u_i^{n+1} can be found using the very efficient Thomas algorithm to solve the resulting set of algebraic equations.

The modified equivalent partial differential equation of this technique is as follows:

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} - \frac{\beta(c^4 - 5c^2 + 4)(\Delta x)^4}{720} \frac{\partial^5 u}{\partial x^5} + \mathrm{O}\{(\Delta x)^6\} = 0. \tag{68}$$

This equation indicates that the formula (67) is fourth-order accurate and incorporates no numerical diffusion.

Note that this method is solvable only if $0 \leqslant c \leqslant 1$.

6. Numerical test

A problem for which the exact solution is known is now used to test the methods described for solving the advection equation. These techniques are applied to solve (1)–(4) with $g_0(t)$, $g_1(t)$, and $f(x)$ known and u unknown.

We take our test from the following function:

$$u(x,t) = \exp\left\{-1.8 \times 10^{-4}\left(x - 0.2 - \frac{t}{10}\right)^2\right\}, \quad 0 < x \leqslant 1, \ 0 < t \leqslant T. \tag{69}$$

Tests were carried out for several values of the Courant number.

The results obtained for $u(0.4, 0.8)$ computed for various values of c, using the Fromm's finite-difference method, the upwind explicit scheme, the FTBS type formula, the Leith's explicit procedure, the Rusanov's third-order formula and the Carpenter's finite-difference technique described in this paper, are shown in Table 1.

Table 1
Numerical test results

Δx	c	Error					
		Fromm	Upwind first-order	FTBS	Leith	Rusanov third-order	Carpenter
0.01000	0.50	2.2×10^{-2}	2.6×10^{-2}	2.4×10^{-2}	2.8×10^{-2}	2.4×10^{-2}	2.0×10^{-2}
0.00500	0.50	1.2×10^{-2}	1.5×10^{-2}	1.3×10^{-2}	1.5×10^{-2}	3.0×10^{-3}	1.3×10^{-3}
0.00250	0.50	5.0×10^{-3}	7.0×10^{-3}	7.0×10^{-3}	8.0×10^{-3}	4.0×10^{-4}	1.0×10^{-4}
0.00125	0.50	2.4×10^{-4}	4.0×10^{-3}	3.3×10^{-3}	4.1×10^{-3}	5.0×10^{-5}	9.0×10^{-6}
0.01200	0.75	4.5×10^{-2}	2.5×10^{-2}	2.6×10^{-2}	3.2×10^{-2}	4.0×10^{-2}	3.0×10^{-2}
0.00600	0.75	2.2×10^{-2}	1.2×10^{-2}	1.4×10^{-2}	1.7×10^{-2}	5.0×10^{-3}	2.0×10^{-3}
0.00300	0.75	1.0×10^{-2}	6.0×10^{-3}	7.0×10^{-3}	8.0×10^{-3}	6.0×10^{-4}	1.2×10^{-4}
0.00150	0.75	5.0×10^{-3}	3.0×10^{-3}	3.6×10^{-3}	3.9×10^{-3}	7.0×10^{-5}	7.5×10^{-6}

The results obtained for $u(0.4, 0.8)$ computed for various values of c, using the second-order upwind finite-difference formula, the Lax–Friedrichs explicit procedure, the third-order upwind method, the Rusanov's fourth-order scheme, the Gadd's explicit formula and the Crowley's finite-difference technique described in this paper, are shown in Table 2.

Table 3 contains the obtained results for the box quasi-implicit procedure, the Robert and Weiss finite-difference formula, the BTCS type formula, the Crank–Nicolson technique, the second-order implicit method and the new implicit finite-difference formula. As it can be seen from Subsection 3.1, the FTCs method is of no practical use. Hence, we did not present values for this method in Tables 1–3.

These results reflect the fourth-order convergence of the new implicit finite-difference formula (67). Clearly the new fourth-order method is the most accurate and efficient of the fourth-order schemes tested.

Table 2
Numerical test results

Δx	c	Error					
		Lax–Friedrichs	Upwind second-order	Upwind third-order	Rusanov fourth-order	Gadd	Crowley
0.01000	0.50	6.6×10^{-2}	5.2×10^{-2}	8.9×10^{-2}	6.8×10^{-2}	6.4×10^{-2}	5.6×10^{-2}
0.00500	0.50	3.2×10^{-2}	1.3×10^{-2}	1.2×10^{-2}	4.2×10^{-3}	4.0×10^{-3}	3.9×10^{-3}
0.00250	0.50	1.6×10^{-2}	3.0×10^{-3}	1.5×10^{-3}	2.6×10^{-4}	2.5×10^{-4}	2.2×10^{-4}
0.00125	0.50	7.9×10^{-3}	7.3×10^{-3}	2.0×10^{-4}	1.5×10^{-5}	1.5×10^{-5}	1.5×10^{-5}
0.01200	0.75	8.0×10^{-2}	6.4×10^{-2}	9.6×10^{-2}	7.2×10^{-2}	7.0×10^{-2}	7.0×10^{-2}
0.00600	0.75	4.1×10^{-2}	1.6×10^{-2}	1.2×10^{-2}	4.5×10^{-3}	4.4×10^{-3}	4.3×10^{-3}
0.00300	0.75	2.0×10^{-2}	4.0×10^{-3}	1.5×10^{-3}	3.0×10^{-4}	2.6×10^{-4}	2.5×10^{-4}
0.00150	0.75	1.0×10^{-3}	1.0×10^{-3}	2.0×10^{-3}	2.5×10^{-5}	1.6×10^{-5}	1.6×10^{-5}

Table 3
Numerical test results

Δx	c	Error					
		BTCS	Box	Roberts–Weiss	Crank–Nicolson	Implicit third-order	New implicit
0.01000	0.50	5.5×10^{-2}	4.8×10^{-2}	7.9×10^{-2}	8.0×10^{-2}	6.5×10^{-2}	7.0×10^{-2}
0.00500	0.50	2.7×10^{-2}	1.2×10^{-2}	1.0×10^{-2}	5.0×10^{-3}	4.0×10^{-3}	4.8×10^{-3}
0.00250	0.50	1.4×10^{-2}	3.0×10^{-3}	1.3×10^{-3}	3.0×10^{-4}	2.2×10^{-4}	3.0×10^{-4}
0.00125	0.50	7.1×10^{-3}	7.6×10^{-3}	1.6×10^{-4}	2.0×10^{-5}	1.4×10^{-5}	1.9×10^{-5}
0.01200	0.75	6.6×10^{-2}	5.4×10^{-2}	8.4×10^{-2}	9.6×10^{-2}	7.5×10^{-2}	9.5×10^{-2}
0.00600	0.75	3.5×10^{-2}	1.4×10^{-2}	1.0×10^{-2}	8.0×10^{-3}	5.0×10^{-3}	6.0×10^{-3}
0.00300	0.75	1.7×10^{-2}	3.5×10^{-3}	1.2×10^{-3}	5.0×10^{-4}	3.0×10^{-4}	3.7×10^{-4}
0.00150	0.75	8.0×10^{-3}	9.0×10^{-4}	1.4×10^{-4}	2.9×10^{-5}	2.0×10^{-5}	2.2×10^{-5}

Note that the values chosen for c are in the range of the stability of all explicit finite-difference schemes considered in this article.

The results obtained showed that the FTBS type scheme and the upwind explicit technique produced errors of similar magnitude.

When the results obtained for the new fourth-order inplicit technique are compared with those of the second-order Rusanov's method, the average error of the former is generally found to be at least two orders of magnitude smaller than the latter.

For large values of c such as $c = 0.90$ errors of the second-order methods are nearly the same and are much larger than those of the third-order schemes.

Inspection of the tables shows that the size of the average error obtained is closely related to the size of the dominant error term in the modified equivalent equation of the method used.

When the results for the implicit schemes are compared with those for the explicit techniques, it is found that the first-order upwind explicit formula and the first-order upwind implicit method are very similar. It is notable that the most accurate second-order explicit method is the Fromm's technique.

Comparison of the leading error terms (in the modified equivalent partial differential equation) shows that for $c = 1$ the numerical solution to the advection equation obtained using the Lax–Friedrichs technique or the Leith's

Table 4
The order of schemes with respect to the space variable

Methods	The order of accuracy	Type
FTCS type technique	1	Explicit
Lax–Friedrichs method	1	Explicit
Upwind explicit scheme	1	Explicit
FTBS type procedure	1	Explicit
Leith's technique	2	Explicit
Second-order upwind scheme	2	Explicit
Fromm's procedure	2	Explicit
Third-order upwind formula	3	Explicit
Rusanov's third-order method	3	Explicit
Rusanov's fourth-order scheme	4	Explicit
Carpenter's procedure	4	Explicit
Gadd's technique	4	Explicit
Crowley's formula	4	Explicit
Box technique	2	Quasi-implicit
Roberts and Weiss method	2	Quasi-Implicit
BTCS type procedure	1	Implicit
Crank–Nicolson type formula	2	Implicit
Second-order implicit scheme	2	Implicit
New implicit method	4	Implicit

explicit scheme or the upwind formulae or the Fromm's technique or the Rusanov's schemes or the box quasi-implicit method or the new implicit technique is more accurate than using the Roberts and Weiss quasi-implicit scheme or the BTCS type formula or the Crank–Nicolson scheme or the second-order implicit procedure.

It is worth noting that the time needed using the second-order explicit finite-difference methods employed was shorter than using the second-order implicit finite-difference schemes. Note that there was little difference in the CPU time required by each of the two-level explicit methods when the parameters used were the same.

Table 4 contains the order of accuracy of the different techniques discussed in this article. All methods are divided into three different groups, explicit, quasi-implicit and implicit procedures.

Overall the most efficient schemes (among methods of the same order) were the quasi-implicit techniques. The main advantage of the unconditionally stable quasi-implicit procedures is that even though they are unconditionally stable they use less central processor time in their implementation than the implicit schemes.

7. Conclusions and future directions

In this article several high-order accurate weighted-based finite-difference schemes were discussed and compared for solving the one-dimensional advection equation. Most of the proposed numerical schemes solved this model quite satisfactory. The two-level explicit finite-difference schemes are very simple to implement and economical to use. They are very efficient and they need less CPU time than the implicit finite-difference methods. The implicit finite-difference schemes developed in this paper are unconditionally stable. A comparison with the implicit schemes for the numerical solution of the advection problem shows that the implicit finite-difference methods, even though they have extended range of stability, use large central processor times. The quasi-implicit finite-difference schemes discussed in this article are the most efficient second-order techniques. The main advantage of these procedures is that they are unconditionally stable and use less central process time than the implicit methods. The explicit finite-difference schemes are very easy to implement for similar higher dimensional problems, but it may be more difficult when dealing with the implicit finite-difference schemes. When comparing the implicit finite-difference techniques described in this report, it was found that the most accurate method is the fourth-order implicit formula. When comparing the two-level three-point explicit finite-difference formulae described in this paper, it was found that the most accurate method is the fourth-order explicit procedures. For each of the finite-difference methods investigated the modified equivalent partial differential equation is employed which permits the order of accuracy of the numerical methods to be determined. Also from the truncation error of the modified equivalent equation, it is possible to eliminate the dominant error terms associated with the finite-difference equations that contain free weights, thus leading to more accurate methods. Further work is being done to develop techniques suitable for use with variable coefficients, non-uniform grid spacing and more general boundary conditions. The use of modified equivalent partial differential equation for further improving the solution accuracy is being investigated. The extension of the method for solving higher dimensional problems is straightforward and is being investigated.

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