Optimization with two parameters in Gaussian distribution

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Example of Optimization with two parameters in Gaussian distribution

We assumed Gaussian distributions, so the densities are:

$$f_{Z_(z;\mu,\sigma)} = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2}(z-\mu)^2/\sigma^2\}$$

We will illustrate in the following the idea of find parameters apply MLE (maximum Likelihood Estimation) we will use log in both side the above equation.

$$\ell(z;\mu,\sigma) = \log(f_{Z_(z;\mu,\sigma)}) = \log\left(\frac{1}{\sigma\sqrt{2\pi}} \exp\{-(z-\mu)^2/2\sigma^2\}\right)$$

$$\ell(z;\mu,\sigma) = \log\left(\frac{1}{\sigma}\right) + \log\left(\frac{1}{\sqrt{2\pi}}\right) + \log\left(\exp\{-\frac{1}{2}(z-\mu)^2/\sigma^2\}\right)$$

$$\ell(z;\mu,\sigma) = \log\left(\sigma^{-1}\right) + \log\left(\sqrt{2\pi}^{-1}\right) - \frac{1}{2\sigma^2}(z-\mu)^2$$

$$\ell(z;\mu,\sigma) = -\log\left(\sigma\right) - \log\left(\sqrt{2\pi}\right) - \frac{1}{2\sigma^2}(z-\mu)^2$$

First, we obtain a strip of data from database (CO_2) uptake of six plants from Quebec and six plants from Mississippi was measured at several levels of ambient (CO_2) concentration. Half the plants of each type were chilled overnight before the experiment was conducted. More detail at r documentation

The methods MLE is used to (CO_2) database, where n is the lenght of the database, and z_i an element,

$$\mathcal{L}(\mu,\sigma) = \sum_{i=1}^{n} \ell(z_i;\mu,\sigma)$$

$$\mathcal{L}(\mu,\sigma) = \sum_{i=1}^{n}\left[-\log\left(\sigma\right) - \log\left(\sqrt{2\pi}\right) - \frac{1}{2\sigma^2}(z_i-\mu)^2\right]$$

$$\mathcal{L}(\mu,\sigma) = -\sum_{i=1}^{n}\log\left(\sigma\right) - \sum_{i=1}^{n}\log\left(\sqrt{2\pi}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[\frac{1}{2\sigma^2}(z_i-\mu)^2\right]$$

$$\mathcal{L}(\mu,\sigma) = -n\log\left(\sigma\right) - n\log\left(\sqrt{2\pi}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(z_i-\mu)^2$$

```r
n <- nrow(CO2)
z <- CO2$uptake
loglik <- function(theta) {
  mu <- theta[1]
  sigma <- theta[2]
  loglik <- -n*log(sqrt(2*pi)) - n*log(sigma) - 0.5*sum((z - mu)^2/sigma^2)
}
```

We can do the bidimentional plot for this function:

```r
n_mu <- 100
n_si <- 100
mu_i <- 1
mu_f <- 30
si_i <- 1
si_f <- 40
h_mu <- (mu_f - mu_i) / ( n_mu - 1)
h_si <- (si_f - si_i) / ( n_si - 1)
ne   <- n_mu * n_si
mesh_log_like <- matrix(0, n_mu, n_si)
eixo_mu <- rep(0,n_mu)
eixo_si <- rep(0, 1, n_si)
for (i in 1 : n_mu){
  eixo_mu[i] <- mu_i + (i - 1) * h_mu
  for (j in 1 : n_si){
    eixo_si[j] <- si_i + (j - 1) * h_si
    mesh_log_like[i, j] <- loglik(c(eixo_mu[i], eixo_si[j]))
    #mesh_log_like[i, j] <- 1
  }
}
#p <- plot_ly(x = eixo_mu, y = eixo_si, z = mesh_log_like , type = "surface")
#p
```

We apply optimization package to find the maximum point.

```r
res <- maxLik(loglik, start=c(mu=30, sigma=10), method = "BFGS")
res1 <- maxBFGS(loglik, start=c(mu=30, sigma=10))
pt  <- res$estimate
pt1 <- res1$estimate
```