MEK4300

Oblig 2

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Problem 1

`i_FSa.py`

a)

the Falkner-Skan equation reads

$$f''' + ff'' + \beta(1 - (f')^2) = 0 \tag{1}$$

where

$$\beta = \frac{2m}{1+m}. \tag{2}$$

By defining $H = f'$, we get two equations

$$\begin{aligned} H &= f' \\ H'' + fH' + \beta(1 - H^2) &= 0 \end{aligned} \tag{3}$$

Implementing this FENics gives us results (figures below) that corresponds well with Fig. 4-11 in White, for all values of β.

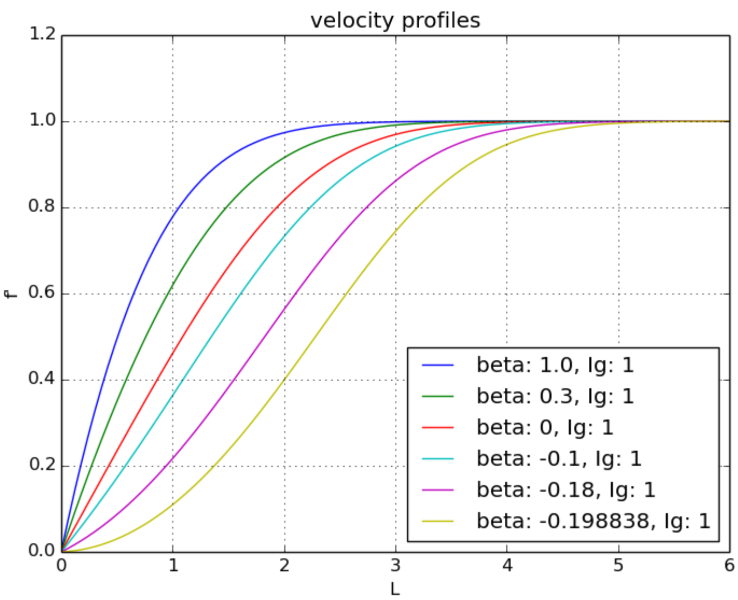

Figure 1: velocity profiles

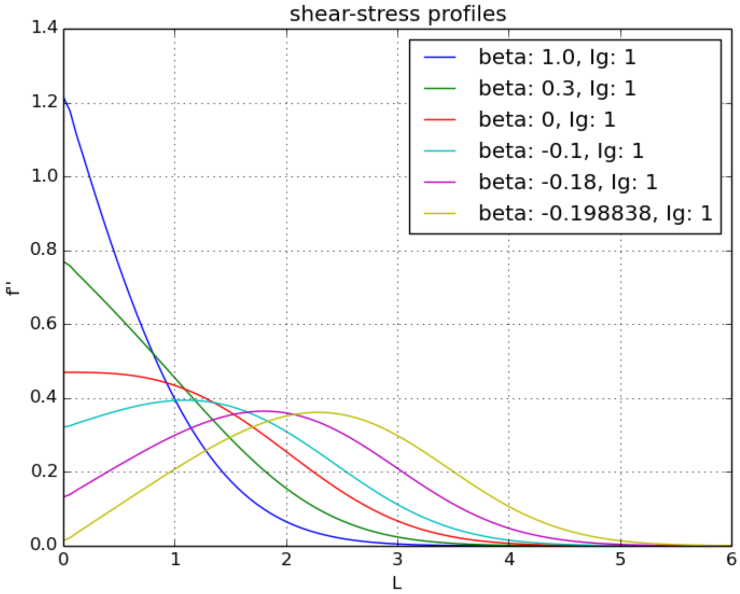

Figure 2: Shear-stress profiles

b)

According to table 4-2 in White the shear stress at $\eta = 0$ and with $\beta = -0.19884$ should be 0.0. I get 0.0097.
Furthermore the initial guess in the solver is important to get the wanted results in a). According to the guess we may acquire two solutions for $-0.19884 \leq \beta \leq 0$. One we've already found, the other shows backflow at the wall. Both solutions are shown in the figures below for $\beta = -0.1$.

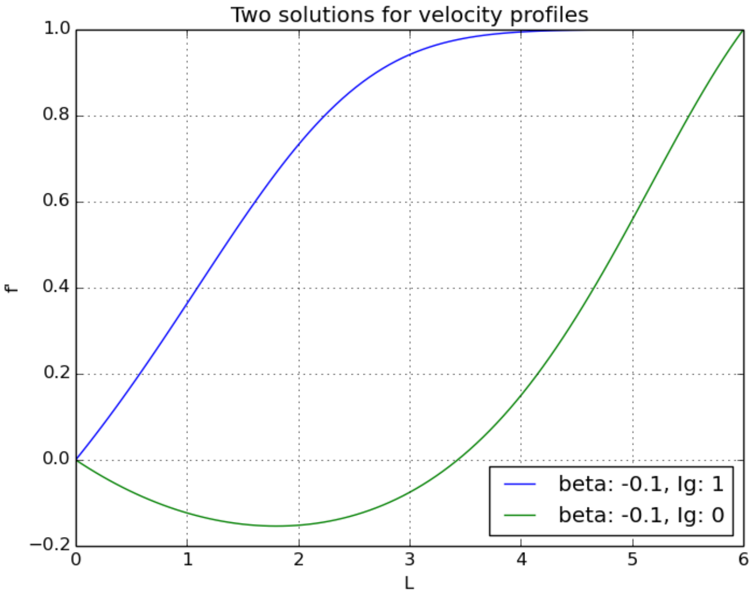

Figure 3: Velocity profiles: two solutions

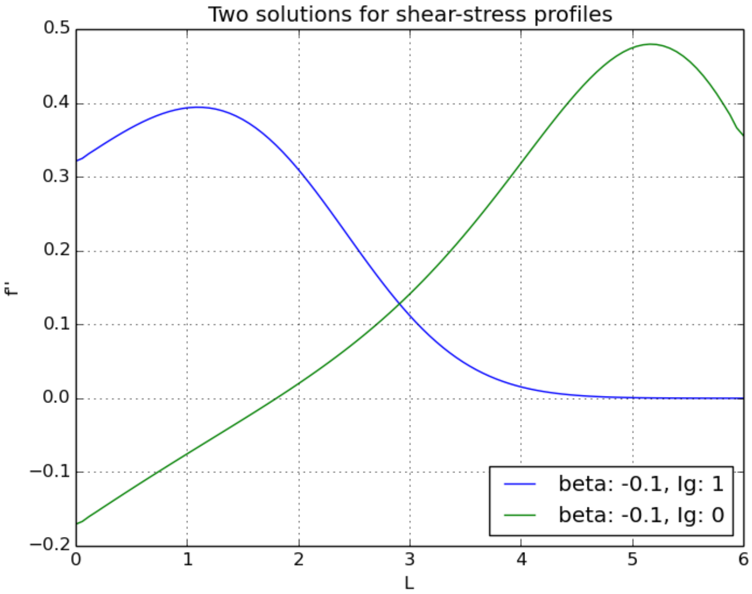

Figure 4: Shear-stress: Two solutions

Problem 2

2D-1

`ii_comparea.py`

The steady-state Navier-Stokes equation is solved in a mixed functionspace. The results and lower- and upper bound are displayed in the table below

Unknowns	c_D	c_L	L_a	ΔP
86484	5.5772	0.0106	0.0846	0.1175
343446	5.5767	0.0106	0.0846	0.1175
lower bound	5.5700	0.0104	0.0842	0.1172
upper bound	5.5900	0.0110	0.0852	0.1176

All computed parameters are in the "allowed" interval.

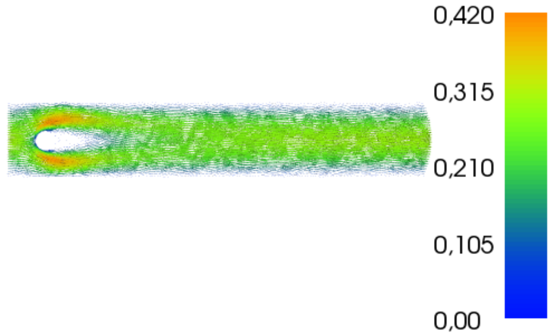

Figure 5: Steady state velocity field

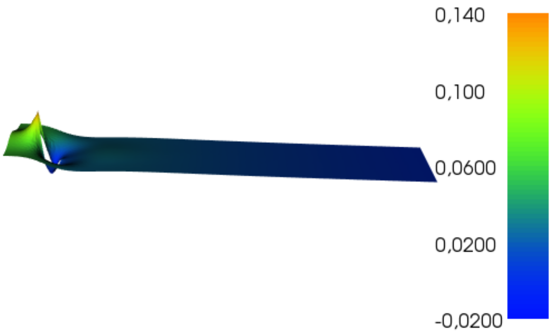

Figure 6: Steady state, pressure field

2D-2

`ii_compareb.py` For the unstedy Navier-Stokes equation, I've used FEM in space and FDM in time.
I've used a splitting scheme where, first, the tentative velocity is computed using an approximation/guess, then we need to correct the pressure and last compute a corrected velocity.

Unknowns	c_{Dmax}	c_{Lmax}	St	ΔP
16134	3.2057	0.9808	0.2994	2.4729
63111	3.2495	1.0730	0.3030	2.4965
lower bound	3.2200	0.9900	0.2950	2.4600
upper bound	5.2400	1.0100	0.3050	2.5000

We can see from the table above that the results are fairly close or correct according to the supposed results. Except maybe c_{Lmax} for de finest mesh.

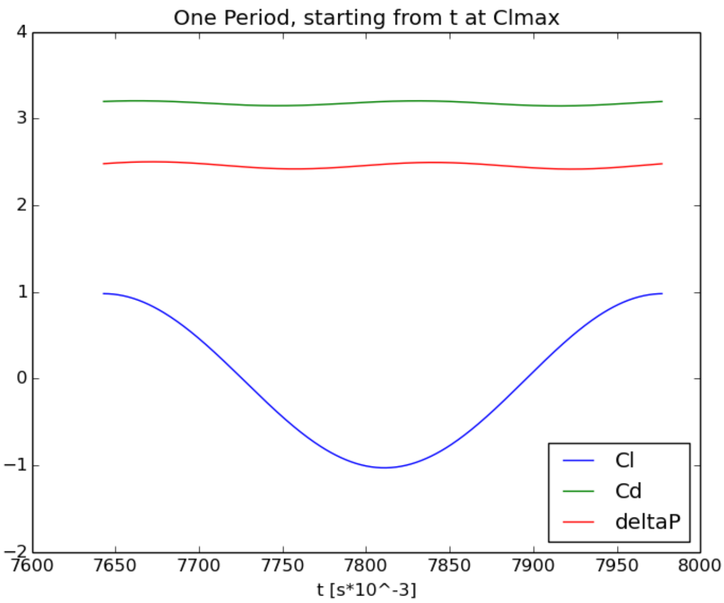

Figure 7: One preiod starting form t = t_0 at $c_L max$

The parameters are calculated in a separate file, `ii_parameters.py`.

Problem 3

`iii_MixingLength.py`

a)

Since the pressure gradient is constant, we may integrate the momentum equation across the channel to compute the value using v^*.

$$0 = \nu \int_0^H \frac{\partial^2 \overline{u}}{\partial y^2} dy - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}\int_0^H dy - \int_0^H \frac{\partial \overline{u'v'}}{\partial y} dy \tag{4}$$

There are n fluctuations at the wall, so the last term becomes zero.

$$\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} H = \nu \frac{\partial \overline{u}}{\partial y}\bigg|_0^H \tag{5}$$

From the identity $v^* = \sqrt{\nu \frac{\partial \bar{u}}{\partial y_{wall}}}$ we have that at the upper wall, the friction will be "pointing" upwards, so $\frac{\partial \bar{u}}{\partial y_{wall}} = \frac{v^{*2}}{\nu}$. At the bottom wall $\frac{\partial \bar{u}}{\partial y_{wall}} = -\frac{v^{*2}}{\nu}$. This gives us the pressure gradient, with the hight equal to 1

$$\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} = 2v^{*2} \tag{6}$$

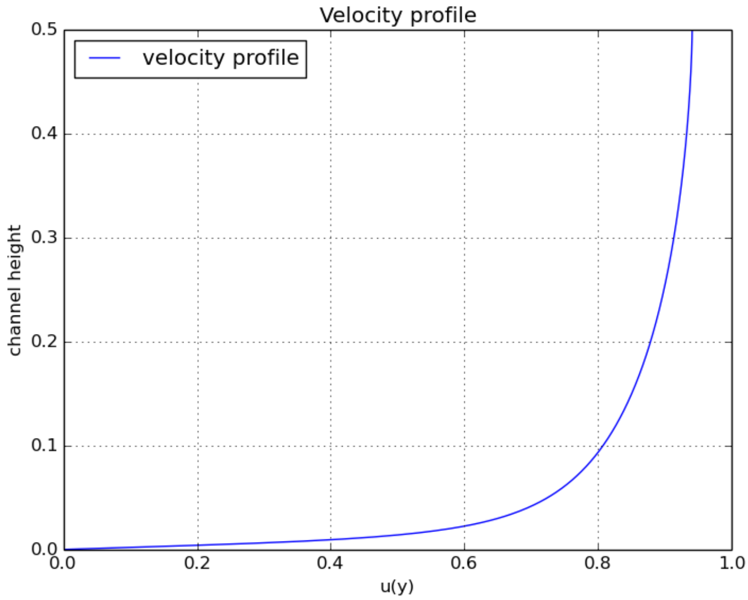

Figure 8: Numerical velocity profile for half the channel

To ensure that the first node is less than ν/v^* ($y^+ < 1$) we need atleast 1001 nodes.

b)

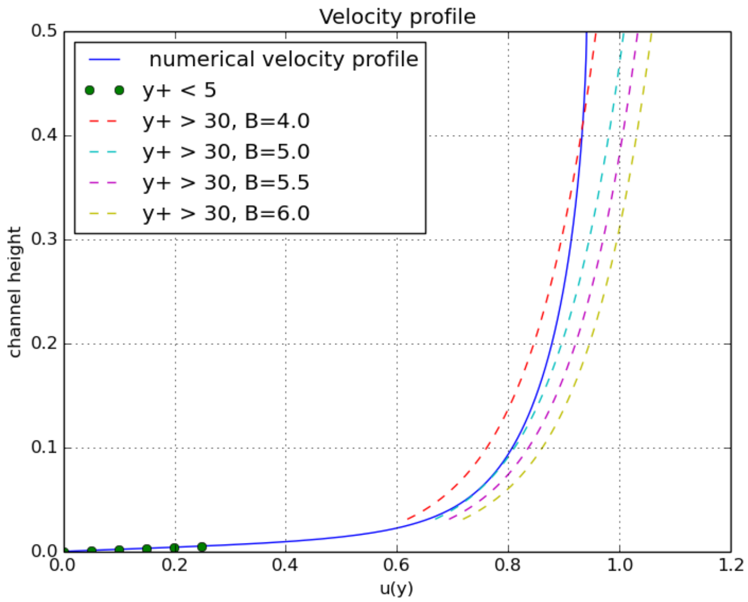

Figure 9: Numerical and theoretical velocity profiles

For $y^+ < 5$ the approximation is good, for $y^+ > 30$ modifying $B = 5$ gives me somewhat the better results, but not great.
ν_T at center is 0.000246 for least amount of nodes. Should be zero.

Problem 4

Momentum and continuity equations reads

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \tag{7}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{8}$$

RANS

Defining u and p in terms of mean flow and fluctuations

$$u = \bar{u} + u'$$
$$p = \bar{p} + p'$$

Now if we average (**??**) we get

$$\overline{\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}} \tag{9}$$

Some similarities are necessary to show if we are to continue

$$\overline{u_i + u_j} = \overline{u_i} + \overline{u_j}$$
$$\overline{\frac{\partial u_i}{\partial x_j}} = \frac{\partial \overline{u_i}}{\partial x_j} \tag{10}$$

Now (**??**) looks like

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \overline{(\mathbf{u}\cdot\nabla)\mathbf{u}} = -\frac{1}{\rho}\nabla\overline{p} + \nu\nabla^2\overline{\mathbf{u}} \tag{11}$$

The convection term now may be written as, using commuting

$$\overline{(\mathbf{u}\cdot\nabla)\mathbf{u}} = \overline{\frac{\partial}{\partial x_j}u_i u_j} = \frac{\partial}{\partial x_j}\overline{u_i u_j} \tag{12}$$

(**??**) becomes

$$\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial}{\partial x_j}\overline{u_i u_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{x_i} + \nu\frac{\partial^2 \overline{u_i}}{\partial x_j^2} \tag{13}$$

$$\frac{\partial \overline{\overline{u_i} + u_i'}}{\partial t} + \frac{\partial}{\partial x_j}\overline{(\overline{u_i} + u_i')(u_j + u_j')} = -\frac{1}{\rho}\frac{\partial \overline{\overline{p} + p'}}{x_i} + \nu\frac{\partial^2 \overline{\overline{u_i} + u_i'}}{\partial x_j^2} \tag{14}$$

Since $\overline{u_i'} = 0$, $\frac{\partial u_i u_j}{\partial x_j} = u_j\frac{\partial u_i}{\partial x_j}$ because of continuity

$$\rho\frac{\partial \overline{u_i}}{\partial t} + \rho\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \mu\frac{\partial^2 \overline{u_i}}{\partial x_j^2} - \rho\frac{\partial \overline{u_i' u_j'}}{\partial x_j} \tag{15}$$

Turbulent kinetic energy equation

Subtracting NS form RANS, dot with u_i' and time average the whole thing and we end up with

$$\overline{u_i'\left(\frac{\partial u_i'}{\partial t} + \overline{u_j}\frac{\partial u_i'}{\partial x_j} = -\frac{1}{\rho}\frac{p'}{\partial x_j} + \nu\frac{\partial^2 u_i'}{\partial x_j^2} + \underset{\mathrm{I}}{\frac{\partial \overline{u_i' u_j'}}{\partial x_j}} - \underset{\mathrm{II}}{\frac{\partial \overline{u_i} u_j'}{\partial x_j}} - \underset{\mathrm{III}}{\frac{\partial u_i' u_j'}{\partial x_j}}\right)} \tag{16}$$

The left hand side can be rewritten as

$$\overline{u_i'\frac{\partial u_i'}{\partial t} + u_i'\overline{u_j}\frac{\partial u_i'}{\partial x_j}}$$
$$\frac{1}{2}(\frac{\partial}{\partial t}(\overline{u_i' u_i'} + \overline{u_j}\frac{\partial}{\partial x_j}(\overline{u_i' u_i'}) = \frac{DK}{Dt} \tag{17}$$

where $\overline{u_i' u_i'} = 2K$, kinetic energy.

Looking at the right hand side

$$
\begin{aligned}
\mathrm{I} &\rightarrow 0 \\
\mathrm{II} &\rightarrow \overline{-u_i' \frac{\partial \overline{u_i} u_j'}{\partial x_j}} = -\overline{u_i' u_j'} \frac{\partial \overline{u_i}}{\partial x_j} \\
\mathrm{III} &\rightarrow \overline{-u_i' \frac{\partial u_i' u_j'}{\partial x_j}} = -\frac{1}{2} \frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j}
\end{aligned}
\tag{18}
$$

The pressure term and III becomes II from White, II become III from White, then we are left with $\nu u_i' \frac{\partial^2 u_i'}{\partial x_j^2}$. Going from White to our solution instead, using product rule and continuity

$$
\begin{aligned}
&\frac{\partial}{\partial x_j}\left[\nu u_i' \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)\right] - \nu \frac{\partial u_i'}{\partial x_j}\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right) \\
&= \nu \left[\cancel{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}} + \cancel{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_j'}{\partial x_i}} + u_i' \frac{\partial^2 u_i'}{\partial x_j^2} + u_i' \cancel{\frac{\partial^2 u_j'}{\partial x_j \partial x_i}} - \cancel{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_j'}{\partial x_i}} - \cancel{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}\right]
\end{aligned}
$$

The en result is now

$$
\frac{DK}{Dt} = -\frac{\partial}{\partial x_j}\left[\overline{u_i'\left(\frac{1}{2}u_i' u_i + \frac{\partial p'}{\rho}\right)}\right] - \overline{u_i' u_j'}\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\overline{\nu u_i' \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}\right] - \nu \overline{\frac{\partial u_i'}{\partial x_j}\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right)}
$$

which is the same as in the White, almost. Might have mixed up the indexes.

Reynolds Stress Equation

We will get two equations. First from subtracting NS with RANS and dot with u_k'

$$
u_k' \left(\frac{\partial u_i'}{\partial t} + \overline{u_j}\frac{\partial u_i'}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_j^2} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j} - \frac{\partial \overline{u_i} u_j'}{\partial x_j} - \frac{\partial u_i' u_j'}{\partial x_j}\right) \tag{19}
$$

second we switch indexes i and k in (**??**)

$$
u_i' \left(\frac{\partial u_k'}{\partial t} + \overline{u_j}\frac{\partial u_k'}{\partial x_j} = -\frac{1}{\rho}\frac{p'}{\partial x_k} + \nu \frac{\partial^2 u_k'}{\partial x_j^2} + \frac{\partial \overline{u_k' u_j'}}{\partial x_j} - \frac{\partial \overline{u_k} u_j'}{\partial x_j} - \frac{\partial u_k' u_j'}{\partial x_j}\right) \tag{20}
$$

Now we have two free indexes when adding (**??**) and (**??**).
Adding the two left sides and time averaging

$$
\begin{aligned}
&\overline{u_k' \frac{\partial u_i'}{\partial t} + u_k' \overline{u_j}\frac{\partial u_i'}{\partial x_j} + u_i' \frac{\partial u_k'}{\partial t} + u_i' \overline{u_j}\frac{\partial u_k'}{\partial x_j}} \\
=&\overline{\frac{\partial u_k' u_i'}{\partial t} + \overline{u_j}\left(\frac{\partial u_i' u_k'}{\partial x_j}\right)} \\
=&\frac{\overline{D u_k' u_i'}}{Dt} = \frac{D\overline{u_k' u_i'}}{Dt}
\end{aligned}
\tag{21}
$$

Next adding the right hand sides

$$\nu \overline{\left(u'_k \frac{\partial^2 u'_i}{\partial x_j^2} + u'_i \frac{\partial^2 u'_k}{\partial x_j^2} \right)} = \nu \left(\frac{\partial^2 \overline{u'_i u'_k}}{\partial x_j^2} - 2 \overline{\frac{\partial u'_i}{\partial x_j} \frac{\partial u'_k}{\partial x_j}} \right) \tag{22}$$

We may commute the first term on RHS. (**??**) is term III and term two in V in the book.

The last terms, except pressure we are able to rewrite using (**??**)

$$\begin{aligned}
\mathrm{I}: &\quad \overline{u'_k \frac{\partial \overline{u'_i u'_j}}{\partial x_j} + u'_i \frac{\partial \overline{u'_k u'_j}}{\partial x_j}} = 0 \\
\mathrm{II}: &\quad \overline{-u'_i \frac{\partial \overline{u_k} u'_j}{\partial x_j} - u'_k \frac{\partial \overline{u_i} u'_j}{\partial x_j}} = -\overline{u'_i u'_j} \frac{\partial \overline{u_k}}{\partial x_j} - \overline{u'_k u'_j} \frac{\partial \overline{u_i}}{\partial x_j} \\
\mathrm{III}: &\quad u'_i \frac{\partial u'_k u'_j}{\partial x_j} + u'_k \frac{\partial u'_i u'_j}{\partial x_j} = \frac{1}{2} \frac{\partial \overline{u'_j u'_i u'_k}}{\partial x_j} + \frac{1}{2} \frac{\partial \overline{u'_j u'_k u'_i}}{\partial x_j} = \frac{\partial \overline{u'_j u'_k u'_i}}{\partial x_j}
\end{aligned} \tag{23}$$

Now all we need is the pressure term.

$$\begin{aligned}
\overline{-u'_k \frac{1}{\rho} \frac{\partial p'}{\partial x_i} - u'_i \frac{1}{\rho} \frac{\partial p'}{\partial x_k}} &= -\frac{1}{\rho} \overline{\left(\frac{\partial u'_k p'}{\partial x_i} + \frac{\partial u'_i p'}{\partial x_k} - p' \frac{\partial u'_k}{\partial x_i} - p' \frac{\partial u'_i}{\partial x_k} \right)} \\
&= -\overline{\frac{\partial}{\partial x_k} \left[\frac{p'}{\rho} (\delta_{jk} u'_i + \delta_{ik} u'_j) \right]} + \overline{\frac{p'}{\rho} \left(\frac{\partial u'_k}{\partial x_i} + \frac{\partial u'_i}{\partial x_k} \right)}
\end{aligned} \tag{24}$$