Maths for CS - Linear Algebra
Practical #2 (week 4)

This practical is about solving linear systems, Gaussian and Gauss-Jordan elimination, elementary matrices and the inversion algorithm. See lecture slides for the basic definitions and examples.

1. Solve the following linear system by Gauss-Jordan elimination

$$\begin{array}{rrrrl} 3x & -y & +z & +7w & = 13 \\ -2x & +y & -z & -3w & = -9 \\ -2x & +y & & -7w & = -8 \end{array}$$

2. Determine all values of a for which the following linear system has (a) no solutions, (b) one solution, or (c) infinitely many solutions.

$$\begin{array}{rrrl} x & +2y & +z & = 2 \\ 2x & -2y & +3z & = 1 \\ x & +2y & -az & = a \end{array}$$

3. Consider the following homogeneous system of linear equations (where a and b are non-zero constants):

$$\begin{array}{rrrl} x & +2y & & = 0 \\ ax & +8y & +3z & = 0 \\ & by & +5z & = 0 \end{array}$$

   (a) Find a value for a which will make it necessary during Gaussian elimination to interchange rows in the coefficient matrix.

   (b) Suppose that a does *not* have the value you found in part (a). Find a value for b so that the system has a nontrivial solution.

   (c) Suppose that a does not have the value you found in part (a) and that $b = 100$. Suppose further that a is chosen so that the solution to the system is *not* unique. Find the general solution to the system.

4. Consider the following matrices A, B, C, D, and E:

$$A = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 & -4 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 0 & 28 \\ 0 & -5 & 25 \end{pmatrix}, \quad F = \begin{pmatrix} 4 & 7 & 9 \\ 0 & 5 & 3 \\ 1 & 3 & -4 \end{pmatrix}$$

   For each of the following equations, find an elementary matrix E that satisfies the equation:

   (a) $EA = B$ (b) $EB = A$ (c) $EA = C$ (d) $EC = A$

   (e) $EB = D$ (f) $ED = B$ (g) $EC = F$ (h) $EF = C$

5. Use the inversion algorithm to find the inverse (if it exists) of each of the following matrices:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 6 & 4 \\ 0 & -2 & 2 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$$

6. (hard) An $n \times n$ matrix $P = (p_{ij})$ is called a *permutation matrix* if there is a bijection $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $p_{ij} = 1$ if $j = \pi(i)$ and $p_{ij} = 0$ otherwise. (Such a bijection is called a permutation of $\{1, \ldots, n\}$.) In other words, every entry of P is either 0 or 1, and there is exactly one 1 in every row and exactly one 1 in every column. Use elementary matrices to prove that $P^{-1} = P^T$.