Homework 8

Michael Anderson

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CS517

Prof. Cull

1

Does:

$$\frac{a}{b} \leq \frac{p}{q} \quad \text{and} \quad \frac{p}{q} \leq \frac{a}{b} \quad \text{imply} \quad \frac{a}{b} = \frac{p}{q}?$$

Yes. a/b and p/q are then (possibly two different ways of writing) the same rational number.

a) If S is not the null set, then $\leq$ is well-defined because all of the elements of the equivalence classes are the same rational number. Iff a rational number (equivalence class in our construction) is less than or equal to a second rational, then the corresponding ordered pair representations of those rationals have that same relationship.

   If S is the null set, then the claim is true because $x \leq y$ and $[x] \leq [y]$ are both always false for any x and y imaginable. False is always the boolean equivalent of false, and iff is the same as double implication or boolean equivalence.

b) Here an equivalence class is simply the set of ordered pairs of naturals that can be used to write some particular positive rational. So $\leq$ must be reflexive and transitive on our equivalence classes because, from grade school math, it is reflexive and transitive on rational numbers.

c) By definition of terms, a number cannot be both greater than and less than some other number. It also cannot be both greater than and equal, or both less than and equal, some other number. So if A is greater than or equal B, and A is less than or equal B, $A = B$.

d) No, because in this construction a positive rational represents an equivalence class in the set of ordered pairs, not an element. A positive rational has an infinite number of ordered pair representations, not any particular one.

e) Since we are defining anti-symmetry with equivalence classes, the definition we want here is:

$$x \leq y \;\text{ or }\; y \leq x \;\text{ or }\; x \equiv y$$

f) Generally, we know that given two rationals x and y, either the first is less than the second, the second is less than the first, or they are equal. If two ordered pairs from our construction are not in the same equivalence class, then they are unequal and one is less than the other. If two ordered pairs from our construction represent the same rational number, then they are in the same equivalence class. So we have that two ordered pairs, by the usual ordering of the rationals given in the problem statement, will either be $<$, $>$, or $\equiv$ to one another.

g)

2

Dividing a list into two lists about a pivot is a subproblem of quicksort.

The set of primes is a *subset* of the set of naturals.

2-SAT is no harder than 3-SAT.

If A is a subproblem of B, then A can be no harder than B, because A must be solved in order to solve B. None of the other 5 possible implications hold.

3

The first relationship holds because we can recognize the complement of a set by simply flipping the output of a recognizer for the set from YES to NO and vice versa. This means that recognizing a set has the same degree as recognizing the complement of a set. By similar reasoning, the second relationship holds as well.

4

a) Want to show that

$$(A \leq_{log} B \quad \text{and} \quad B \leq_{log} C) \quad \rightarrow \quad A \leq_{log} C$$

Since we can bound time as exponential in the amount of space required, we have that log-space reductions are also polynomial time reductions, because a logarithm exponentiated is a polynomial. We already know that polynomial time reductions used in NP, for example, are transitive.

b)

5

6

7

a) Substitute the given solution into the difference equation to obtain:

$$\left(\frac{1-\beta}{\beta}\right)^n \stackrel{?}{=} \beta\left(\frac{1-\beta}{\beta}\right)^{n+1} + (1-\beta)\left(\frac{1-\beta}{\beta}\right)^{n-1}$$

$$\left(\frac{1-\beta}{\beta}\right)^n \stackrel{?}{=} (1-\beta)\left(\frac{1-\beta}{\beta}\right)^n + \beta\left(\frac{1-\beta}{\beta}\right)^n$$

$$\left(\frac{1-\beta}{\beta}\right)^n \stackrel{?}{=} \left(\frac{1-\beta}{\beta}\right)^n - \beta\left(\frac{1-\beta}{\beta}\right)^n + \beta\left(\frac{1-\beta}{\beta}\right)^n$$

$$\left(\frac{1-\beta}{\beta}\right)^n = \left(\frac{1-\beta}{\beta}\right)^n$$

b) Substitution into the solution gives that $p_0 = 1$, and that it is certain n must equal 0. So the algorithm must terminate.

c) The substition in a) holds for any $\beta \neq 0$, which shows that the same solution is still valid here, and that the algorithm must terminate eventually.