① Notes on Barankin Lower Bound (BLB)

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* Let $\boxed{S = \int_{\tilde{\theta} \in \Theta} \frac{f(x|\tilde{\theta}) - f(x|\theta)}{f(x|\theta)} \rho(\tilde{\theta}) d\tilde{\theta}}$.

* Note that when $\rho(\tilde{\theta}) = \frac{1}{h}\delta(\tilde{\theta} - (\theta+h))$, $S = \frac{f(x|\theta+h) - f(x|\theta)}{h \cdot f(x|\theta)}$.

  If $h \to 0$, $S = \lim_{h \to 0} \frac{f(x|\theta+h) - f(x|\theta)}{h \, f(x|\theta)} = \frac{1}{f} \frac{df(x|\theta)}{d\theta} = \frac{d \log f(x|\theta)}{d\theta}$ as

  in the CRLB proof.

* This is a more general result which does not require regularity of $f(x|\theta)$.

① If $\hat{\theta}(\cdot)$ is unbias and $x \sim f(x|\theta)$, then

$$\boxed{E[(\hat{\theta}-\theta)^2] \geq BLB, \quad BLB = \cancel{\max} \frac{A^2}{E[s^2]} \quad A = \int (\tilde{\theta}-\theta)\rho(\tilde{\theta}) d\tilde{\theta}.}$$

(max w.r.t $\rho(\cdot)$ can be applied to the RHS)

Pf: $E_x[s] = 0$ $\left(\int_x s f(x|\theta) dx = 0 \to \text{check} \right) \Rightarrow E_x[\theta s] = 0$ ①

$E_x[\hat{\theta}(x) s] = \int (\tilde{\theta} - \theta)\rho(\tilde{\theta}) d\tilde{\theta} = A$ ② (check..)

② - ① $\Rightarrow$ $E_x[(\hat{\theta}-\theta)s] = A$

By CS $\rightarrow$ $A^2 = (E[(\hat{\theta}-\theta)s])^2 \leq E[(\hat{\theta}-\theta)^2] \cdot E[s^2] \Rightarrow E[(\hat{\theta}-\theta)^2] \leq \frac{A^2}{E[s^2]}$ ☐

②

② Iff $\dfrac{S}{A} = k(\hat{\theta}(X) - \theta) \iff \hat{\theta}(X)$ is MVUE $\left(E[(\hat{\theta}-\theta)^2] = \dfrac{A^2}{E[S^2]}\right)$

and $\boxed{k = \dfrac{1}{BLB}}$.

Pf: Use CS equality result

$1 = E\left[(\hat{\theta}-\theta)\dfrac{S}{A}\right] \Rightarrow$ Iff $\dfrac{S}{A} = k(\hat{\theta}-\theta) \iff E[(\hat{\theta}-\theta)^2] = \dfrac{A^2}{E[S^2]}$.

(3) <u>Example</u>: Estimate $\theta > 0$ from $x_1, \dots x_n \overset{iid}{\sim} U[0, \theta]$.

$$\Rightarrow f(x_1 \cdots x_n | \theta) = \prod_{i=1}^{n} f(x_i | \theta) = \prod_{i=1}^{n} \frac{1}{\theta} I(0 \le x_i \le \theta)$$

$$= \frac{1}{\theta^n} \prod_{i=1}^{n} I(0 \le x_i) \prod_{i=1}^{n} (x_i \le \theta)$$

$$= \frac{1}{\theta^n} I(\theta \ge \max_i x_i) \prod_{i=1}^{n} I(x_i \ge 0)$$

$*$ Set $\mathcal{X} = [0, \infty)^n$
and $\underline{x} = [x_1, \cdots x_n]^T \in \mathcal{X}$. $\Rightarrow f(\underline{x} | \theta) = \frac{1}{\theta^n} I(\theta \ge \max_i x_i)$

$*$ Let $\rho(\tilde{\theta}) = \begin{cases} \tilde{\theta}^n & \tilde{\theta} \le \theta \\ 0 & \text{o.w.} \end{cases}$

$$\Rightarrow A = \int (\tilde{\theta} - \theta) \rho(\tilde{\theta}) d\tilde{\theta} = \int_0^\theta (\tilde{\theta} - \theta) \rho(\tilde{\theta}) d\tilde{\theta} = \int_0^\theta (\tilde{\theta} - \theta) \tilde{\theta}^n d\tilde{\theta} = \int_0^\theta \tilde{\theta}^{n+1} d\tilde{\theta} - \theta \int_0^\theta \tilde{\theta}^n d\tilde{\theta}$$

$$= \frac{\tilde{\theta}^{n+2}}{n+2}\Big|_0^\theta - \theta \left(\frac{\tilde{\theta}^{n+1}}{n+1}\Big|_0^\theta\right)$$

$$= \frac{\theta^{n+2}}{n+2} - \frac{\theta^{n+2}}{n+1}$$

$$= \boxed{- \frac{\theta^{n+2}}{(n+1)(n+2)}}$$

$$S = \int \left(\frac{f(x|\tilde{\theta}) - f(x|\theta)}{f(x|\theta)} \right) \rho(\tilde{\theta}) d\tilde{\theta} = \int \left(\frac{f(x|\tilde{\theta})}{f(x|\theta)} - 1 \right) \rho(\tilde{\theta}) d\tilde{\theta}$$

$$= \int_0^\theta \left(\frac{\theta^n}{\tilde{\theta}^n} I(\tilde{\theta} \ge \max_i x_i) - 1 \right) \tilde{\theta}^n d\tilde{\theta} = \int_0^\theta \theta^n I(\tilde{\theta} \ge \max x_i) d\tilde{\theta} - \int_0^\theta \tilde{\theta}^n d\tilde{\theta}$$

$$= \theta^n (\theta - \max_i(x_i)) - \frac{\theta^{n+1}}{n+1} = \theta^{n+1} - \theta^n \max_i(x_i) - \frac{\theta^{n+1}}{n+1} = \frac{n}{n+1} \theta^{n+1} - \theta^n \max_i x_i$$

$$\frac{S}{A} = \theta^{-2} (n+1)(n+2) \max_i x_i - \theta^{-1} (n+2) \cancel{\tfrac{(n+1)}{(n+1)}} n = \frac{n(n+2)}{\theta^2} \left(\frac{(n+1)}{n} \max_i x_i - \theta \right)$$

(4)

Since,

$$\frac{S}{A} = \underbrace{\frac{n(n+2)}{\theta^2}}_{k} \left(\hat{\theta}(\underline{x}) - \theta\right)$$

$$\boxed{\hat{\theta}(\underline{X}) = \left(\frac{n+1}{n}\right) \cdot \max_i X_i}$$ is MVUE

with $\boxed{MSE = E[(\hat{\theta}-\theta)^2] = BLB = \frac{1}{k} = \frac{\theta^2}{n(n+2)}}$

* Note MSE is $O(n^{-2})$. This a <u>faster rate</u> as compared to the CRLB for n iid samples from a regular $f(x|\theta)$: $CRLB_n = \frac{CRLB_1}{n} \sim O(n^{-1})$.

* Note that $E[x_i] = \frac{\theta}{2}$, here one can consider the MoM estimator for θ : $\hat{\theta}_{mom} = 2\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{2}{n}\sum_{i=1}^{n} X_i$

- The $\hat{\theta}_{mom}$ is unbiased (check...)
- The $\boxed{MSE(\hat{\theta}_{mom}) = \frac{4}{n^2} n \, VAR(X_i) = \frac{4}{n} \frac{\theta^2}{12} = \frac{\theta^2}{3n}} \sim O(n^{-1})$

  - $\hat{\theta}_{mom} = \hat{\theta}$ at $n=1$ $\left(\begin{array}{l} \hat{\theta} = \frac{n+1}{n} \max_i X_i \overset{n=1}{=} 2X_1 \\ \hat{\theta}_{mom} = \frac{2}{1} X_1 = 2X_1 \end{array} \right) \rightarrow MSE_{Mom} = MSE_{opt}$
  - for $n \geq 2$ $MSE_{mom} > MSE_{opt}$
- <u>Average</u> is significantly worse than the <u>max</u> (in this setup).