Final Project Report

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ECE565

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$$y = \begin{bmatrix} y\left([1 - \frac{N+1}{2}]\frac{2f_o}{N}\right) \\ y\left([2 - \frac{N+1}{2}]\frac{2f_o}{N}\right) \\ \vdots \\ y\left([N - \frac{N+1}{2}]\frac{2f_o}{N}\right) \end{bmatrix}$$

The general pdf for a complex normal is:

$$f(y|\theta) = \frac{1}{det(\pi C)} \exp\{-(x-\mu)^T C^{-1}(x-\mu)\}$$

Since μ in this problem is 0, this simplifies to:

$$f(y|\theta) = \frac{1}{det(\pi C)} \exp\{-x^T C^{-1} x\}$$

Since the n draws from the distribution are all independent, with variances as given in the problem statement, C and C^{-1} are the following diagonal matrices:

$$C = diag\left[\frac{SNR \cdot \sigma^2}{\sqrt{f_d^2 - f_1^2}} + \sigma^2, \frac{SNR \cdot \sigma^2}{\sqrt{f_d^2 - f_2^2}} + \sigma^2, \quad \dots \quad, \frac{SNR \cdot \sigma^2}{\sqrt{f_d^2 - f_N^2}} + \sigma^2\right]$$

$$C^{-1} = diag\left[\left(\frac{SNR \cdot \sigma^2}{\sqrt{f_d^2 - f_1^2}} + \sigma^2\right)^{-1}, \left(\frac{SNR \cdot \sigma^2}{\sqrt{f_d^2 - f_2^2}} + \sigma^2\right)^{-1}, \quad \dots \quad, \left(\frac{SNR \cdot \sigma^2}{\sqrt{f_d^2 - f_N^2}} + \sigma^2\right)^{-1}\right]$$

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It is given that the Fisher Information Matrix of a complex normal distribution is:

$$FIM_{ij} = 2R\left[\left(\frac{d\mu}{d\theta_i}\right)^T C^{-1}\frac{d\mu}{d\theta_j}\right] + tr\left[C^{-1}\frac{dC}{d\theta_i}C^{-1}\frac{dC}{d\theta_j}\right]$$

The mean of our complex normal is 0, so the left term is 0, leaving:

$$FIM_{ij} = tr\left[C^{-1}\frac{dC}{d\theta_i}C^{-1}\frac{dC}{d\theta_j}\right]$$

Need to calculate $dC/d\theta$:

$$\frac{d}{df_d}\left(\frac{SNR\cdot\sigma^2}{\sqrt{f_d^2-f_i^2}}+\sigma^2\right) = SNR{\cdot}\sigma^2\ \frac{d}{df_d}(f_d^2{-}f_1^2)^{-1/2} = -\frac{1}{2}2f_d{\cdot}SNR{\cdot}\sigma^2(f_d^2{-}f_i^2)^{-3/2} = -\frac{f_d\cdot SNR\cdot\sigma^2}{(f_d^2-f_i^2)^{3/2}}$$

$$\text{So}\quad \frac{dC}{df_d} = diag\left[-\frac{f_d\cdot SNR\cdot\sigma^2}{(f_d^2-f_1^2)^{3/2}}, -\frac{f_d\cdot SNR\cdot\sigma^2}{(f_d^2-f_2^2)^{3/2}},\ \ldots\ , -\frac{f_d\cdot SNR\cdot\sigma^2}{(f_d^2-f_N^2)^{3/2}}\right]$$

$$\frac{dC}{dSNR} = diag\left[\frac{\sigma^2}{\sqrt{f_d^2-f_1^2}}, \frac{\sigma^2}{\sqrt{f_d^2-f_2^2}},\ \ldots\ , \frac{\sigma^2}{\sqrt{f_d^2-f_N^2}},\right]$$

$$\frac{dC}{d\sigma^2} = diag\left[\frac{SNR}{\sqrt{f_d^2-f_1^2}}+1, \frac{SNR}{\sqrt{f_d^2-f_2^2}}+1,\ \ldots\ , \frac{SNR}{\sqrt{f_d^2-f_N^2}}+1\right]$$

Now to calculate the Fisher Information Matrix. To save a little time, let $k = \sqrt{f_d^2+f_i^2}$ to get:

$$FIM_{11} = \sum_{i=1}^{N}\left(\frac{SNR\cdot\sigma^2}{k}+\sigma^2\right)^{-2}\left(-\frac{f_d\cdot SNR\cdot\sigma^2}{k^{3/2}}\right)^2 = \sum_{i=1}^{n}\frac{f_d^2\cdot SNR^2\cdot\sigma^2\sigma^2}{k^6\left(\frac{SNR^2\cdot\sigma^2\sigma^2}{k^2}+2\frac{SNR^2\cdot\sigma^2\sigma^2}{k}+\sigma^2\sigma^2\right)} =$$

$$\sum_{i=1}^{n}\frac{f_d^2\cdot SNR^2}{k^4(SNR+k)^2}$$

$$FIM_{12} = FIM_{21} = \sum_{i=1}^{n}\left(\frac{SNR\cdot\sigma^2}{k}+\sigma^2\right)^{-2}\left(-\frac{f_d\cdot SNR\cdot\sigma^2}{k^{3/2}}\right)\left(\frac{\sigma^2}{k}\right) =$$

$$\sum_{i=1}^{n}-\frac{f_d^2\cdot SNR\cdot\sigma^2\sigma^2}{k^4\left(\frac{SNR^2\cdot\sigma^2\sigma^2}{k^2}+2\frac{SNR^2\cdot\sigma^2\sigma^2}{k}+\sigma^2\sigma^2\right)} = \sum_{i=1}^{n}-\frac{f_d^2\cdot SNR}{k^2(SNR+k)^2}$$

$$FIM_{13} = FIM_{31} = \sum_{i=1}^{n} \left(\frac{SNR \cdot \sigma^2}{k} + \sigma^2 \right)^{-2} \left(-\frac{f_d \cdot SNR \cdot \sigma^2}{k^{3/2}} \right) \left(\frac{SNR}{k} + 1 \right) =$$

$$\sum_{i=1}^{n} \frac{-\frac{f_d \cdot SNR^2 \cdot \sigma^2}{k^4} - \frac{f_d \cdot SNR \cdot \sigma^2}{k^3}}{\sigma^2 \sigma^2 \left(\frac{SNR}{k} + 1 \right)^2} = \sum_{i=1}^{n} -\frac{f_d \cdot SNR}{k^2 \sigma^2 (SNR + k)^2}$$

$$FIM_{22} = \sum_{i=1}^{n} \left(\frac{SNR \cdot \sigma^2}{k} + \sigma^2 \right)^{-2} \left(\frac{\sigma^2}{k} \right)^2 = \sum_{i=1}^{n} \frac{1}{(SNR + k)^2}$$

$$FIM_{23} = FIM_{32} = \sum_{i=1}^{n} \left(\frac{SNR \cdot \sigma^2}{k} + \sigma^2 \right)^{-2} \left(\frac{\sigma^2}{k} \right) \left(\frac{SNR}{k} + 1 \right) =$$

$$\sum_{i=1}^{n} \frac{\frac{SNR}{k^2} + \frac{1}{k}}{\sigma^2 (SNR/k + 1)^2} = \sum_{i=1}^{n} \frac{1}{\sigma^2 (SNR + k)}$$

$$FIM_{33} = \sum_{i=1}^{n} \left(\frac{SNR \cdot \sigma^2}{k} + \sigma^2 \right)^{-2} \left(\frac{SNR}{k} + 1 \right)^2 = \sum_{i=1}^{n} \frac{1}{\sigma^2 \sigma^2} = \frac{n}{\sigma^2 \sigma^2}$$

To calculate the $CRLB$ we need to invert the FIM, and the result of such an inversion takes quite a bit of time and space to display. The closed form rule for inverting 3 by 3 matrices, along with the Rule of Sarrus for calculating the determinant, makes the process easy but tedious.

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Forthcoming.

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Attached is the code, along with a plot of the CRLB as a function of the SNR. The only portion of the code that is missing is the line that defines the Maximum Likelihood Estimator of f_d.