ECE 565 Estimation, Filtering and Detection - Fall 2011 Homework 1+2 - Dr. Raich*

November 17, 2011

1. Given problem is

$$y(n) = \begin{cases} a & n \text{ is even} \\ b & n \text{ is odd} \end{cases} + e(n) \tag{1}$$

   (a) Fitting the Least Square Model for y, we get

$$y = \mathrm{H} \cdot \theta + e \tag{2}$$

   Writing y in matrix notation, we get

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e(1) \\ e(2) \\ \vdots \\ e(N) \end{bmatrix} \tag{3}$$

   From the above equation, we can show that

$$\theta = \begin{bmatrix} a \\ b \end{bmatrix} \tag{4}$$

   and

$$\mathrm{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \\ 0 & 1 \end{bmatrix} \tag{5}$$

   Note here it is assumed that N is even. If N is odd, then the final row of H would be [1 0].

*I would like to thank Reddy Karthikeyan for his help with scribing these notes.

(b) The solution for the above θ by LS method is given by

$$\hat{\theta}_{LS} = (\mathrm{H}^T\mathrm{H})^{-1}\mathrm{H}^T y \tag{6}$$

- N is even

$$(\mathrm{H}^T\mathrm{H})^{-1} = \begin{bmatrix} 1 & 0 & \ldots 0 \\ 0 & 1 & \ldots 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{N}{2} & 0 \\ 0 & \frac{N}{2} \end{bmatrix} \tag{7}$$

and

$$\mathrm{H}^T y = \begin{bmatrix} 1 & 0 & \ldots 0 \\ 0 & 1 & \ldots 1 \end{bmatrix} \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} \sum_{n=1,3}^{N-1} y(n) \\ \sum_{n=2,4}^{N} y(n) \end{bmatrix} \tag{8}$$

Therefore,

$$\hat{\theta}_{LS} = (\mathrm{H}^T\mathrm{H})^{-1}\mathrm{H}^T y = \begin{bmatrix} \frac{2}{N} & 0 \\ 0 & \frac{2}{N} \end{bmatrix} \begin{bmatrix} \sum_{n=1,3}^{N-1} y(n) \\ \sum_{n=2,4}^{N} y(n) \end{bmatrix} \tag{9}$$

which gives

$$a = \frac{2}{N} \sum_{n=1,3}^{N-1} y(n) \quad \text{and} \quad b = \frac{2}{N} \sum_{n=2,4}^{N} y(n) \tag{10}$$

- If N is odd, it can be shown that a and b are obtained from the above computations as

$$a = \frac{2}{N+1} \sum_{n=1,3}^{N} y(n) \quad \text{and} \quad b = \frac{2}{N-1} \sum_{n=2,4}^{N-1} y(n) \tag{11}$$

(c) The expression for y in terms of θ' can be given as

$$y = \mathrm{H}\mathrm{M}^{-1}\theta' + e(n) \tag{12}$$

The Least Square solution for the above θ' is then given by

$$\hat{\theta}'_{LS} = \left(\left(\mathrm{H}\mathrm{M}^{-1}\right)^T \mathrm{H}\mathrm{M}^{-1} \right)^{-1} \left(\mathrm{H}\mathrm{M}^{-1}\right)^T y \tag{13}$$

Simplifying the above equation by applying the matrix transpose and inverse properties we get,

$$\hat{\theta}'_{LS} = \left(\left(\mathrm{M}^{-1}\right)^T \mathrm{H}^T\mathrm{H}\mathrm{M}^{-1} \right)^{-1} \left(\mathrm{M}^{-1}\right)^T \mathrm{H}^T y \tag{14}$$

$$= \left(\mathrm{M}\mathrm{H}^{-1}\left(\mathrm{H}^T\right)^{-1}\left(\left(\mathrm{M}^{-1}\right)^T\right)^{-1} \right) \left(\mathrm{M}^{-1}\right)^T \mathrm{H}^T y \tag{15}$$

$$= \mathrm{M}\left(\mathrm{H}^T\mathrm{H}\right)^{-1}\mathrm{M}^T\left(\mathrm{M}^{-1}\right)^T\mathrm{H}^T y \tag{16}$$

$$= \mathrm{M}\big(\mathrm{H}^T\mathrm{H}\big)^{-1}\mathrm{H}^T y \tag{17}$$

Replacing $\hat{\theta}_{LS}$ from equation 6, we get

$$\hat{\theta}'_{LS} = \mathrm{M}\hat{\theta}_{LS} \tag{18}$$

(d)

$$\theta' = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{19}$$

Therefore

$$M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \tag{20}$$

- From Equations 10, 20, we can get the LS estimator for c and d for even values of N as

$$\hat{\theta'}_{LS} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{2}{N}\sum_{n=1,3}^{N-1} y(n) \\ \frac{2}{N}\sum_{n=2,4}^{N} y(n) \end{bmatrix} \tag{21}$$

$$c = \frac{1}{N}\sum_{n=1}^{N} y(n) \quad \text{and} \quad d = \frac{1}{N}\Big(\sum_{n=1,3}^{N-1} y(n) - \sum_{n=2,4}^{N} y(n)\Big) \tag{22}$$

- For odd N,

$$c = \frac{1}{N+1}\Big(\sum_{n=1,3}^{N} y(n) + \sum_{n=2,4}^{N-1} y(n)\Big) \quad \text{and} \quad d = \frac{1}{N+1}\Big(\sum_{n=1,3}^{N} y(n) - \sum_{n=2,4}^{N-1} y(n)\Big) \tag{23}$$

2. The relation between y and x is given by

$$y = \sum_{k=0}^{4} a_k x^k \tag{24}$$

(a) Assuming that y_i for i=1,2 $\cdots$ 5 is the observation for the inputs x_i for i=1,2$\cdot\cdots$, we get θ and H from the equation below:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 \\ 1 & x_4 & x_4^2 & x_4^3 & x_4^4 \\ 1 & x_5 & x_5^2 & x_5^3 & x_5^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} \tag{25}$$

Therefore ,

$$\mathrm{H} = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 \\ 1 & x_4 & x_4^2 & x_4^3 & x_4^4 \\ 1 & x_5 & x_5^2 & x_5^3 & x_5^4 \end{bmatrix} \quad \text{and} \quad \theta = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{26}$$

(b) From the constrainst given, we have

$$a_0 = 1 \quad \text{and} \quad a_0 + a_1 + a_2 + a_3 + a_4 = 0 \tag{27}$$

Therefore

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathrm{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{28}$$

(c) The constrained LS estimator for θ is obtained by differentiating the below expression and setting it to zero.

$$||\mathrm{H}\theta - y||^2 + \lambda^T(\mathrm{A}\theta - \mathrm{b}) \tag{29}$$

which gives the following equation

$$2\mathrm{H}^T\mathrm{H}\theta - 2\mathrm{H}^T y = \mathrm{A}^T\lambda = 0 \tag{30}$$

Simplifying it further, we get,

$$\hat{\theta}_{CLS} = \left(\mathrm{H}^T\mathrm{H}\right)^{-1}\mathrm{H}^T y - \frac{1}{2}\left(\mathrm{H}^T\mathrm{H}\right)^{-1}\mathrm{A}^T\lambda \tag{31}$$

Now, if we substitute θ from the above equation into the constraint, $\mathrm{A}\theta = \mathrm{b}$, we get

$$\lambda = 2\left(\mathrm{A}\left(\mathrm{H}^T\mathrm{H}\right)^{-1}\mathrm{A}^T\right)^{-1}\left(\mathrm{A}\left(\mathrm{H}^T\mathrm{H}\right)^{-1}\mathrm{H}^T y - b\right) \tag{32}$$

Substituting the above, we get,

$$\hat{\theta}_{CLS} = \left(\mathrm{H}^T\mathrm{H}\right)^{-1}\mathrm{H}^T y - \left(\mathrm{H}^T\mathrm{H}\right)^{-1}\mathrm{A}^T\left(\mathrm{A}\left(\mathrm{H}^T\mathrm{H}\right)^{-1}\mathrm{A}^T\right)^{-1}\left(\mathrm{A}\left(\mathrm{H}^T\mathrm{H}\right)^{-1}\mathrm{H}^T y - b\right) \tag{33}$$

(d) The code for evaluating the constrained LS estimator is included below:

```
clear all
clc
%%% Generate five (x,y) measurements
c=0;
x=(0.3:0.1:0.7)';
y=1+(c-3)*x.^2+2*(1-c)*x.^3+c*x.^4+0.05*randn(5,1);
%%% Estimate the polynomial coefficients
H=[ones(5,1),x,x.^2,x.^3,x.^4];
a_ls=inv(H'*H)*(H'*y); %%% LS
% Constrainted Least Square Method
A = [1 0 0 0 0; 1 1 1 1 1];
b = [1 0]';
a_cls=a_ls-inv(H'*H)*A'*inv(A*inv(H'*H)*A')*(A*inv(H'*H)*H'*y-b);
%%%% uncomment one for plotting
```

```
a_est=a_ls;
a_est=a_cls;
%%% Plotting true polynomial, noisy measurements, estimated polynomial
xt=(0.01:0.01:1)';
Ht=[ones(size(xt)),xt,xt.^2,xt.^3,xt.^4];
yt=Ht*[1 0 (c-3) (2-2*c) c]';
ye=Ht*a_est;
plot(x,y,'x',xt,yt,xt,ye)
axis([0 1 -0.5 1.5])
legend('Noisy measurements','True polynomial','Estimated polynomial')
```

The estimator vector for the least square and the constrained least squares are obtained respectively as
$\hat{\theta}_{LS} = [-4.674347.1494\ -144.2574182.6038\ -83.5550]'$ and
$\hat{\theta}_{CLS} = [1.0000\ 0.3411\ -5.4946\ 6.3861\ -2.2326]'$
Figures 1 and 2 are curves of the estimated polynomial for least squares and constrained least squares. Even though the estimated curve for the least square exactly matches with the observed measurement, it doesnt satisfy the polynomial we are looking for. On the other hand, the constrained least square has minimised the noise between the observation and model such that the model satisfies our constraints and hence its close to the actual model.

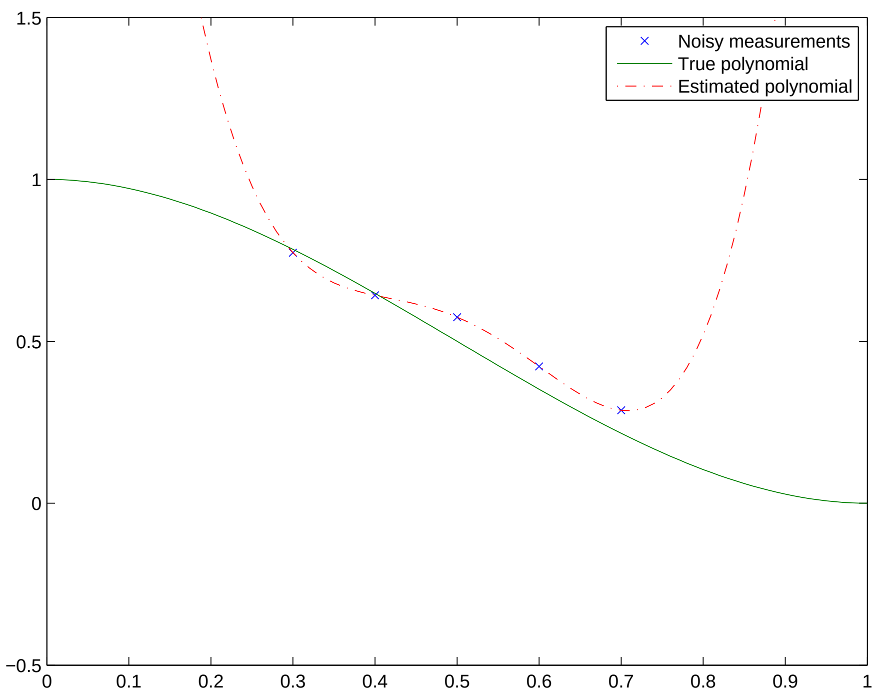

Figure 1: Curve estimated with Least Square Method

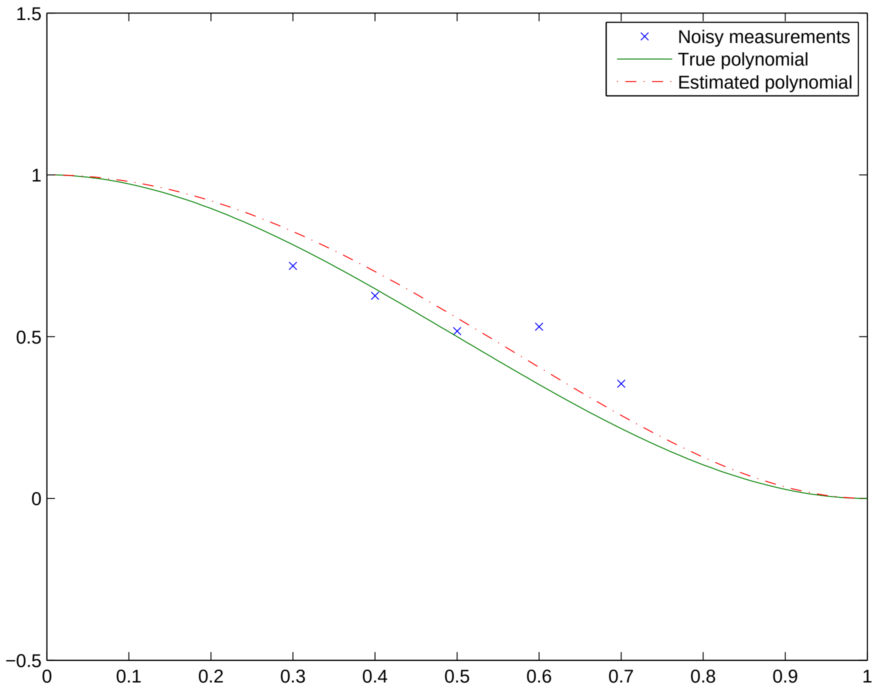

Figure 2: Curve estimated with Constrained Least Square Method

3. The expression for y can be given as

$$\begin{bmatrix} y(n-N+1) \\ y(n-N+2) \\ \vdots \\ y(n) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \cdot [A] + \begin{bmatrix} e(n-N+1) \\ e(n-N+2) \\ \vdots \\ e(n) \end{bmatrix} \tag{34}$$

The minimising expression given as $\sum_{i=n-N+1}^{n} \lambda^{i-n} e(i)^2$ can be written as

$$\begin{bmatrix} e(n-N+1) & e(n-N+2) & \cdots & e(n) \end{bmatrix} \begin{bmatrix} \lambda^{(-N+1)} & 0 & \cdots & 0 \\ 0 & \lambda^{(-N+2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} e(n-N+1) \\ e(n-N+2) \\ \vdots \\ e(n) \end{bmatrix} \tag{35}$$

(a) From the above two equations, we get W, y, H and θ as

$$W = \begin{bmatrix} \lambda^{(-N+1)} & 0 & \cdots & 0 \\ 0 & \lambda^{(-N+2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \tag{36}$$

$$y = \begin{bmatrix} y(n-N+1) \\ y(n-N+2) \\ \vdots \\ y(n) \end{bmatrix} \tag{37}$$

$$H = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{38}$$

$$\theta = \begin{bmatrix} A \end{bmatrix} \tag{39}$$

(b) The WLS estimator for A is given by

$$\hat{\theta}_{WLS} = \left(H^T W H\right)^{-1} H^T W y \tag{40}$$

Substituting for H, W and y, we get the estimator as

$$\hat{\theta}_{WLS} = \frac{\sum_{i=n-N+1}^{n} \lambda^{i-n} y(i)}{\sum_{i=n-N+1}^{n} \lambda^{i-n}} \tag{41}$$

Substituting, i $\Rightarrow$ n-i in the above equation and assigning $x(i) = y(n-i)$, we get,

$$\hat{\theta}_{WLS} = \frac{\sum_{i=0}^{N-1} \lambda^{-i} x(i)}{\sum_{i=0}^{N-1} \lambda^{-i}} \tag{42}$$

As N $\rightarrow \infty$, we know that $\sum_{i=0}^{\infty} \lambda^{-i} = \frac{1}{1-\lambda^{-1}}$. Substituting it in the above equation, we have

$$\hat{\theta}_{WLS} = \frac{1}{1-\lambda^{-1}} \sum_{i=0}^{N-1} \lambda^{-i} x(i), \text{ where } x(i) = y(n-i) \text{ as } n \rightarrow \infty \tag{43}$$

4. (a) Let us denote the $z(t) = x(t) + \jmath\, y(t)$. The observation vector is then given by

$$\underline{z} = \begin{bmatrix} x(\frac{T}{N}) + \jmath\, y(\frac{T}{N}) \\ x(\frac{2T}{N}) + \jmath\, y(\frac{2T}{N}) \\ \vdots \\ x(T) + \jmath\, y(T) \end{bmatrix} \tag{44}$$

Let $e_x(t)$ and $e_y(t)$ be the errors in the observation for $x(t)$ and $y(t)$ respectively. Then the observation $z(t)$ can be written as

$$z(t) = R\mathrm{e}^{\jmath\,\omega\,t} + e(t), \quad \text{where} \quad e(t) = e_x(t) + \jmath\, e_y(t) \tag{45}$$

The $\underline{h}(v)$ is then given by

$$\underline{h}(v) = \begin{bmatrix} R\mathrm{e}^{\jmath \frac{v}{R} \frac{T}{N}} \\ R\mathrm{e}^{\jmath \frac{v}{R} \frac{2T}{N}} \\ \vdots \\ R\mathrm{e}^{\jmath \frac{v}{R}\, T} \end{bmatrix} \tag{46}$$

The cost function that has to be minimised is

$$||\underline{z} - \underline{h}(v)||^2 = [\overline{\underline{z} - \underline{h}(v)}]^T[\underline{z} - \underline{h}(v)] \tag{47}$$

Simplyfying the above equation, we get

$$[\overline{\underline{z} - \underline{h}(v)}]^T[\underline{z} - \underline{h}(v)] = \sum_{n=1}^{N} \overline{\Big(z(n) - R\mathrm{e}^{\jmath \frac{v}{R} \frac{nT}{N}}\Big)}\Big(z(n) - R\mathrm{e}^{\jmath \frac{v}{R} \frac{nT}{N}}\Big) \tag{48}$$

$$= \sum_{n=1}^{N} \bigg(\overline{z(n)}z(n) + R^2 - R\Big(\overline{z(n)}\mathrm{e}^{\jmath \frac{v}{R} \frac{nT}{N}} + z(n)\mathrm{e}^{-\jmath \frac{v}{R} \frac{nT}{N}}\Big)\bigg) \tag{49}$$

$$= \sum_{n=1}^{N} \bigg(x^2(n) + y^2(n) + R^2 - R\Big(\overline{z(n)}\mathrm{e}^{\jmath \frac{v}{R} \frac{nT}{N}} + z(n)\mathrm{e}^{-\jmath \frac{v}{R} \frac{nT}{N}}\Big)\bigg) \tag{50}$$

$$= \sum_{n=1}^{N} \Big(x^2(n) + y^2(n) + R^2\Big) - \sum_{n=1}^{N} R\Big(\overline{z(n)}\mathrm{e}^{\jmath \frac{v}{R} \frac{nT}{N}} + z(n)\mathrm{e}^{-\jmath \frac{v}{R} \frac{nT}{N}}\Big) \tag{51}$$

The first summation in the above expression is a constant, hence to minimise the above expression, we need to maximise the second summation term.

$$\text{maximise} \sum_{n=1}^{N} R\Big(\overline{z(n)}\mathrm{e}^{\jmath \frac{v}{R} \frac{nT}{N}} + z(n)\mathrm{e}^{-\jmath \frac{v}{R} \frac{nT}{N}}\Big) \tag{52}$$

The above expresion on simplification is

$$= \sum_{n=1}^{N} R \cdot 2\mathrm{Re}\Big(z(n)\mathrm{e}^{-\jmath \frac{v}{R} \frac{nT}{N}}\Big) \tag{53}$$

Finally, we get the cost function as

$$\text{maximise}\ \left[2R\sum_{n=1}^{N}\left(x(n)\cos\left(\frac{vnT}{RN}\right)+y(n)\sin\left(\frac{vnT}{RN}\right)\right)\right] \tag{54}$$

(b) The fourier transform of a signal z(t) is given by

$$F(Z)=\sum_{n=1}^{N}z(n)\mathrm{e}^{(-\jmath\omega n)} \tag{55}$$

Expanding it we get,

$$F(Z)=\sum_{n=1}^{N}\Big(x(n)\cos(\omega n\frac{T}{N})+y(n)\sin(\omega n\frac{T}{N})\Big)-\jmath\Big(x(n)\sin(\omega n\frac{T}{N})-y(n)\cos(\omega n\frac{T}{N})\Big) \tag{56}$$

The real part gives the projection of $x(t)$ and $y(t)$ on $\cos(\omega t)$ and $\sin(\omega t)$ respectively. Similarly, the imaginary part gives the projection of $x(t)$ and $y(t)$ on $\sin(\omega t)$ and $\cos(\omega t)$. Since ideally, $x(t) = R\cos(\omega t)$ and $y(t) = R\sin(\omega t)$ we are interested in the projection of $x(t)$ and $y(t)$ on $\cos(\omega t)$ and $\sin(\omega t)$. Therefore, the cost function for the estimation of $v = \omega/R$ involves in maximising the real part of the fourier transform of the observation $z(t)$.

(c) Assuming, $\omega T << 1$, we have

$$\begin{bmatrix} x(1)+\jmath y(1)-R \\ x(2)+\jmath y(2)-R \\ \vdots \\ x(N)+\jmath y(N)-R \end{bmatrix} = \begin{bmatrix} \jmath\frac{T}{N} \\ \jmath\frac{2T}{N} \\ \vdots \\ \jmath T \end{bmatrix} v + \begin{bmatrix} e(1) \\ e(2) \\ \vdots \\ e(N) \end{bmatrix} \tag{57}$$

The Least square estimator is given by,

$$\hat{v}_{LS} = \frac{1}{2}(\overline{H}^T H)^{-1}(\overline{H}^T z + \overline{z}^T H) \tag{58}$$

$$= (\overline{H}^T H)^{-1}\mathrm{Re}(\overline{H}^T z) \tag{59}$$

Computing the above expressions, we get

$$\overline{H}^T H = \left(\frac{T}{N}\right)^2 \sum_{n=1}^{N} n^2 \tag{60}$$

$$\mathrm{Re}(\overline{H}^T z) = \frac{T}{N}\sum_{n=1}^{N} y(n)\cdot n \tag{61}$$

Therefore

$$\hat{v}_{LS} = \frac{N}{T}\frac{\sum_{n=1}^{N} y(n)\cdot n}{\sum_{n=1}^{N} n^2} \tag{62}$$

5. (a) The matrix multiplication, $H_{n+1}^T H_{n+1}$ can be expressed as

$$H_{n+1}^T H_{n+1} = \sum_{i=1}^{n+1} \underline{h}_i \underline{h}_i^T \tag{63}$$

$$= \sum_{i=1}^{n} \underline{h}_i \underline{h}_i^T + \underline{h}_{n+1} \underline{h}_{n+1}^T \tag{64}$$

Therefore,

$$H_{n+1}^T H_{n+1} = H_n^T H_n + \underline{h}_{n+1} \underline{h}_{n+1}^T \tag{65}$$

Also,

$$H_{n+1}^T \underline{y}_{n+1} = \sum_{i=1}^{n+1} \underline{h}_i y(i) \tag{66}$$

$$= \sum_{i=1}^{n} \underline{h}_i y(i) + \underline{h}_i y(n+1) \tag{67}$$

$$H_{n+1}^T \underline{y}_{n+1} = H_n^T y_n + \underline{h}_i y(n+1) \tag{68}$$

Using the results obtained above, we get the LS estimator for θ_{n+1} as

$$\hat{\theta}_{n+1} = \Big(H_n^T H_n + \underline{h}_{n+1} \underline{h}_{n+1}^T\Big)^{-1} \Big(H_n^T y_n + \underline{h}_{n+1} y(n+1)\Big) \tag{69}$$

(b) Matrix inverse lemma is given by

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} \tag{70}$$

Here $A = H_n^T H_n$, $B = -\underline{h}_{n+1}$, $D^{-1} = I$ and $C = \underline{h}_{n+1}^T$. Substituting these in the Lemma, we get

$$(H_n^T H_n)^{-1} + (H_n^T H_n)^{-1}(-\underline{h}_{n+1})\Big(I + \underline{h}_{n+1}^T (H_n^T H_n)^{-1} \underline{h}_{n+1}\Big)^{-1} \underline{h}_{n+1}^T (H_n^T H_n)^{-1} \tag{71}$$

Since $\underline{h}_{n+1}^T (H_n^T H_n)^{-1} \underline{h}_{n+1}$ is a scalar, we can rewrite the above equation as,

$$\Big(H_n^T H_n + \underline{h}_{n+1} \underline{h}_{n+1}^T\Big)^{-1} = (H_n^T H_n)^{-1} - \frac{(H_n^T H_n)^{-1} \underline{h}_{n+1} \underline{h}_{n+1}^T (H_n^T H_n)^{-1}}{1 + \underline{h}_{n+1}^T (H_n^T H_n)^{-1} \underline{h}_{n+1}} \tag{72}$$

(c) Substituting this equation in the LS estimator,

$$\hat{\theta}_{n+1} = \left((H_n^T H_n)^{-1} - \frac{(H_n^T H_n)^{-1} \underline{h}_{n+1} \underline{h}_{n+1}^T (H_n^T H_n)^{-1}}{1 + \underline{h}_{n+1}^T (H_n^T H_n)^{-1} \underline{h}_{n+1}}\right) \Big(H_n^T y_n + \underline{h}_{n+1} y(n+1)\Big) \tag{73}$$

$$= (H_n^T H_n)^{-1} \Big(H_n^T y_n + \underline{h}_{n+1} y(n+1)\Big) - \frac{(H_n^T H_n)^{-1} \underline{h}_{n+1} \underline{h}_{n+1}^T (H_n^T H_n)^{-1} \Big(H_n^T y_n + \underline{h}_{n+1} y(n+1)\Big)}{1 + \underline{h}_{n+1}^T (H_n^T H_n)^{-1} \underline{h}_{n+1}} \tag{74}$$

$$= \hat{\theta}_n + (H_n^T H_n)^{-1}\underline{h}_{n+1} y(n+1) - \frac{(H_n^T H_n)^{-1}\underline{h}_{n+1}\underline{h}_{n+1}^T\Big(\hat{\theta}_n + (H_n^T H_n)^{-1}\underline{h}_{n+1} y(n+1)\Big)}{1 + \underline{h}_{n+1}^T (H_n^T H_n)^{-1}\underline{h}_{n+1}} \tag{75}$$

$$= \hat{\theta}_n + \frac{(H_n^T H_n)^{-1}\underline{h}_{n+1}(y(n+1) - \underline{h}_{n+1}^T\hat{\theta}_n)}{1 + \underline{h}_{n+1}^T (H_n^T H_n)^{-1}\underline{h}_{n+1}} \tag{76}$$

Thus proved

(d) Substituting for P_n, k_n and $e(n)$ which are given in the question in the above equation, we get

$$\hat{\theta}_{n+1} = \hat{\theta}_n + \frac{P_n \underline{h}_{n+1}}{1 + \underline{h}_{n+1}^T P_n \underline{h}_{n+1}}(y(n+1) - \underline{h}_{n+1}^T\hat{\theta}_n) \tag{77}$$

which again on simplification gives

$$\hat{\theta}_{n+1} = \hat{\theta}_n + \underline{k}_n e(n) \tag{78}$$

From equation 72

$$P_{n+1} = P_n - \frac{P_n \underline{h}_{n+1}\underline{h}_{n+1}^T P_n}{1 + \underline{h}_{n+1}^T P_n \underline{h}_{n+1}} \tag{79}$$

Thus

$$P_{n+1} = (I - \underline{k}_n \underline{h}_{n+1}^T) P_n \tag{80}$$