# ECE 565 Estimation, Filtering and Detection - Fall 2011 Homework 3 - Dr. Raich*

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1. **Problem 4.9 in classnotes**
   Given

$$f(x;\theta) = \frac{1}{2}\frac{(1+3\theta x^2)}{1+\theta} \tag{1}$$

   (a) The p.d.f of the n i.i.d sequence is given by

$$P(x_1, x_2, \ldots x_n;\theta) = \prod_{i=1}^{n} \frac{1}{2}\frac{(1+3\theta x_i^2)}{1+\theta} = \frac{1}{[2(1+\theta)]^n}\prod_{i=1}^{n}(1+3\theta x_i^2) \tag{2}$$

   From Equation (2), it can be seen that it is not of the form $h(x)\ G(\theta)\ \mathrm{e}^{A(\theta)\ \hat{\theta}(x)}$. Hence it is does not belong to the exponential family. This can also be tested formally by checking whether $\frac{\partial^2 P}{\partial x \partial \theta}$ can be written as the product $f_1(x) f_2(\theta)$:

$$\frac{\partial^2 P}{\partial x \partial \theta} = \frac{\partial^2 \sum_{i=1}^{n} \log(1+3\theta x_i^2)}{\partial x \partial \theta} = \sum_{i=1}^{n} \frac{6x_i}{(1+\theta x_i^2)^2} \tag{3}$$

   Since the result cannot be factorized into a product of a function of $x$ and a function of $\theta$, $P$ is not a member of the exponential family.

   (b) CR Bound:
   The logarithm of the probability function gives

$$\log(P(x_1, x_2, \ldots x_n;\theta)) = \sum_{i=1}^{n} \log f(x_i;\theta) \tag{4}$$

$$= \sum_{i=1}^{n} \log(1+3\theta\ x_i^2) - n\log(2) - n\log(1+\theta) \tag{5}$$

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The score of the p.d.f is given by

$$\frac{\mathrm{d}\log(P(X;\theta)}{\mathrm{d}\theta} = \sum_{i=1}^{n} \frac{3x_i^2}{(1+3\theta x_i^2)} - \frac{n}{1+\theta} \tag{6}$$

Differentiating it once again, gives

$$\frac{\mathrm{d}^2\log(P(X;\theta)}{\mathrm{d}\theta^2} = -\sum_{i=1}^{n} \frac{9x_i^4}{(1+3\theta x_i^2)^2} + \frac{n}{(1+\theta)^2} \tag{7}$$

The Fischer information matrix is given by

$$FIM = CRLB^{-1} = -\mathrm{E}\Big[\frac{\mathrm{d}^2\log P(X;\theta)}{\mathrm{d}\theta^2}\Big] \tag{8}$$

Taking the expectation of equation 7,

$$\mathrm{E}\Big[\frac{\mathrm{d}^2\log(f(x;\theta)}{\mathrm{d}\theta^2}\Big] = \frac{n}{(1+\theta)^2} - \sum_{i=1}^{n} \mathrm{E}\Big(\frac{9x_i^4}{(1+3\theta x_i^2)^2}\Big) \tag{9}$$

The $\mathrm{E}\Big(\frac{9x_i^4}{(1+3\theta x_i^2)^2}\Big)$ is then given by

$$= \int_{-1}^{1} \Big(\frac{9x^4}{(1+3\theta x^2)^2}\Big) \frac{1}{2} \frac{(1+3\theta x^2)}{1+\theta}\,\mathrm{d}x \tag{10}$$

$$= \frac{9}{2(1+\theta)} \int_{-1}^{1} \frac{x^4}{(1+3\theta x^2)}\,\mathrm{d}x \tag{11}$$

$$= \frac{9}{2(1+\theta)} \left[\frac{\tan^{-1}(\sqrt(3\theta)\ x)}{9\theta^2\sqrt(3\theta)} - \frac{x}{9\theta^2} + \frac{x^3}{9\theta}\right]_{-1}^{1} \tag{12}$$

$$= \frac{1}{\theta^2(1+\theta)} \left[\frac{\tan^{-1}\sqrt(3\theta)}{\sqrt(3\theta)} - (1-\theta)\right] \tag{13}$$

$$\mathrm{E}\Big[\frac{\mathrm{d}^2\log(f(x;\theta)}{\mathrm{d}\theta^2}\Big] = -\frac{n}{\theta^2(1+\theta)^2} \left[\frac{(1+\theta)\tan^{-1}\sqrt(3\theta)}{\sqrt(3\theta)} - 1\right] \tag{14}$$

The CRLB is thus given by

$$CRLB = \frac{\theta^2(1+\theta)^2}{n} \left[\frac{\sqrt(3\theta)}{(1+\theta)\tan^{-1}\sqrt(3\theta) - \sqrt(3\theta)}\right] \tag{15}$$

2. **Problem 4.13 given in classnotes**
Given,

$$f(x_i;\theta) = \frac{1}{\Gamma(\theta)} x^{(\theta-1)} \mathrm{e}^{-x} \tag{16}$$

The p.d.f of the n i.i.d sequence is given by

$$P(x_1, x_2, \ldots x_n; \theta) = \prod_{i=1}^{n} f(x_i;\theta) \tag{17}$$

The logarithm of the probability function gives

$$\log(P(X;\theta)) = \sum_{i=1}^{n} \log f(x_i;\theta) \tag{18}$$

$$= -n \log \Gamma(\theta) + \sum_{i=1}^{n} (\theta \log(x_i) - \log(x_i) - x_i) \tag{19}$$

The score is then given by

$$\frac{\mathrm{d} \log(P(X;\theta)}{\mathrm{d}\theta} = -n \frac{\Gamma'(\theta)}{\Gamma(\theta)} + \sum_{i=1}^{n} \log(x_i) \tag{20}$$

The polygamma function is given by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. Substituting it in the above equation and again taking the derivative, we get

$$\frac{\mathrm{d}^2 \log(P(X;\theta)}{\mathrm{d}\theta^2} = -n\ \psi'(\theta) \tag{21}$$

$$-\mathrm{E}(\frac{\mathrm{d}^2 \log(P(X;\theta)}{\mathrm{d}\theta^2}) = n\ \psi'(\theta) \tag{22}$$

Therefore the CR Lower Bound is given by

$$CRLB = \frac{1}{n\ \psi'(\theta)} \tag{23}$$

3. **Problem 4.18 given in classnotes**

$$p(k;\theta) = \begin{cases} \left(\frac{\theta}{1+\theta}\right)^{k-k_o} \frac{1}{1+\theta} & k = k_o, k_o+1, \dots \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

(a) Yes the density is in the exponential family. The above expression can be rewritten as

$$p(k;\theta) = \frac{1}{(1+\theta)}\ \mathrm{e}^{(k-k_o)\log(\frac{\theta}{1+\theta})} \tag{25}$$

Similarly, for a $P(x_1, x_2, \dots; k)$ can be written as

$$P(X;\theta) = \frac{1}{(1+\theta)^n}\ \mathrm{e}^{\left((\frac{1}{n}\sum_{i=1}^{n} k_i - k_o)\log(\frac{\theta}{1+\theta})^n\right)} \tag{26}$$

This is of the form, $h(k)G(\theta)\mathrm{e}^{A(\theta)\ \hat{\theta(k)}}$, where $h(k) = 1$, $G(\theta) = \frac{1}{(1+\theta)^n}$, $A(\theta) = \log(\frac{\theta}{1+\theta})^n$ and $\hat{\theta(k)} = \frac{1}{n}\sum_1^n k_i - k_o$. Let us derive to see if the score of the p.d.f satisfies the third condition of the cramer rao lower bound, and if we get the estimator for $\theta$ as given here.

(b) The P(X) is given by

$$P(X;\theta) = \prod_1^n \frac{\theta^{k_i - k_o}}{(1+\theta)^{k_i - k_o + 1}} \tag{27}$$

$$\log P(X;\theta) = \sum_1^n \Big((k_i - k_o)\log(\theta) - (k_i - k_o + 1)\log(1+\theta)\Big) \tag{28}$$

The score of the above p.d.f is given by

$$\frac{\mathrm{d}\log P(X)}{\mathrm{d}\theta} = \sum_1^n \frac{(k_i - k_o)}{\theta} - \frac{k_i - k_o + 1}{1+\theta} \tag{29}$$

Simplifying it, we get

$$\frac{\mathrm{d}\log P(X)}{\mathrm{d}\theta} = \frac{n}{\theta\ (1+\theta)}\Big[\Big(\frac{1}{n}\sum_1^n k_i - k_o\Big) - \theta\Big] \tag{30}$$

This is of the form $I(\theta)\big[\hat{\theta}(k) - \theta\big]$ Therefore, the estimator for $\theta$ is

$$\hat{\theta}(k) = \frac{1}{n}\sum_1^n k_i - k_o \tag{31}$$

The CR Lower Bound is then given by $I(\theta)^{-1}$ which is

$$CRLB = \frac{\theta\ (1+\theta)}{n} \tag{32}$$

4. **Problem 4.23 given in classnotes**

(a) Let us consider the p.d.f $f_\theta(x)$ for the parameter $\theta$. For an unbiased estimator, we know that

$$\mathrm{E}[\hat{\theta}(x)] = \int_x \hat{\theta}(x) f_\theta(x)\,\mathrm{d}x = \theta \tag{33}$$

Similarly

$$\int_x \hat{\theta}(x) f_{\theta+\Delta}(x)\,\mathrm{d}x = \theta + \Delta \tag{34}$$

Subtracting (34)-(33), we get

$$\int_x \hat{\theta}(x)\Big(f_{\theta+\Delta}(x) - f_\theta(x)\Big)\,\mathrm{d}x = \Delta \tag{35}$$

$$\int_x \hat{\theta}(x)\frac{\big(f_{\theta+\Delta}(x) - f_\theta(x)\big)}{\Delta}\,\mathrm{d}x = 1 \tag{36}$$

Dividing and multiplying the integral function by $f_\theta(x)$ and replacing $\frac{\big(f_{\theta+\Delta}(x) - f_\theta(x)\big)}{\Delta} = \delta f_\theta$ in (36), we get

$$\int_x \hat{\theta}(x)\big(\frac{\delta f_\theta}{f_\theta}\big) f_\theta(x)\,\mathrm{d}x = 1 = \mathrm{E}[\hat{\theta}(x)\big(\frac{\delta f_\theta}{f_\theta}\big)] \tag{37}$$

We also know that the area under the p.d.f is 1, i.e.,

$$\int_x f_\theta(x)\,\mathrm{d}x = 1 \qquad \text{and} \qquad \int_x f_{\theta+\Delta}(x)\,\mathrm{d}x = 1 \tag{38}$$

Subtracting the above two equations, we get

$$\int_x (f_{\theta+\Delta}(x) - f_\theta(x))\,\mathrm{d}x = 0 \tag{39}$$

Dividing and multiplying the above equation 39 integration function by $f_\theta(x)$, and also dividing the equation by $\Delta$; multiplying it with $\theta$, we get

$$\int_x \theta\big(\frac{\delta f_\theta}{f_\theta}\big) f_\theta(x)\,\mathrm{d}x = 0 = \mathrm{E}[\theta\big(\frac{\delta f_\theta}{f_\theta}\big)] \tag{40}$$

Subtracting equation (37) and (40), we get

$$\int_x (\hat{\theta}(x) - \theta)\big(\frac{\delta f_\theta}{f_\theta}\big) f_\theta(x)\,\mathrm{d}x = 1 = \mathrm{E}[(\hat{\theta}(x) - \theta)\big(\frac{\delta f_\theta}{f_\theta}\big)] \tag{41}$$

By Cauchy-Schwartz inequality, $E[XY]^2 \leq E[X^2]E[Y^2]$. Using this inequality in the equation (41), we get

$$\Big(\mathrm{E}[(\hat{\theta}(x) - \theta)\big(\frac{\delta f_\theta}{f_\theta}\big)]\Big)^2 = 1 \leq \mathrm{E}[(\hat{\theta}(x) - \theta)^2]\mathrm{E}[\big(\frac{\delta f_\theta}{f_\theta}\big)^2] \tag{42}$$

Thus, we have the Chapman-Robbins bound given by

$$\mathrm{E}[(\hat{\theta}(x) - \theta)^2] \geq \frac{1}{\mathrm{E}[\left(\frac{\delta f_\theta}{f_\theta}\right)^2]} \tag{43}$$

(b) Take the function $\frac{\delta f_\theta}{f_\theta}$, and applying the limit as $\Delta \to 0$, if the function is continous in $\theta$, then we get

$$\lim_{\Delta \to 0} \frac{f_{(\theta+\Delta)} - f_\theta}{f_\theta \; \Delta} = \frac{1}{f_\theta} \lim_{\Delta \to 0} \frac{f_{(\theta+\Delta)} - f_\theta}{\Delta} = \frac{f'_\theta}{f_\theta} = (\log f_\theta)' \tag{44}$$

Hence,

$$\lim_{\Delta \to 0} \frac{\delta f_\theta}{f_\theta} = \frac{\mathrm{d} \log(f_\theta)}{\mathrm{d}\theta} \tag{45}$$

Thus in the limiting case as $\Delta \to 0$, the Chapman Robbins bound implies the Cramér-Rao Bound.