Homework 1

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ECE565

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(a) $y(1) = b + e(1)$
$y(2) = a + e(2)$
$y(3) = b + e(3)$
$y(4) = a + e(4)$
$\vdots$
$y(N) = \{a \ or \ b\} + e(N)$

So the transform H that describes this set of equations, and the parameter list θ, are given by:

$$y(n) = H\theta + e(n) \rightarrow$$

$$y(n) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ \vdots & \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + e(n)$$

(b)

$$\hat{\theta}_{LS} = (H^T H)^{-1} H^T y =$$

$$\left(\begin{bmatrix} 0 & 1 & 0 & 1 & \\ 1 & 0 & 1 & 0 & \dots \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ \vdots & \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 & 1 & \\ 1 & 0 & 1 & 0 & \dots \end{bmatrix} \begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \\ \vdots \end{bmatrix} =$$

$$\begin{bmatrix} \lfloor N/2 \rfloor & 0 \\ 0 & \lceil N/2 \rceil \end{bmatrix}^{-1} \begin{bmatrix} y(2) + y(4) + y(6) + \cdots \\ y(1) + y(3) + y(5) + \cdots \end{bmatrix} =$$

$$\begin{bmatrix} \frac{y(2)+y(4)+y(6)+\cdots}{\lfloor n/2 \rfloor} \\ \\ \frac{y(1)+y(3)+y(5)+\cdots}{\lceil n/2 \rceil} \end{bmatrix}$$

(c) With the change of parameters we now have:

$$y(n) = HM\theta + e(n)$$

To return to the canonical form, let $\theta' = M\theta$, so $\theta = M^{-1}\theta'$. Absorb the M^{-1} term into H' to get $H' = HM^{-1}$. So to estimate in terms of θ':

$$\hat{\theta}'_{LS} = (H'^T H')^{-1} H'^T y = ((HM^{-1})^T HM^{-1})^{-1} (HM)^{-1T} y =$$

$$(M^{-1T}(H^T H)M^{-1})^{-1} M^{-1T} H^T y = M(H^T H)^{-1} M^T M^{-1T} H^T y =$$

$$M(H^T H)^{-1} H^T y = M\hat{\theta}_{LS}$$

(d) $c = a/2 + b/2$ and $d = a/2 - b/2$ written as a linear transform M is:

$$M = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Now:

$$\hat{\theta}'_{LS} = M\hat{\theta}_{LS} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} \frac{y(2)+y(4)+y(6)+\cdots}{\lfloor N/2 \rfloor} \\ \\ \frac{y(1)+y(3)+y(5)+\cdots}{\lceil N/2 \rceil} \end{bmatrix} = \begin{bmatrix} \frac{\sum_{i=1}^{N} y(i)}{N} \\ \\ \frac{(y(2)+y(4)+y(6)+\cdots)-(y(1)+y(3)+y(5)+\cdots)}{N} \end{bmatrix}$$

The result fits with intuition. Our c is estimated by the average of all observations, and d is estimated by the average difference of the even (a valued) and odd (b valued) observations.

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(a) $y(n-N+1) = A + \lambda^{-N+1} e(n-N+1)$
$y(n-N+2) = A + \lambda^{-N+2} e(n-N+2)$
$y(n-N+3) = A + \lambda^{-N+3} e(n-N+3)$
$\vdots$
$y(n) = A + \lambda^0 e(N) = A + e(N)$

The problem states that the y values in question are between $y(n-N+1)$ and $y(n)$ inclusive, A appears alone in the $H\theta$ term of each equation as the only parameter, and the error weights are simply the diagonals of W, so the setup is:

$$y(n) = H\theta + We(n) \rightarrow$$

$$\begin{bmatrix} y(n-N+1) \\ y(n-N+2) \\ y(n-N+3) \\ \vdots \\ y(n) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [A] + \begin{bmatrix} \lambda^{-N+1} & 0 & 0 & & 0 \\ 0 & \lambda^{-N+2} & 0 & & 0 \\ 0 & 0 & \lambda^{-N+3} & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} e(n-N+1) \\ e(n-N+2) \\ e(n-N+3) \\ \vdots \\ e(n) \end{bmatrix}$$

(b)

$$\hat{\theta}_{WLS} = (H^T W H)^{-1} H^T W y =$$

$$\left(\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \lambda^{-N+1} & 0 & 0 & & 0 \\ 0 & \lambda^{-N+2} & 0 & & 0 \\ 0 & 0 & \lambda^{-N+3} & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right)^{-1}$$

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \lambda^{-N+1} & 0 & 0 & & 0 \\ 0 & \lambda^{-N+2} & 0 & & 0 \\ 0 & 0 & \lambda^{-N+3} & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y(n-N+1) \\ y(n-N+2) \\ y(n-N+3) \\ \vdots \\ y(n) \end{bmatrix} =$$

$$\left(\frac{1}{\sum_{i=0}^{N-1} \lambda^{-i}} \right) \left(\sum_{i=n-N+1}^{n} \lambda^{i-n} y(i) \right)$$

Since $\lambda^{-i} = (\frac{1}{\lambda})^i$, and recalling that $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$, and letting N go to infinity, yields:

$$\frac{1}{\frac{1}{1-\frac{1}{\lambda}}} \left(\sum_{i=-\infty}^{n} \lambda^{i-n} y(i) \right) = \left(1 - \frac{1}{\lambda} \right) \left(\sum_{i=-\infty}^{n} \lambda^{i-n} y(i) \right)$$