Homework 1

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ST561

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1 Problem 5

Proof by induction:

- **Base case**. It is given in the hint to this problem that we should start at the $n = 3$ case, which we already know works:

$$P(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} P(A_i) \; - \sum_{i=1,j>i}^{3} P(A_i \cap A_j) \; + \sum_{i=1,k>j>i}^{3} P(A_i \cap A_j \cap A_k) =$$

$$P(A_1)+P(A_2)+P(A_3)-P(A_1 \cap A_2)-P(A_1 \cap A_3)-P(A_2 \cap A_3)+P(A_1 \cap A_2 \cap A_3)$$

- **Inductive case**. Want to assume that:

$$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{j>i,i=1}^{n} P(A_i \cap A_j) + \sum_{k>j>i,i=1}^{n} P(A_i \cap A_j \cap A_k) \cdots + (-1)^{n-1} P(\cap_{i=1}^{n} A_i)$$

To show that:

$$P(\cup_{i=1}^{n+1} A_i) = \sum_{i=1}^{n+1} P(A_i) - \sum_{j>i,i=1}^{n+1} P(A_i \cap A_j) + \sum_{k>j>i,i=1}^{n+1} P(A_i \cap A_j \cap A_k) \cdots + (-1)^{n} P(\cap_{i=1}^{n+1} A_i)$$

Here we must consider how to calculate the probability of the union of A_{n+1} with the other n sets. If we simply add in the probability, then we will be double-counting all of the elements in the intersection of A_{n+1} with the other n sets. If we then subtract off that difference, we will subtract off all of the places in which A_{n+1} intersects with two other sets twice, which is one time too many. So we must add those intersections back in, but then we are double-counting all of the places in which A_{n+1} intersects with three sets. So we must subtract that off... and this pattern continues up to n. This insight, along with the inductive assumption, gives:

$$P(\cup_{i=1}^{n} A_i) \cup P(A_{n+1}) = \sum_{i=1}^{n} P(A_i) - \sum_{j>i,i=1}^{n} P(A_i \cap A_j) + \sum_{k>j>i,i=1}^{n} P(A_i \cap A_j \cap A_k) \cdots + (-1)^{n-1} P(\cap_{i=1}^{n} A_i)$$

$$+P(A_{n+1}) - \sum_{i=1}^{n} P(A_{n+1} \cap A_i) + \sum_{j>i,i=1}^{n} P(A_{n+1} \cap A_i \cap A_j) \cdots + (-1)^{n} P(\cap_{i=1}^{n+1} A_i)$$

The two terms on the left side of the equation are easy to combine, and the terms on the right side can be combined in pairs by like number of intersections to give:

$$P(\cup_{i=1}^{n+1} A_i) = \sum_{i=1}^{n+1} P(A_i) - \sum_{j>i,i=1}^{n+1} P(A_i \cap A_j) + \sum_{k>j>i,i=1}^{n+1} P(A_i \cap A_j \cap A_k) \cdots (-1)^n P(\cap_{i=1}^{n+1} A_i)$$

2 Problem 6

Assume that each of the possible 4 combinations of two children (BB, BG, GB, GG) are equally likely.

In the case of the man, we are given the information that at least one of his children is a boy, or alternatively that he does not have two girls. We have then 3 equally likely possibilities remaining (BG, GB, BB), of which only one is two boys. So the probability that he has two boys is $\frac{1}{3}$.

In the case of the woman, we are given that her second child is a boy. We have then 2 equally likely possibilities remaining (BB, BG), of which only one is two boys. So the probability that she has two boys is $\frac{1}{2}$.

So no, the probabilities are not equal.

3 Problem 7

(a) Let:

$P(C) = \frac{1}{5}$	be the probability that a taxpayer cheated
$P(C^C) = \frac{4}{5}$	be the probability that a taxpayer did not cheat
$P(Y\|R) = 1$	be the probability that a taxpayer said "Yes" given that they rolled red
$P(Y\|C) = 1$	be the probability that a taxpayer said "Yes" given that they cheated
$P(Y\|C^C) = \frac{1}{5}$	be the probability that a taxpayer said "Yes" given that they did not cheat

Want to find $P(C|Y)$, the probability that a taxpayer cheated given that they said "Yes". By Bayes' Theorem:

$$P(C|Y) = \frac{P(Y|C)P(C)}{P(Y)} = \frac{P(Y|C)P(C)}{P(Y|C)P(C) + P(Y|C^C)P(C^C)} = \frac{1(1/5)}{1(1/5) + (1/3)(4/5)} = \frac{3}{7}$$

(b) We get no information from the approximately $120 \times \frac{1}{3} = 40$ taxpayers who said said "Yes" automatically because they rolled red. Out of the approximately $120 - 40 = 80$ taxpayers who rolled green, $54 - 40 = 14$ said "Yes" and therefore cheated. So the estimate is then $14/80$.