# Homework 3

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ST561

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# 1

Since $F(\infty) = 1$, and since the only places where the p.d.f is not equal to zero are the intervals [0,1] and [2,3], we have:

$$F(\infty) = \int_0^1 f(x)dx + \int_2^3 f(x)dx = 1$$

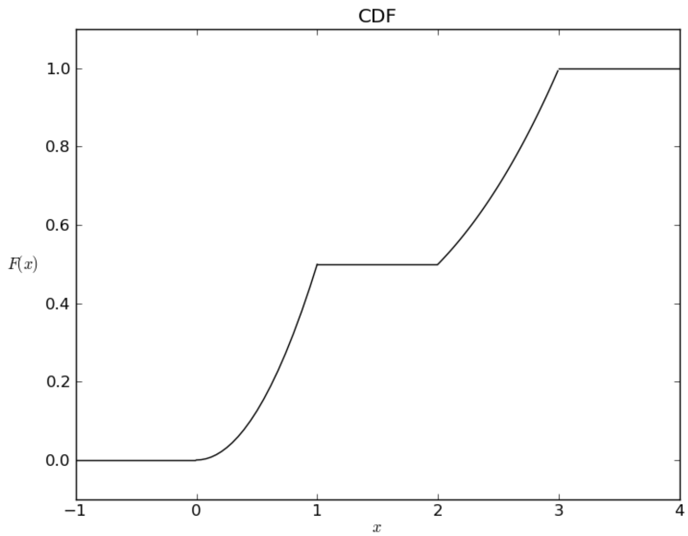
$$\int_2^3 cx^2\,dx = 1 - \int_0^1 x\,dx$$

$$\left.\frac{cx^3}{3}\right|^{x=3} - \left.\frac{cx^3}{3}\right|^{x=2} = 1 - \left(\left.\frac{x^2}{2}\right|^{x=1} - \left.\frac{x^2}{2}\right|^{x=0}\right)$$

$$\frac{19}{3}c = \frac{1}{2} \quad \Longrightarrow \quad c = \frac{3}{38}$$

Using these results to find the c.d.f. yields:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2/2 & 0 \le x < 1 \\ 1/2 & 1 \le x < 2 \\ 1/2 + (1/38)x^3 - 8/38 & 2 \le x < 3 \\ 1 & 3 \le x \end{cases}$$




As can be seen from the plot, $F(x)$ is continuous everywhere. $F(x)$ is split piecewise at the 4 points $x = 0$, $x = 1$, $x = 2$, $x = 3$. It is only differentiable at $x = 0$, where the derivative approaches 0 from both the right and left side. It is not differentiable at the other 3 points.

$$P(-1.5 < X < 1.8) \;=\; F(1.8) - F(-1.5) \;=\; \frac{1}{2} - 0 \;=\; \frac{1}{2}$$

We have that $m$ is the median of $X$ iff $P(x \leq m) = 1/2$ and $P(x \geq m) = 1/2$. These equalities hold for values of x where $F(x) = 1/2$, which is in the interval [1,2].

# 2

It is given by symmetry that for every possible $f(x)$, $x > a$, there is one and only one $f(2a - x)$, $(2a - x) < a$ such that $f(x) = f(2a - x)$. This means that:

$$\int_{-\infty}^{a} f(x)dx = \int_{a}^{\infty} f(x)dx$$

Since each value of df of the integral on the left side of the equation has a corresponding and equal value of df in the integral on the right, and vice versa.

Now since the intervals of these integrals partition the real numbers about $a$, and since $\int_{-\infty}^{\infty} f(x)dx = 1$, we have:

$$\int_{-\infty}^{a} f(x)dx = \int_{a}^{\infty} f(x)dx = 1/2$$

So $a$ satisfies the definition of median given at the end of Problem 1.

## 3

Yes, each can be evaluated uniquely.

Since Gaussian distributions are symmetric about their median, the mean and median are equal. Therefore, $\mu = 50$.

Standardizing gives:

$$1 - 0.025 = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$0.975 = \Phi\left(\frac{100-50}{\sigma}\right)$$

$$1.96 = \frac{100-50}{\sigma}$$

$$\sigma = \frac{100-50}{1.96} \approx 25.51$$

## 4

Since $|X|$ cannot take on negative values, $F(x)_{|X|}$ and $f(x)_{|X|}$ will equal 0 for any $x < 0$. For $x \geq 0$, $f(x)_{|X|}$ will be the sum of $f(x)$ and $f(-x)$. $F(x)_{|X|}$ will include not only the area under $f(x)$ from 0 to $x$, but also the area under $f(x)$ from $-x$ to 0. So:

$$f(x)_{|X|} = \begin{cases} 0 & x < 0 \\ f(x) + f(-x) & x \geq 0 \end{cases}$$

$$F(x)_{|X|} = \begin{cases} 0 & x < 0 \\ \int_{-x}^{x} f(x)dx & x \geq 0 \end{cases}$$

# 5

Standardizing gives:

$$1 - 0.015 = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$0.985 = \Phi\left(\frac{8-\mu}{0.25}\right)$$

$$2.17 = \frac{8-\mu}{0.25}$$

$$\mu = 8 - (2.17)(0.25) \approx 7.46$$

# 6

Since we can think of the mean of a random variable as the average of all possible values it could take, weighted by the probability of their occurence, we have:

$$\bar{x} = \int_{-\infty}^{\infty} xf(x)$$

Where $f(x)$ is the p.d.f. of the continuous random variable $X$.

(a)

$$\bar{x} = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{1} ax^{a}dx = \left.\frac{ax^{a+1}}{a+1}\right|^{x=1} - \left.\frac{ax^{a+1}}{a+1}\right|^{x=0} = \frac{a}{a+1} - 0 = \frac{a}{a+1}$$

(b)

$$\bar{x} = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{2} x(3/2)(x-1)^2dx = \int_{0}^{2}\left(\frac{3x^3}{2} - 3x^2 + \frac{3x}{2}\right)dx =$$

$$\frac{3x^4}{8} - x^3 + \frac{3x^2}{4}\Bigg|^{x=2} - \frac{3x^4}{8} - x^3 + \frac{3x^2}{4}\Bigg|^{x=0} = (6 - 8 + 3) - 0 = 0$$

# 7

$F(x) = 1$ for arbitrarily large values of $x$. So for arbitrarily large $x$ the area under $F(x)$ over the interval $(x, x + k) = 1k = k$. Similarly, as $x$ goes to infinity the area under $F(x)$ over the interval $(x + a, x + b)$ is simply $1(b - a) = b - a$. This can be restated as:

$$\int_{-\infty}^{\infty} [F(x + b) - F(x + a)]dx = b - a$$