ST 561 Homework Solution 1 (Total pts: 30)

Due Monday Oct. 3, 2011

1.
(a) $A_1 \cup A_2 = \{x : x = 0 < x < 3\}$, $A_1 \cap A_2 = \emptyset$.
(b) You need to draw a diagram, and $A_1 \cup A_2 =$ both squares, and $A_1 \cap A_2 = \{(x, y) : 1 < x < 3, 1 < y < 3\}$, the overlapped area.
(c) You also need to draw a diagram for this question: $A_1 \cup A_2 = A_1$ the triangle in the first quadrant, and $A_1 \cap A_2 = A_2$, the quarter circle in the first quadrant.

2.
(a) $\bigcup_{k=1}^{\infty} A_k = \{x : 0 < x < 2\}$.
(b) $\bigcup_{k=1}^{\infty} A_k = \{(x, y) : 0 < x^2 + y^2 < 4\}$.

3.
(a) $\bigcap_{k=1}^{\infty} A_k = \{x : x = 2\}$.
(b) $\bigcap_{k=1}^{\infty} A_k = \emptyset$.
(c) $\bigcap_{k=1}^{\infty} A_k = \{(0, 0)\}$.

4.
$P(A \cup B \cup C) = P([A \cup B] \cup C) = P(A \cup B) + P(C) - P([A \cup B] \cap C)$
$= P(A) + P(B) - P(A \cap B) + P(C) - P([A \cap C] \cup [B \cap C])$
$= P(A) + P(B) + P(C) - P(A \cap B) - \{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)\}$
$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

5. (10 points)
Using the result for $n = 2$, we have
$P(\bigcup_{i=1}^{n} A_i) = P(\bigcup_{i=1}^{n-1} A_i) + P(A_n) - P(\bigcup_{i=1}^{n-1} (A_i \cap A_n) \equiv part(1) + part(2) + part(3)$
Use the induction assumption for $n - 1$ on part (1) and part (3) to get

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n-1} \sum_{j}^{n-1} P(A_i \cap A_j) + \sum_{i<j}^{n-1} \sum_{j<k}^{n-1} \sum_{k}^{n-1} P(A_i \cap A_j \cap A_k) + \cdots$$

$$+(-1)^{n-2} P(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) - \{\sum_{i=1}^{n-1} P(A_i \cap A_n) - \sum_{i<j}^{n-1} \sum_{j}^{n-1} P(A_i \cap A_j \cap A_n) +$$

$$\sum_{i<j}^{n-1} \sum_{j<k}^{n-1} \sum_{k}^{n-1} (A_i \cap A_j \cap A_k \cap A_n) - \cdots + (-1)^{n-2} P(A_1 \cap A_2 \cap \cdots \cap A_n)\}$$

Now put these pieces together to get

$$\sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} \sum_{j}^{n} P(A_i \cap A_j) + \sum_{i<j}^{n} \sum_{j<k}^{n} \sum_{k}^{n} P(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

This completes the proof.

6. (10 points)
P(man has 2 boys | at least one boy) = P((2 boys) $\cap$ (at least one boy))/P(at least one boy) = P(2 boys)/P(at least one boy) = $\frac{1}{4} / \frac{3}{4} = \frac{1}{3}$.

P(woman has 2 boys | younger is boy) = P((2 boys) $\cap$ (younger is a boy))/P(younger is a boy) = P(2 boys)/P(younger is a boy) = $\frac{1}{4}/\frac{1}{2} = \frac{1}{2}$.

7. (10 points)
(a) P(cheat) = 0.2, P(no cheat) = 1-P(cheat) = 0.8. P(green) = 4/6 = 2/3, P(red) = 2/6 = 1/3. We'll find P(cheat | say Yes), which equals to P(cheat $\cap$ say Yes)/P(say Yes).

First calculate the denominator: P(say Yes) = P(say Yes | green)P(green) + P(say Yes | red)P(red) = $0.2 \times 2/3 + 1 \times 1/3 = 7/15$. Note that the tricky part here is P(say Yes | green) = P(cheat) = 0.2.

Next calculate the numerator: P(cheat $\cap$ say Yes) = P(cheat) - P(cheat $\cap$ say No) = P(cheat) - 0 = P(cheat) = 0.2.

Thus P(cheat | say Yes) = 0.2/(7/15) = 3/7 = 42.86%.

(b) Note that P(say Yes) = P(say Yes | green)P(green) + P(say Yes | red)P(red). Since P(say Yes) = 54/120, we have

$$54/120 = P(sayYes|green) \times (2/3) + 1 \times (1/3),$$

so P(cheat) = P(say Yes | green) = 7/40 = 17.5%.

The grades are available on Blackboard. If you have any questions about these exercises, or if you find any grading errors, please stop by Kidder 113B, Thursday 3-4pm.