

Homework 1

Michael Anderson

January 17, 2012

ST562

1 6.2.1

$$f_X(x) = p^2(1-p)^{x_1}(1-p)^{x_2} = p^2(1-p)^{x_1+x_2}$$

$$f_T(t) = \sum_0^t p(1-p)^i p(1-p)^{t-i} = p^2(t+1)(1-p)^t$$

$$\frac{f_X(x)}{f_T(t)} = \frac{p^2(1-p)^{x_1+x_2}}{p^2(t+1)(1-p)^{x_1+x_2}} = \frac{1}{t+1}$$

$\frac{f_X(x)}{f_T(t)}$ is not a function of p .

2 6.2.2

Substituting in $1-p$ for q , and letting $\hat{x} = \sum_0^m x_i$ and $\hat{y} = \sum_0^n y_i$ gives:

$$f_{X,Y}(x,y) = \left(\prod_1^m p^{x_i} (1-p)^{1-x_i} \right) \left(\prod_1^n p^{1-y_i} (1-p)^{y_i} \right) = p^{n+\hat{x}-\hat{y}} (1-p)^{m-\hat{x}+\hat{y}}$$

\hat{X} is distributed $Bin(p, m)$, \hat{Y} is distributed $Bin(1-p, n)$, $T = \hat{X} - \hat{Y}$. Using the discrete version of convolution for subtracting random variables we have:

$$f_T(t) = \sum_0^n f_{\hat{Y}}(\hat{y}_i) f_{\hat{X}}(t + \hat{y}_i) =$$

$$\sum_0^n \left[C_{\hat{y}_i}^n p^{n-\hat{y}_i} (1-p)^{\hat{y}_i} C_{t+\hat{y}_i}^m p^{t+\hat{y}_i} (1-p)^{m-(t+\hat{y}_i)} \right] =$$

$$p^{n+t} (1-p)^{m-t} \sum_0^n C_{\hat{y}_i}^n C_{t+\hat{y}_i}^m$$

$$\frac{f_{X,Y}(x,y)}{f_T(t)} = \frac{p^{n+\hat{x}-\hat{y}} (1-p)^{m-\hat{x}+\hat{y}}}{p^{n+t} (1-p)^{m-t} \sum_0^n C_{\hat{y}_i}^n C_{t+\hat{y}_i}^m} = \frac{1}{\sum_0^n C_{\hat{y}_i}^n C_{t+\hat{y}_i}^m}$$

$\frac{f_{X,Y}(x,y)}{f_T(t)}$ is not a function of p .

3 6.2.5

Using the resulting form of $f_{X,Y}(x, y)$ from the previous problem:

$$f_{X,Y}(x, y) = p^{3+(x_1+x_2)-(y_1+y_2+y_3)}(1-p)^{2-(x_1+x_2)+(y_1+y_2+y_3)}$$

$Y_1 Y_2 = 1$ iff $Y_1 = Y_2 = 1$, so:

$$f_{Y_1 Y_2}(y) = (q^2)^y(1-q^2)^{1-y} = (1-p)^{2y} [1 - (1-p)^2]^{1-y}$$

Now by discrete convolution, where $T = X_1 + Y_1 Y_2$:

$$f_T(t) = f_X(t)f_{Y_1 Y_2}(0) + f_X(t-1)f_{Y_1 Y_2}(1) = p^t(1-p)^{1-t}p^2 + p^{t-1}(1-p)^{1-(t-1)}(1-p^2) =$$

$$p^{t+2}(1-p)^{1-t} + p^{t-1}(1-p)^{2-t}(1-p^2)$$

$$\frac{f_{X,Y}(x, y)}{f_T(t)} = \frac{p^{3+(x_1+x_2)-(y_1+y_2+y_3)}(1-p)^{2-(x_1+x_2)+(y_1+y_2+y_3)}}{p^{t+2}(1-p)^{1-t} + p^{t-1}(1-p)^{2-t}(1-p^2)}$$

Cannot cancel out p 's out of the above, $\frac{f_{X,Y}(x,y)}{f_T(t)}$ is a function of p .

4 6.2.6

$$f_{X,Y}(x, y) = \prod_1^m \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \prod_1^n \frac{2^{y_i} \lambda^{y_i} e^{-2\lambda}}{y_i!} = \frac{2^{(\sum_0^n y_i)} \lambda^{(\sum_0^m x_i) + (\sum_0^n y_i)} e^{-\lambda(m+2n)}}{(\prod_1^m x_i!)(\prod_1^n y_i!)}$$

Since $Poisson(\lambda_1) + Poisson(\lambda_2) = Poisson(\lambda_1 + \lambda_2)$:

$$f_T(t) = \frac{(\lambda[m+2n])^t e^{-\lambda(m+2n)}}{t!}$$

$$\frac{f_{X,Y}(x, y)}{f_T(t)} = \frac{\frac{2^{(\sum_0^n y_i)} \lambda^{(\sum_0^m x_i) + (\sum_0^n y_i)} e^{-\lambda(m+2n)}}{(\prod_1^m x_i!)(\prod_1^n y_i!)}}{\frac{(\lambda[m+2n])^t e^{-\lambda(m+2n)}}{t!}} = \frac{2^{(\sum_0^n y_i)} t!}{(\prod_1^m x_i!)(\prod_1^n y_i!)(m+2n)^t}$$

$\frac{f_{X,Y}(x,y)}{f_T(t)}$ is not a function of λ .

5 6.2.7

$$f_{X,Y}(x,y) = \frac{2(\sum_1^5 y_i) \lambda (\sum_1^4 x_i) + (\sum_1^5 y_1) e^{-14\lambda}}{(\prod_1^4 x_i!) (\prod_1^5 y_i!)}$$

$$f_T(t) = \frac{(3\lambda)^t e^{-3\lambda}}{t!}$$

$$\frac{f_{X,Y}(x,y)}{f_T(t)} = \frac{\frac{2(\sum_1^5 y_i) \lambda (\sum_1^4 x_i) + (\sum_1^5 y_1) e^{-14\lambda}}{(\prod_1^4 x_i!) (\prod_1^5 y_i!)}}{\frac{(3\lambda)^t e^{-3\lambda}}{t!}} =$$

$$\frac{2(\sum_1^5)}{3^t} \lambda (\sum_2^m x_i) + (\sum_2^n y_i) e^{-11\lambda} \frac{t!}{(\prod_1^m x_i) (\prod_1^n y_i)}$$

Cannot cancel p 's from above, $\frac{f_{X,Y}(x,y)}{f_T(t)}$ is a function of p .

6 6.2.10

(i)

$$\begin{aligned} f_X(x) &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp \left\{ -\frac{\sum_1^n (x_i - \mu)^2}{2\sigma^2} \right\} = \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp \left\{ -\frac{\sum_1^n (x_i^2 - 2x_i\mu + \mu^2)}{2\sigma^2} \right\} = \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp \left\{ -\frac{\sum_1^n x_i^2}{2\sigma^2} \right\} \exp \left\{ n \frac{2\mu n^{-1} \sum_1^n x_i - \mu^2}{2\sigma^2} \right\} \end{aligned}$$

(ii)

$$f_X(x) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp \left\{ -\frac{nn^{-1} \sum_1^n (x_i - \mu)^2}{2\sigma^2} \right\} 1$$

7 6.2.11

(i)

$$\begin{aligned} f_X(x) &= \sigma^{-n} \exp \left\{ - \sum_1^n (x_i - \mu) / \sigma \right\} I(x_{(1)} > \mu) = \\ &= \sigma^{-n} \exp \left\{ - \sum_1^n (x_i - \mu) / \sigma \right\} \exp \{ \mu n / \sigma \} I(x_{(1)} > \mu) \end{aligned}$$

(ii)

$$f_X(x) = \sigma^{-n} \exp \left\{ - n n^{-1} \sum_1^n (x_i - \mu) / \sigma \right\} I(x_{(1)} > \mu)$$

(iii)

$$\begin{aligned} f_X(x) &= \sigma^{-n} \exp \left\{ - \sum_1^n (x_i - \mu) / \sigma \right\} I(x_{(1)} > \mu) = \\ &= \sigma^{-n} \exp \left\{ - \sum_1^n [(x_i - x_{(1)}) - (\mu - x_{(1)})] / \sigma \right\} I(x_{(1)} > \mu) = \\ &= \sigma^{-n} \exp \left\{ - \left[\left(\sum_1^n x_i - x_{(1)} \right) - n\mu + nx_{(1)} \right] / \sigma \right\} I(x_{(1)} > \mu) \end{aligned}$$

8 6.2.13

$$f_X(x) = I \left(\theta - \frac{1}{2} < X_{(1)} \right) I \left(X_{(n)} < \theta + \frac{1}{2} \right) 1$$

9 6.2.16

$$f_X(x) = 2^n \theta^{-n} \exp \left\{ - \left(\sum_1^n x_i^2 \right) / \theta \right\} \left(\prod_1^n x_i \right) I(x_{(1)} > 0)$$