Homework 2

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ST562

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1 6.3.2

$$f_X(x) = \sigma^{-n} \exp\left\{ -\sum_1^n (x_i - \mu)/\sigma \right\} I(x_{(1)} > \mu)$$

(i)

$$f_X(x)/f_X(y) = \exp\left\{ \left(\sum_{i=1}^n y_i - \sum_{j=1}^n x_j \right) /\sigma \right\} \frac{I(x_{(1)} > \mu)}{I(y_{(1)} > \mu)}$$

Let $A = I(x_{(1)} > \mu)/I(y_{(1)} > \mu)$. Suppose $x_{(1)} > y_{(1)}$. Then consider the following three possible cases for the value of μ relative to the order statistics:

$$\begin{cases} A = 0/0 & \text{if } \mu > x_{(1)} \\ A = 1/0 & \text{if } x_{(1)} > \mu > y_{(1)} \\ A = 1 & \text{if } y_{(1)} > \mu \end{cases}$$

Since the above shows that in this instance the value of A, and by extension the value of the entire likelihood ratio, is dependent on μ, $x_{(1)} \leq y_{(1)}$ to make the likelihood ratio not dependent on μ. Now supposing $x_{(1)} < y_{(1)}$, a symmetric argument to the above will show that $x_{(1)} \geq y_{(1)}$. So $x_{(1)} = y_{(1)}$ to make the likelihood ratio not depend on μ, and so $x_{(1)}$ is minimally sufficient for μ.

(ii) Again:

$$f_X(x)/f_X(y) = \exp\left\{\left(\sum_{i=1}^n y_i - \sum_{j=1}^n x_j\right)/\sigma\right\}\frac{I(x_{(1)} > \mu)}{I(y_{(1)} > \mu)}$$

$\sum_{j=1}^n x_i = \sum_{i=1}^n y_j$ in order to make σ drop out. Since

$$n^{-1}\sum_1^n (x_i - \mu) = n^{-1}\left(\left[\sum_1^n x_i\right] - \mu n\right)$$

and since n and μ are known, $n^{-1}\sum_1^n (x_i - \mu)$ is a function of $\sum_1^n x_i$ and is minimally sufficient for σ.

(iii) Again:

$$f_X(x)/f_X(y) = \exp\left\{\left(\sum_{i=1}^n y_i - \sum_{j=1}^n x_j\right)/\sigma\right\}\frac{I(x_{(1)} > \mu)}{I(y_{(1)} > \mu)}$$

Clearly need $\sum_{i=1}^n x_i = \sum_{j=1}^n x_j$ and $x_{(1)} = y_{(1)}$ to eliminate both μ and σ, making $(x_{(1)}, \sum_1^n x_i)$ minimally sufficient. Since

$$\left(x_{(1)}, \sum_1^n [x_i - x_{(1)}]\right) = \left(x_{(1)}, \left[\sum_1^n x_i\right] - n x_{(1)}\right)$$

$\left(x_{(1)}, \sum_1^n [x_i - x_{(1)}]\right)$ is a function of $(x_{(1)}, \sum_1^n x_i)$ and is minimally sufficient.

2 6.3.8

$$f_X(x) = \frac{1}{(2\pi\theta^2)^{-n/2}}\exp\left\{-\sum_1^n (x_i - \theta)^2/(2\theta^2)\right\}$$

$$= \frac{1}{(2\pi\theta^2)^{-n/2}}\exp\left\{-\sum_1^n (x_i^2 - 2x_i\theta + \theta^2)/(2\theta^2)\right\}$$

$$= \frac{1}{(2\pi\theta^2)^{-n/2}}\exp\left\{-\frac{\sum_1^n x_i^2}{2\theta^2} + \frac{2\sum_1^n x_i\theta}{2\theta^2} - \frac{n\theta^2}{2\theta^2}\right\}$$

$$= A(\theta) \exp\left\{-\frac{\sum_1^n x_i^2}{2\theta^2} + \frac{\sum_1^n x_i}{\theta}\right\}$$

A distribution with the given pdf is not a member of the 2-parameter exponential family, because the parameter vector contains only one element. It is however part of the *curved exponential family*, as defined in the class notes.

To apply Theorem 3 from the notes, let $R_1 = \sum_1^n x_i^2$, and $\eta_1 = -1/(2\theta)$, and $R_2 = \sum_1^n x_i$, and $\eta_2 = \theta$. Since R_1 and R_2 are linearly independent, we get the following minimal sufficient statistic T:

$$T = (R_1, R_2) = \left(\sum_1^n x_i^2, \sum_1^n x_i\right)$$

3 6.3.9

$$f_X(x) = \frac{1}{(2\pi\theta)^{-n/2}} \exp\left\{-\sum_1^n (x_i - \theta)^2/(2\theta)\right\}$$

$$= \frac{1}{(2\pi\theta)^{-n/2}} \exp\left\{-\sum_1^n (x_i^2 - 2x_i\theta + \theta^2)/(2\theta)\right\}$$

$$= \frac{1}{(2\pi\theta)^{-n/2}} \exp\left\{-\frac{\sum_1^n x_i^2}{2\theta} + \frac{2\sum_1^n x_i\theta}{2\theta} - \frac{n\theta^2}{2\theta}\right\}$$

$$= A(\theta)h(x) \exp\left\{-\frac{\sum_1^n x_i^2}{2\theta^2}\right\}$$

As in the last problem, this distribution is not a 2-parameter exponential.

Here Theorem 3 applies because the family is full rank. Let $R_1 = \sum_1^n x_i^2$ and $\eta_1 = -1/(2\theta^2)$. Then the statistic $T = \sum_1^n x_i^2$ is minimally sufficient.

4 6.3.12

$$f_X(x) = (2\theta)^{-n}\ I(-\theta < x_{(1)} < \theta)\ I(-\theta < x_{(n)} < \theta)$$

$$f_X(x)/f_X(y) = \frac{I(-\theta < x_{(1)} < \theta)\ I(-\theta < x_{(n)} < \theta)}{I(-\theta < y_{(1)} < \theta)\ I(-\theta < y_{(n)} < \theta)}$$

$$= \frac{I(|x_{(1)}| < \theta)\ I(|x_{(n)}| < \theta)}{I(|y_{(1)}| < \theta)\ I(|y_{(n)}| < \theta)}$$

$$\frac{I(\max(|x_{(1)}|, |x_{(n)}|) < \theta)}{I(\max(|y_{(1)}|, |y_{(n)}|) < \theta)}$$

This suggests the minimum statistic $T = \max(|x_{(1)}|, |x_{(n)}|)$

5 6.3.13

$$f_{X,Y}(x, y) = \frac{1}{(2\pi\sigma^2)^{(m+n)/2}} \exp\left\{-\left[\sum_{i=1}^{m}(x_i - \mu_1)^2 + \sum_{j=1}^{n}(y_j - \mu_2)^2\right]/(2\sigma^2)\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{(m+n)/2}} exp\left\{-\frac{\sum_{i=1}^m x_i^2 + \sum_{j=1}^n y_j^2}{2\sigma^2} + \frac{\sum_{i=1}^m x_i\mu_1}{\sigma^2} + \frac{\sum_{j=1}^n y_j\mu_2}{\sigma^2} - \frac{\mu_1^2}{2\sigma^2} - \frac{\mu_2^2}{2\sigma^2}\right\}$$

$$= A(\theta)\exp\left\{-\frac{\sum_{i=1}^m x_i^2 + \sum_{j=1}^n y_j^2}{2\sigma^2} + \frac{\sum_{i=1}^m x_i\mu_1}{\sigma^2} + \frac{\sum_{j=1}^n y_j\mu_2}{\sigma^2}\right\}$$

This family is full rank, so apply Theorem 3. Let $R_1 = \sum_{i=1}^m x_i^2 + \sum_{j=1}^n y_j^2$, and $R_2 = \sum_{i=1}^m x_i$, and $R_3 = \sum_{j=1}^n y_j$ to get the following minimal sufficient statistic by Theorem 3:

$$T = \left(\sum_{i=1}^{m} x_i^2 + \sum_{j=1}^{n} y_j^2,\ \ \sum_{i=1}^{m} x_i,\ \ \sum_{j=1}^{n} y_j\right)$$

6 6.3.15

$$f_{X,Y}(x, y) = [\beta^\alpha\Gamma(\alpha)]^{-m}[k\beta^\alpha\Gamma(\alpha)]^{-n}\exp\left\{-\frac{\sum_{i=1}^n x_i}{\beta} - \frac{\sum_{j=1}^n y_j}{k\beta}\right\}\prod_{i=1}^{m} x_i^{\alpha-1}\prod_{j=1}^{n} y_j^{\alpha-1}$$

$$= A(\theta)\exp\left\{\left[-\frac{k\sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j}{k\beta}\right] + \log\left[\prod_{i=1}^{m} x_i^{\alpha-1}\prod_{j=1}^{n} y_j^{\alpha-1}\right]\right\}$$

$$= A(\theta)\exp\left\{\left[-\frac{k\sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j}{k\beta}\right] + (\alpha-1)\log\left[\prod_{i=1}^{m} x_i\prod_{j=1}^{n} y_j\right]\right\}$$

This family is full rank, so apply Theorem 3. Factoring functions of θ out of each term in the exponent, and treating *log* as an invertible function can give the following minimal sufficient statistic:

$$T = \left(k\sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j,\ \ \prod_{i=1}^{m} x_i \prod_{j=1}^{n} y_j\right)$$

7 6.3.18

$$f_{X,Y}(x,y) = (2\pi\sigma^2)^{-m/2}(2\pi k\sigma^2)^{-n/2}\exp\left\{-\frac{\sum_{i=1}^{m}(x_i-\mu)^2}{2\sigma^2} - \frac{\sum_{j=1}^{n} y_j^2}{2k\sigma^2}\right\}$$

$$= (2\pi\sigma^2)^{-m/2}(2\pi k\sigma^2)^{-n/2}\exp\left\{-\frac{\sum_{i=1}^{m} x_i^2}{2\sigma^2} + \frac{\mu\sum_{i=1}^{m} x_i}{\sigma^2} - \frac{m\mu^2}{2\sigma^2} - \frac{\sum_{j=1}^{n} y_j^2}{2k\sigma^2}\right\}$$

$$= A(\theta)\exp\left\{-\frac{k\sum_{i=1}^{m} x_i^2 + \sum_{j=1}^{n} y_i^2}{2k\sigma^2} + \frac{\mu\sum_{i=1}^{m} x_i}{\sigma^2}\right\}$$

This family is full rank, so apply Theorem 3. Factoring functions of θ out of the terms in the exponential gives the following minimally sufficient statistic:

$$T = \left(k\sum_{i=1}^{m} x_i^2 + \sum_{j=1}^{n} y_i^2,\ \ \sum_{i=1}^{m} x_i\right)$$