Homework 1

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ST562

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1 6.2.1

$$f_X(x) = p^2(1-p)^{x_1}(1-p)^{x_2} = p^2(1-p)^{x_1+x_2}$$

$$f_T(t) = \sum_0^t p(1-p)^i p(1-p)^{t-i} = p^2(t+1)(1-p)^t$$

$$\frac{f_X(x)}{f_T(t)} = \frac{p^2(1-p)^{x_1+x_2}}{p^2(t+1)(1-p)^{x_1+x_2}} = \frac{1}{t+1}$$

$\frac{f_X(x)}{f_T(t)}$ is not a function of p.

2 6.2.2

Substituting in $1-p$ for q, and letting $\hat{x} = \sum_0^m x_i$ and $\hat{y} = \sum_0^n y_i$ gives:

$$f_{X,Y}(x,y) = \left(\prod_1^m p^{x_i}(1-p)^{1-x_i}\right)\left(\prod_1^n p^{1-y_i}(1-p)^{y_i}\right) = p^{n+\hat{x}-\hat{y}}(1-p)^{m-\hat{x}+\hat{y}}$$

$\hat{X}$ is distributed $Bin(p,m), \hat{Y}$ is distributed $Bin(1-p,n), T = \hat{X} - \hat{Y}$. Using the discrete version of convolution for subtracting random variables we have:

$$f_T(t) = \sum_0^n f_{\hat{Y}}(\hat{y}_i) f_{\hat{X}}(t+\hat{y}_i) =$$

$$\sum_0^n \left[C_{\hat{y}_i}^n p^{n-\hat{y}_i}(1-p)^{\hat{y}_i} C_{t+\hat{y}_i}^m p^{t+\hat{y}_i}(1-p)^{m-(t+y_i)}\right] =$$

$$p^{n+t}(1-p)^{m-t}\sum_0^n C_{\hat{y}_i}^n C_{t+\hat{y}_i}^m$$

$$\frac{f_{X,Y}(x,y)}{f_T(t)} = \frac{p^{n+\hat{x}-\hat{y}}(1-p)^{m-\hat{x}+\hat{y}}}{p^{n+t}(1-p)^{m-t}\sum_0^n C_{\hat{y}_i}^n C_{t+\hat{y}_i}^m} = \frac{1}{\sum_0^n C_{\hat{y}_i}^n C_{t+\hat{y}_i}^m}$$

$\frac{f_{X,Y}(x,y)}{f_T(t)}$ is not a function of p.

3 6.2.5

Using the resulting form of $f_{X,Y}(x, y)$ from the previous problem:

$$f_{X,Y}(x, y) = p^{3+(x_1+x_2)-(y_1+y_2+y_3)}(1-p)^{2-(x_1+x_2)+(y_1+y_2+y_3)}$$

$Y_1Y_2 = 1$ iff $Y_1 = Y_2 = 1$, so:

$$f_{Y_1Y_2}(y) = (q^2)^y(1-q^2)^{1-y} = (1-p)^{2y}\left[1-(1-p)^2\right]^{1-y}$$

Now by discrete convolution, where $T = X_1 + Y_1Y_2$:

$$f_T(t) = f_X(t)f_{Y_1Y_2}(0)+f_X(t-1)f_{Y_1Y_2}(1) = p^t(1-p)^{1-t}p^2+p^{t-1}(1-p)^{1-(t-1)}(1-p^2) =$$

$$p^{t+2}(1-p)^{1-t} + p^{t-1}(1-p)^{2-t}(1-p^2)$$

$$\frac{f_{X,Y}(x, y)}{f_T(t)} = \frac{p^{3+(x_1+x_2)-(y_1+y_2+y_3)}(1-p)^{2-(x_1+x_2)+(y_1+y_2+y_3)}}{p^{t+2}(1-p)^{1-t} + p^{t-1}(1-p)^{2-t}(1-p^2)}$$

Cannot cancel out p's out of the above, $\frac{f_{X,Y}(x,y)}{f_T(t)}$ is a function of p.

4 6.2.6

$$f_{X,Y}(x, y) = \prod_1^m \frac{\lambda^{x_i}e^{-\lambda}}{x_i!}\prod_1^n \frac{2^{y_i}\lambda^{y_i}e^{-2\lambda}}{y_i!} = \frac{2^{(\sum_0^n y_i)}\lambda^{(\sum_0^m x_i)+(\sum_0^n y_i)}e^{-\lambda(m+2n)}}{(\prod_1^m x_i!)(\prod_1^n y_i!)}$$

Since $Poisson(\lambda_1) + Poisson(\lambda_2) = Poisson(\lambda_1 + \lambda_2)$:

$$f_T(t) = \frac{(\lambda[m+2n])^t\, e^{-\lambda(m+2n)}}{t!}$$

$$\frac{f_{X,Y}(x, y)}{f_T(t)} = \frac{\frac{2^{(\sum_0^n y_i)}\lambda^{(\sum_0^m x_i)+(\sum_0^n y_i)}e^{-\lambda(m+2n)}}{(\prod_1^m x_i!)(\prod_1^n y_i!)}}{\frac{(\lambda[m+2n])^t\, e^{-\lambda(m+2n)}}{t!}} = \frac{2^{(\sum_0^n y_1)}t!}{(\prod_1^m x_i!)(\prod_1^n y_i!)(m+2n)^t}$$

$\frac{f_{X,Y}(x,y)}{f_T(t)}$ is not a function of λ.

5 6.2.7

$$f_{X,Y}(x,y) = \frac{2^{(\sum_1^5 y_i)}\lambda^{(\sum_1^4 x_i)+(\sum_1^5 y_1)}e^{-14\lambda}}{(\prod_1^4)x_i!)(\prod_1^5 y_i!)}$$

$$f_T(t) = \frac{(3\lambda)^t e^{-3\lambda}}{t!}$$

$$\frac{f_{X,Y}(x,y)}{f_T(t)} = \frac{\frac{2^{(\sum_1^5 y_i)}\lambda^{(\sum_1^4 x_i)+(\sum_1^5 y_1)}e^{-14\lambda}}{(\prod_1^4)x_i!)(\prod_1^5 y_i!)}}{\frac{(3\lambda)^t e^{-3\lambda}}{t!}} =$$

$$\frac{2^{(\sum_1^5)}}{3^t}\lambda^{(\sum_2^m x_i)+(\sum_2^n y_i)}e^{-11\lambda}\frac{t!}{(\prod_1^m x_i)(\prod_1^n y_i)}$$

Cannot cancel p's from above, $\frac{f_{X,Y}(x,y)}{f_T(t)}$ is a function of p.

6 6.2.10

(i)

$$f_X(x) = \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left\{-\frac{\sum_1^n(x_i-\mu)^2}{2\sigma^2}\right\} =$$

$$\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left\{-\frac{\sum_1^n(x_i^2-2x_i\mu+\mu^2)}{2\sigma^2}\right\} =$$

$$\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left\{-\frac{\sum_1^n x_i^2}{2\sigma^2}\right\}\exp\left\{n\frac{2\mu n^{-1}\sum_1^n x_i-\mu^2}{2\sigma^2}\right\}$$

(ii)

$$f_X(x) = \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left\{-\frac{nn^{-1}\sum_1^n(x_i-\mu)^2}{2\sigma^2}\right\}1$$

7 6.2.11

(i)

$$f_X(x) = \sigma^{-n}\text{exp}\left\{-\sum_1^n (x_i-\mu)/\sigma\right\} I(x_{(1)}>\mu) =$$

$$\sigma^{-n}\text{exp}\left\{-\sum_1^n (x_i-\mu)/\sigma\right\} \text{exp}\{\mu n/\sigma\} I(x_{(1)}>\mu)$$

(ii)

$$f_X(x) = \sigma^{-n}\text{exp}\left\{-nn^{-1}\sum_1^n (x_i-\mu)/\sigma\right\} I(x_{(1)}>\mu)$$

(iii)

$$f_X(x) = \sigma^{-n}\exp\left\{-\sum_1^n (x_i-\mu)/\sigma\right\} I(x_{(1)}>\mu) =$$

$$\sigma^{-n}\exp\left\{-\sum_1^n [(x_i-x_{(1)})-(\mu-x_{(1)})]/\sigma\right\} I(x_{(1)}>\mu) =$$

$$\sigma^{-n}\exp\left\{-\left[\left(\sum_1^n x_i - x_{(1)}\right) - n\mu + nx_{(1)}\right]/\sigma\right\} I(x_{(1)}>\mu)$$

8 6.2.13

$$f_X(x) = I\left(\theta-\frac{1}{2}<X_{(1)}\right) I\left(X_{(n)}<\theta+\frac{1}{2}\right) 1$$

9 6.2.16

$$f_X(x) = 2^n\theta^{-n}\exp\left\{-\left(\sum_1^n x_i^2\right)/\theta\right\}\left(\prod_1^n x_i\right) I(x_{(1)}>0)$$