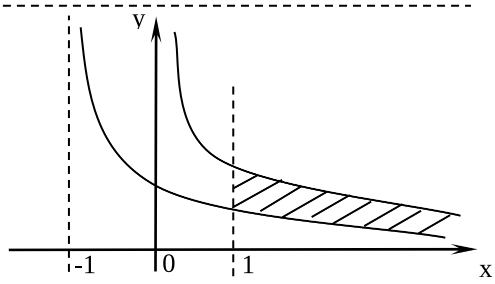
5. adibidea

Izan bedi Ω $y=\dfrac{1}{x}$, $y=\dfrac{1}{x+1}$ kurben artean dagoen planoaren eskualdea, $x=1$ zuzenaren eskuinean.

Ω-ren azalera, finitua ala infinitua da? Justifikatu erantzuna era egokian.

$$A=\int_1^\infty \frac{1}{x}dx-\int_1^\infty \frac{1}{x+1}dx=\int_1^\infty \frac{1}{x(x+1)}dx$$



Beraz integral inpropio bat daukagu.

$f(x)=\dfrac{1}{x(x+1)}\leq\dfrac{1}{x^2}=\varphi(x)$ denez eta $\displaystyle\int_a^\infty \frac{1}{x^2}\,dx$ konbergentea denez ($n=2>1$ delako), orduan

$A=\displaystyle\int_1^\infty \frac{1}{x(x+1)}dx\leq\int_1^\infty \frac{1}{x^2}dx$ ere konbergentea da eta Ω-ren azalera **finitua da**.

Haren balioa hauxe da:

$$A=\int_1^\infty \frac{1}{x(x+1)}dx=\lim_{b\to\infty}\left[\int_1^b \frac{1}{x}dx-\int_1^b \frac{1}{x+1}dx\right]=\lim_{b\to\infty}\Big[\ln|x|-\ln|x+1|\Big]_1^b=$$

$$=\lim_{b\to\infty}\left[\ln\left|\frac{x}{x+1}\right|\right]_1^b=\lim_{b\to\infty}\left[\ln\frac{b}{b+1}\right]-\ln\frac{1}{2}=-\ln 1+\ln 2=\ln 2 \quad u^2$$

6. adibidea

Azaldu hurrengo “kalkuluaren” errorea:

$$\int_{-2}^{1} \frac{dx}{2x+1} = \left[\frac{1}{2} \ln|2x+1| \right]_{-2}^{1} = \frac{1}{2}\ln 3 - \frac{1}{2}\ln 3 = 0$$

Zein da integral honen balio zuzena?

- -

Proposatutako integrala inpropioa da, izendatzailea integrazio tartearen barnean dagoen $x = -\frac{1}{2}$ puntuan nulua egiten delako; hortaz, ezin da kalkulua zuzenean burutu.

$$\int_{-2}^{1} \frac{dx}{2x+1} = \int_{-2}^{-1/2} \frac{dx}{2x+1} + \int_{-1/2}^{1} \frac{dx}{2x+1}$$

$$\int_{-2}^{-1/2} \frac{dx}{2x+1} = \lim_{b \to -1/2^-} \int_{-2}^{b} \frac{dx}{2x+1} = \lim_{b \to -1/2^-} \left[\frac{1}{2} \ln|2x+1| \right]_{-2}^{b} =$$

$$= \lim_{b \to -1/2^-} \frac{1}{2}\left[\ln|2b+1| - \ln 3 \right] = \frac{1}{2}\left[\ln 0 - \ln 3 \right] = -\infty$$

Deskonposaketan agertzen diren bi integraletako lehena dibergentea da eta, horregatik, proposatutako integrala ere dibergentea da.

7. adibidea

Kalkulatu $a \geq 0$ balioen arabera: $\displaystyle\int_0^{\infty} \frac{dx}{x^2+a}$

1.- Baldin $a = 0$, hauxe daukagu:

$$\int_0^{\infty} \frac{dx}{x^2+a} = \int_0^{\infty} \frac{dx}{x^2} = \lim_{t \to 0}\left[\int_t^c \frac{dx}{x^2}\right] + \lim_{t\to\infty}\left[\int_c^t \frac{dx}{x^2}\right]$$

$\displaystyle\int_0^c \frac{dx}{x^2}$ dibergentea da, zeren eta $\varphi(x) = \dfrac{1}{x^2}$; n = 2 $\geq$ 1: dibergitzen du.

$\displaystyle\int_c^{\infty} \frac{dx}{x^2}$ konbergentea da, zeren eta $\varphi(x) = \dfrac{1}{x^2}$; n = 2 > 1: konbergitzen du.

Beraz, baldin $a = 0$, $\displaystyle\int_0^{\infty} \frac{dx}{x^2+a}$ integralak **dibergitzen du**.

2.- Baldin a > 0, hauxe daukagu:

$$\int_0^{\infty} \frac{dx}{x^2+a} = \lim_{t\to\infty}\left[\int_0^t \frac{(1/a)\,dx}{(x/\sqrt{a})^2+1}\right] = \lim_{t\to\infty} \frac{1}{\sqrt{a}}\left[\arctan \frac{x}{\sqrt{a}}\right]_0^t = \frac{\pi}{2\sqrt{a}}$$

Beraz, baldin $a = 0$, $\displaystyle\int_0^{\infty} \frac{dx}{x^2+a}$ integralak **konbergitzen du**.

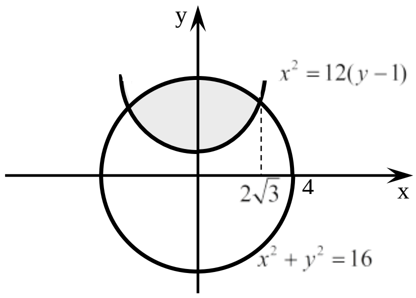
8. adibidea

Kalkulatu $x^2 + y^2 = 16$; $x^2 = 12(y-1)$ kurben arteko azalera.

- -

$C(0,0)$ zentroko eta $r = 4$ erradioko zirkunferentzia bat eta ardatz bertikaleko parabola bat dauzkagu.

$$\left.\begin{array}{l} x^2 + y^2 = 16 \\ y = \dfrac{x^2}{12} + 1 \end{array}\right\} \Rightarrow \begin{array}{l} 12y - 12 + y^2 = 16 \\ y^2 + 12y - 28 = 0 \quad \Rightarrow \quad y = 2 \quad \Rightarrow \quad x = \pm 2\sqrt{3} \end{array}$$



$$A = 2\int_0^{2\sqrt{3}} \left(\sqrt{16 - x^2} - \frac{x^2}{12} - 1 \right) dx = 2\int_0^{2\sqrt{3}} \sqrt{16 - x^2}\, dx - 2\left(\frac{x^3}{36} + x \right) \Bigg]_0^{2\sqrt{3}} =$$

$$= \left[\begin{array}{lll} x = 4\sin t & x = 0 & \Rightarrow t = 0 \\ dx = 4\cos t\, dt & x = 2\sqrt{3} & \Rightarrow t = \pi/3 \end{array} \right] = 32\int_0^{\pi/3} \cos^2 t\, dt - \frac{16\sqrt{3}}{3} =$$

$$= 16\int_0^{\pi/3} (1 + \cos 2t)\, dt - \frac{16\sqrt{3}}{3} = \left[16(t + \sin 2t/2) \right]_0^{\pi/3} - \frac{16\sqrt{3}}{3} =$$

$$= 16\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} \right) - \frac{16\sqrt{3}}{3} = \frac{16\pi - 4\sqrt{3}}{3} \qquad u^2$$

9. adibidea

Kalkulatu $y \le \sqrt{x}$, $y \ge x^2$ eskualdea abzisen ardatzaren inguruan biratzean sortutako solidoaren bolumena

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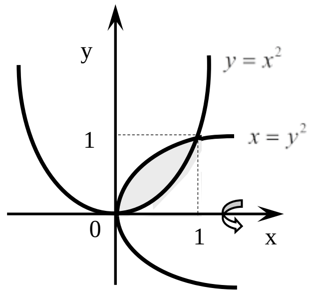
Ebakidura puntuak determinatzen dira eta biraketa eskualdea marrazten da:

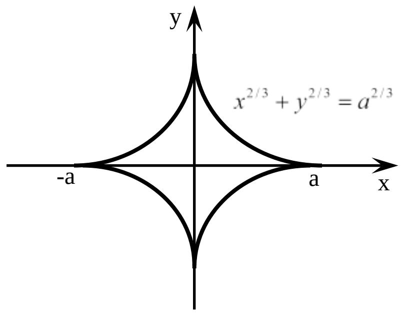
$$\left.\begin{array}{l} y = \sqrt{x} \\ y = x^2 \end{array}\right\} \Rightarrow (x=0;\ y=0) \vee (x=1;\ y=1)$$



$$V = \pi\int_0^1 (\sqrt{x})^2\,dx - \pi\int_0^1 (x^2)^2\,dx = \pi\int_0^1 (x - x^4)dx = \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{3\pi}{10} \quad u^3$$

10. adibidea

Determinatu hurrengo kurbaren luzera: $x^{2/3}+y^{2/3}=a^{2/3}$ (astroidea).



Kurba-arku baten luzeraren formula: $L=\int_a^b \sqrt{1+y'^{\,2}}\,dx$.

Gure kasuan: $y=(a^{2/3}-x^{2/3})^{3/2} \quad \Rightarrow \quad y'=\frac{3}{2}(a^{2/3}-x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)$

$$y'^{\,2}=(a^{2/3}-x^{2/3})\,x^{-2/3}=a^{2/3}\,x^{-2/3}-1 \quad \Rightarrow \quad \sqrt{1+y'^{\,2}}=a^{1/3}\,x^{-1/3}$$

Simetriagatik: $L=4\int_0^a \sqrt{1+y'^{\,2}}\,dx=4\int_0^a a^{1/3}\,x^{-1/3}\,dx=6a^{1/3}\left[x^{2/3}\right]_0^a=6a \quad u$