2. ORRIA (240 puntu)

A) Kalkulatu: $\displaystyle\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

(40 p)

Ebazpena

$$\int \frac{dx}{(x-1)\sqrt{x^2+x+1}} = \left\{ \begin{array}{l} x-1=\frac{1}{t} \quad \Rightarrow \quad dx=-\frac{dt}{t^2} \\ x^2+x+1=\frac{(1+t)^2}{t^2}+\frac{1+t}{t}+1=\frac{3t^2+3t+1}{t^2} \end{array} \right\} = \int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\cdot\frac{\sqrt{3t^2+3t+1}}{t}} =$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{t^2+t+\frac{1}{3}}} = -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{12}}} = \left\{ \begin{array}{l} t+\frac{1}{2}=z \\ dt=dz \end{array} \right\} =$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dz}{\sqrt{z^2+\frac{1}{12}}} = -\frac{1}{\sqrt{3}}\ln\left|z+\sqrt{z^2+\frac{1}{12}}\right|+C = -\frac{1}{\sqrt{3}}\ln\left|t+\frac{1}{2}+\sqrt{t^2+t+\frac{1}{3}}\right|+C =$$

$$= -\frac{1}{\sqrt{3}}\ln\left|\frac{1}{x-1}+\frac{1}{2}+\sqrt{\frac{1}{(x-1)^2}+\frac{1}{(x-1)}+\frac{1}{3}}\right|+C = -\frac{1}{\sqrt{3}}\ln\left|\frac{1}{x-1}+\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{\sqrt{3}(x-1)}\right|+C$$

B) Izan bedi hurrengo $[D]$ domeinua:

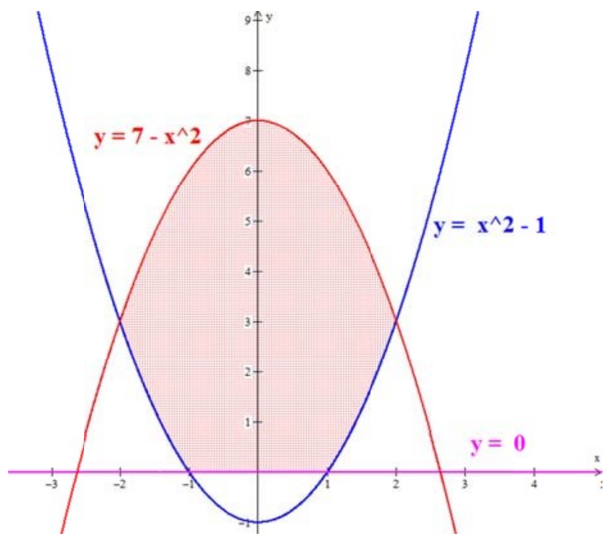
$$D=\left\{(x,y)\in\mathbb{R}^2 / (y\le 7-x^2)\wedge(y\ge x^2-1)\wedge(y\ge 0)\right\}$$

Kalkulatu:

1.-$[D]$ domeinu lauaren perimetroa. **(50 p)**

2.- $[D]$ abzisa ardatzaren inguruan biratzean sortutako bolumena **(50 p)**

Ebazpena



1.- $[D]$ domeinu lauaren perimetroa:

$$P=2\left[1+L_1+L_2\right]$$

Non:

$$L_1=\int_1^2\sqrt{1+4x^2}\,dx=\frac{1}{4}\left[2x\sqrt{1+4x^2}+\ln\left|2x+\sqrt{1+4x^2}\right|\right]_1^2=$$

$$=\frac{1}{4}\left[\left(4\sqrt{17}+\ln\left(4+\sqrt{17}\right)\right)-\left(2\sqrt{5}+\ln\left(2+\sqrt{5}\right)\right)\right]=\sqrt{17}-\frac{\sqrt{5}}{2}+\frac{1}{4}\ln\frac{4+\sqrt{17}}{2+\sqrt{5}}$$

$$L_2=\int_0^2\sqrt{1+4x^2}\,dx=\frac{1}{4}\left[2x\sqrt{1+4x^2}+\ln\left|2x+\sqrt{1+4x^2}\right|\right]_0^2=$$

$$=\frac{1}{4}\left[\left(4\sqrt{17}+\ln\left(4+\sqrt{17}\right)\right)-\ln(1)\right]=\sqrt{17}+\frac{1}{4}\ln\left(4+\sqrt{17}\right)$$

Beraz, perimetroa honako hau da:

$$P = 2\left[1 + L_1 + L_2\right] = 2\left[1 + \sqrt{17} - \frac{\sqrt{5}}{2} + \frac{1}{4}\ln\frac{4+\sqrt{17}}{2+\sqrt{5}} + \sqrt{17} + \frac{1}{4}\ln\left(4+\sqrt{17}\right)\right] =$$

$$= 2 + 4\sqrt{17} - \sqrt{5} + \frac{1}{4}\ln\frac{\left(4+\sqrt{17}\right)^2}{2+\sqrt{5}} \quad u$$

2.- $[D]$ abzisa ardatzaren inguruan biratzean sortutako bolumena.

$$V = 2V_1$$

$$V_1 = \pi\left[\int_0^2 \left(7-x^2\right)^2 dx - \int_1^2 \left(x^2-1\right)^2 dx\right] = \pi\left[\left(49x - \frac{14}{3}x^3 + \frac{x^5}{5}\Big|_0^2\right) - \left(\frac{x^5}{5} - \frac{2}{3}x^3 + x\right)\Big|_1^2\right] =$$

$$= \pi\left[\left(98 - \frac{112}{3} + \frac{32}{5}\right) - \left(\frac{32}{5} - \frac{16}{3} + 2 - \frac{1}{5} + \frac{2}{3} - 1\right)\right] = \pi\left[\frac{1006}{15} - \frac{38}{15}\right] = \frac{968}{15}\pi$$

Beraz, bolumena honako hau da:

$$V = 2V_1 = \frac{1936}{15}\pi \quad u^3$$

C) Izan bedi hurrengo $[D]$ domeinua:

$$D=\left\{(x,y)\in\mathbb{R}^2/(9x^2+25y^2-225\le 0)\wedge(3x-5y+15\le 0)\right\}$$

1.- Bi era desberdinetan, $I=\iint_{[D]} f(x,y)\,dx\,dy$ integralean, integrazio-limiteak zehaztu.

(30 p)

2.- $[D]$ domeinu lauaren grabitate-zentro geometrikoaren abzisa koordenatua kalkulatu, integral bikoitzak erabiliz.

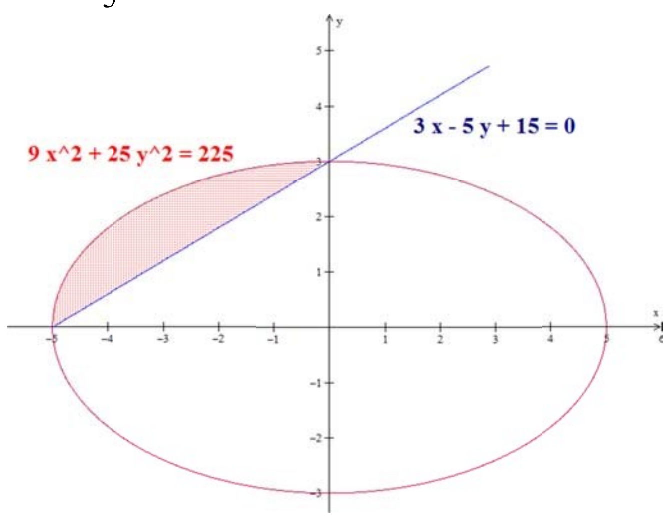
(70 p)

Ebazpena

1.- Bi era desberdinetan, $I=\iint_{[D]} f(x,y)\,dx\,dy$ integralean, integrazio-limiteak zehaztu.

$$9x^2+25y^2-225=0 \quad\rightarrow\quad \frac{x^2}{25}+\frac{y^2}{9}=1 \quad\rightarrow\quad a=5 \quad ; \quad b=3$$ erdi-ardatzetako elipsea

$$3x-5y+15=0 \quad\rightarrow\quad y=\frac{3}{5}x+3 \quad\rightarrow\quad (-5,0)$$ eta $(0,3)$ puntuetatik pasatzen den zuzena



Lehenengo integrazio aldagaitzat (x) hartuz

$$I=\iint_{[D]} f(x,y)\,dx\,dy=\int_0^3 dy\int_{-(5/3)\sqrt{9-y^2}}^{(5/3)(y-3)} f(x,y)\,dx$$

Lehenengo integrazio aldagaitzat (y) hartuz

$$I=\iint_{[D]} f(x,y)\,dx\,dy=\int_{-5}^{0} dx\int_{(3/5)x+3}^{(3/5)\sqrt{25-x^2}} f(x,y)dy$$

2.- $[D]$ domeinu lauaren grabitate-zentro geometrikoaren abzisa koordenatua kalkulatu, integral bikoitzak erabiliz.

$$A=\iint_D dx\,dy=\int_{-5}^{0}dx\int_{(3/5)x+3}^{(3/5)\sqrt{25-x^2}}dy \;=\int_{-5}^{0}\left[\frac{3}{5}\sqrt{25-x^2}-\frac{3}{5}x-3\right]dx\;=$$

$$=\frac{3}{5}\int_{-5}^{0}\sqrt{25-x^2}\,dx\;-\left[\frac{3x^2}{10}+3x\right]_{-5}^{0}\;=\frac{3}{5}J+\frac{75}{10}-15=\frac{3}{5}J-\frac{15}{2}$$

$$J=\int_{-5}^{0}\sqrt{25-x^2}\,dx\;=\begin{bmatrix} x=5\operatorname{sen}t & x=-5 & \Rightarrow t=-\pi/2 \\ dx=5\cos t\,dt & x=0 & \Rightarrow t=0\end{bmatrix}=25\int_{-\pi/2}^{0}\cos^2 t\,dt=$$

$$=25\left[\frac{t}{2}+\frac{\operatorname{sen}2t}{4}\right]_{-\pi/2}^{0}=\frac{25\pi}{4}\quad\Rightarrow\quad\boxed{A}=\frac{3}{5}J-\frac{15}{2}=\frac{15\pi}{4}-\frac{15}{2}=\boxed{\frac{15}{4}(\pi-2)}$$

$$I=\iint_D x\,dx\,dy=\int_{-5}^{0}x\,dx\int_{(3/5)x+3}^{(3/5)\sqrt{25-x^2}}dy\;=\int_{-5}^{0}x\left[\frac{3}{5}\sqrt{25-x^2}-\frac{3}{5}x-3\right]dx=$$

$$=\frac{3}{5}\int_{-5}^{0}x\sqrt{25-x^2}\;dx-\left[\frac{x^3}{5}+\frac{3x^2}{2}\right]_{-5}^{0}=\frac{3}{5}H-25+\frac{75}{2}=\frac{3}{5}H+\frac{25}{2}$$

$$H=\int_{-5}^{0}x\sqrt{25-x^2}\;dx=\begin{bmatrix}25-x^2=t^2 & x=-5 & \Rightarrow t=0\\ -xdx=tdt & x=0 & \Rightarrow t=5\end{bmatrix}=-\int_{0}^{5}t^2dt=\left[-\frac{t^3}{3}\right]_{0}^{5}=-\frac{125}{3}$$

$$I=\frac{3}{5}H+\frac{25}{2}=-25+\frac{25}{2}=-\frac{25}{2}$$

$$\boxed{x_c}=\frac{1}{A}\iint_D x\,dx\,dy=\frac{-25/2}{15(\pi-2)/4}=\boxed{\frac{-10}{3(\pi-2)}}$$