$$H\left[f(-1,-2)\right]=3>0;\quad \frac{\partial^2 f}{\partial x^2}(-1,-2)=4>0 \quad \wedge \quad f(-1,-2)=14$$

$M(-1,-2,14)$ puntua **minimo erlatiboa** da.

D) Integrazio-limiteak bi era desberdinetan jarri integral honetan $I=\iint_D f(x,y)\,dx\,dy$, hurrengo $[D]$ domeinuarentzat:

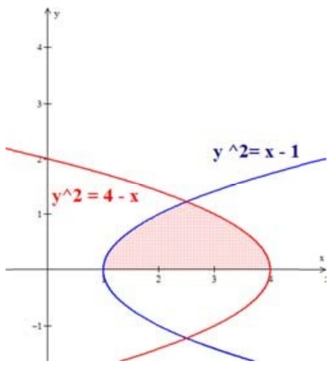
$$D=\left\{(x,y)\in\mathbb{R}^2 \,/\, x\geq 0 \;;\; y\geq 0 \;;\; y^2\leq x-1 \;;\; y^2\leq 4-x\right\}$$

Kalkulatu $[D]$ domeinua x ardatzaren inguruan biratzekoan sorrarazten den bolumena.

(6 p)

Ebazpena

Domeinua bat dator lehenengo koadrantean 0X ardatzeko $y^2=x-1$; $y^2=4-x$ parabolek mugatutako eskualdearekin.

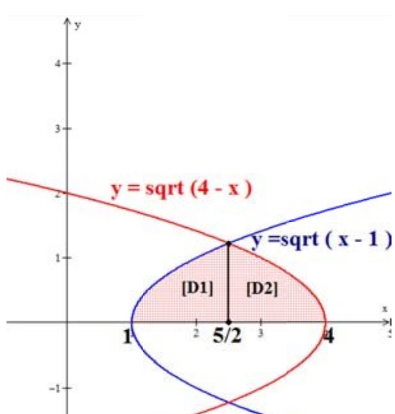


Bi kurba horien mozketa puntua lehenengo koadrantean:

$$\begin{cases} y^2=x-1 \\ y^2=4-x \end{cases} \;\rightarrow\; x-1=4-x \;\rightarrow\; x=\frac{5}{2} \;\rightarrow\; P\left(\frac{5}{2},\sqrt{\frac{3}{2}}\right)$$

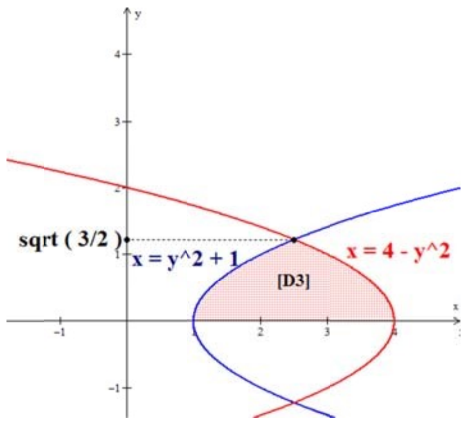
- Lehenengo integrazio aldagaitzat (y) hartuz:

$$I=\int_1^{5/2}dx\int_0^{\sqrt{x-1}}f(x,y)\,dy+\int_{5/2}^{4}dx\int_0^{\sqrt{4-x}}f(x,y)\,dy$$



- Lehenengo integrazio aldagaitzat (x) hartuz:

$$I=\int_0^{\sqrt{\frac{3}{2}}} dy\int_{y^2+1}^{4-y^2} f(x,y)\,dx$$



$[D]$ domeinua x ardatzaren inguruan biratzekoan sorrarazten den bolumena:

$$V=\pi\left[\int_1^{5/2}\left(\sqrt{x-1}\right)^2 dx+\int_{5/2}^{4}\left(\sqrt{4-x}\right)^2 dx\right]=\pi\left[\frac{x^2}{2}-x\Bigg|_1^{5/2}+4x-\frac{x^2}{2}\Bigg\|_{5/2}^{4}\right]=\frac{9}{4}\pi \quad u^3$$

Existitzen den simetria kontuan hartuz, bolumena honela geratzen da:

$$V=2\pi\left[\int_1^{5/2}\left(\sqrt{x-1}\right)^2 dx\right]=2\pi\left[\frac{x^2}{2}-x\Bigg|_1^{5/2}\right]=\frac{9}{4}\pi \quad u^3$$

---

2. ORRIA (20 puntu)

A) Klasifikatu eta ebatzi hurrengo EDA: $\left(xy-2y^2\right)dx-\left(x^2-3xy\right)dy=0$

(4 p)

Ebazpena

EDA **homogeneoa**:

$$y'=\frac{xy-2y^2}{x^2-3xy} \quad\rightarrow\quad y'=\frac{\dfrac{xy-2y^2}{x^2}}{\dfrac{x^2-3xy}{x^2}}=\frac{\dfrac{y}{x}-2\left(\dfrac{y}{x}\right)^2}{1-3\left(\dfrac{y}{x}\right)}$$

Hurrengo aldaketarekin: $\dfrac{y}{x}=u \quad\Rightarrow\quad y=xu \quad\Rightarrow\quad y'=u+xu'$

$$u+xu'=\frac{u-2u^2}{1-3u} \quad\rightarrow\quad x\frac{du}{dx}=\frac{u-2u^2}{1-3u}-u=\frac{u-2u^2-u+3u^2}{1-3u}=\frac{u^2}{1-3u}$$

Aldagai banangarrien EDA lortzen dugu: $\dfrac{1-3u}{u^2}du = \dfrac{1}{x}dx$

Integratuz:

$$-\frac{1}{u} - 3\ln|u| = \ln|x| + C \quad \rightarrow \quad -\frac{x}{y} = \ln|x| + 3\ln\left|\frac{y}{x}\right| + C \quad \rightarrow \quad \boxed{-\frac{x}{y} = \ln\left|\frac{y^3}{x^2}\right| + C}$$

B) Ebatzi hurrengo koefiziente aldakorreko ekuazioa

$$\left(x^2-1\right)y'' - 2x\,y' + 2xy = \left(x^2-1\right)^2$$

jakinda $y_1(x) = x$ ekuazio homogeneoaren soluzio partikularra dela. ***(6 p)***

Ebazpena

$$\left(x^2-1\right)y'' - 2x\,y' + 2xy = \left(x^2-1\right)^2 \quad \rightarrow \quad y'' - \frac{2x}{x^2-1}y' + \frac{2x}{x^2-1}y = x^2-1$$

Elkartutako ekuazio homogeneoaren beste soluzio partikularra (y_2), (y_1)-ekin linealki independentea dena, hurrengo formularen bidez lor daiteke:

$$y_2 = y_1 \int \frac{\exp\left(-\int P(x)dx\right)}{y_1^{\,2}}dx$$

$$P(x) = \frac{-2x}{x^2-1} \quad \Rightarrow \quad \int P(x)\,dx = -\int \frac{2x}{x^2-1}\,dx = -\ln\left|x^2-1\right| + cte$$

$$\exp\left(-\int P(x)\,dx\right) = e^{\ln\left|x^2-1\right|} = x^2-1 \quad \Rightarrow$$

$$\int \frac{\exp\left(-\int P(x)dx\right)}{y_1^{\,2}}dx = \int \frac{(x^2-1)}{x^2}dx = \int\left(1-\frac{1}{x^2}\right)dx = x + \frac{1}{x} + cte$$

$$y_2 = y_1 \int \frac{\exp\left(-\int P(x)dx\right)}{y_1^{\,2}}dx = x\left[x+\frac{1}{x}\right] = x^2+1$$

Beraz, elkartutako ekuazio homogeneoaren soluzio orokorra hurrengoa da:

$$y_h = C_1 \cdot x + C_2(x^2+1)$$

Parametroen aldakuntzaren metodoa aplikatuko dugu:

$$y = L_1(x)\cdot x + L_2(x)\cdot(x^2+1) \qquad [*]$$

$L'_1(x)$ y $L'_2(x)$ hurrengo sistema ebatziz lortuko dira:

$$\begin{cases} L_1' \cdot x + L_2' \cdot (x^2+1) = 0 \\ L_1' \cdot 1 + L_2' \cdot 2x = x^2 - 1 \end{cases}$$

$$L_1'(x) = \frac{\begin{vmatrix} 0 & x^2+1 \\ x^2-1 & 2x \end{vmatrix}}{\begin{vmatrix} x & x^2+1 \\ 1 & 2x \end{vmatrix}} = \frac{-(x^2-1)(x^2+1)}{2x^2-x^2-1} = \frac{-(x^2-1)(x^2+1)}{x^2-1} = -(x^2+1)$$

$$L_2'(x) = \frac{\begin{vmatrix} x & 0 \\ 1 & x^2-1 \end{vmatrix}}{\begin{vmatrix} x & x^2+1 \\ 1 & 2x \end{vmatrix}} = \frac{x(x^2-1)}{x^2-1} = x$$

$$L_1(x) = -\int (x^2+1)\,dx = -\frac{x^3}{3} - x + A \qquad ; \qquad L_2(x) = \int x\,dx = \frac{x^2}{2} + B$$

[*] adierazpenean ordezkatuz, soluzio orokorra lortzen da:

$$\boxed{y} = \left[-\frac{x^3}{3} - x + A\right] \cdot x + \left[\frac{x^2}{2} + B\right] \cdot (x^2+1) = Ax + B(x^2+1) - \frac{x^4}{3} - x^2 + \frac{x^4}{2} + \frac{x^2}{2} =$$

$$= \boxed{Ax + B(x^2+1) + \frac{x^4}{6} - \frac{x^2}{2}}$$

C) Hurrengo EDA ebatzi**:** $y'' + 4y = (t-1)^2\, u_1 \quad ; \quad y(0) = y'(0) = 0$

(6 p)

Ebazpena

Laplace transformatua aplikatuko da.

Hurrengo propietatea kontuan hartuz $\mathfrak{L}\,[f(t)\cdot u_a] = e^{-pa}\mathfrak{L}\,[f(t+a)]$

$$\mathfrak{L}\,[(t-1)^2 \cdot u_1] = e^{-p}\mathfrak{L}\,[(t+1-1)^2] = e^{-p}\mathfrak{L}\,[t^2] = e^{-p} \cdot \frac{2}{p^3}$$

$$\left[p^2 Y(p) - p y(0) - y'(0)\right] + 4Y(p) = \frac{2e^{-p}}{p^3}$$

$$\frac{\partial z}{\partial y}(0,0) = 2 \cdot e^{-1} \cdot 0 + (-1) \cdot e^{-1} \cdot (1) = -\frac{1}{e}$$

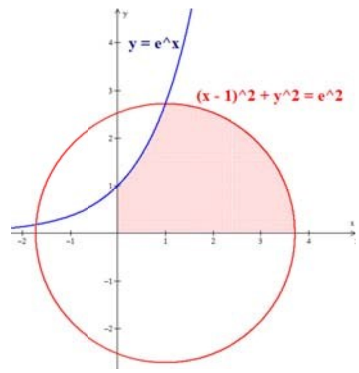
D) Jarri integrazio-limiteak bi era desberdinetan $I = \iint_D f(x,y)\,dx\,dy$ integralean, hurrengo $[D]$ eremurako:

$$D = \left\{ (x,y) \in \mathbb{R}^2 \,/\, x \geq 0 \ ; \ \ y \geq 0 \ ; \ \ y \leq e^x \ ; \ \ (x-1)^2 + y^2 \leq e^2 \right\}$$

$[D]$ eremu lauaren azalera kalkulatu.

(6 p)

Ebazpena



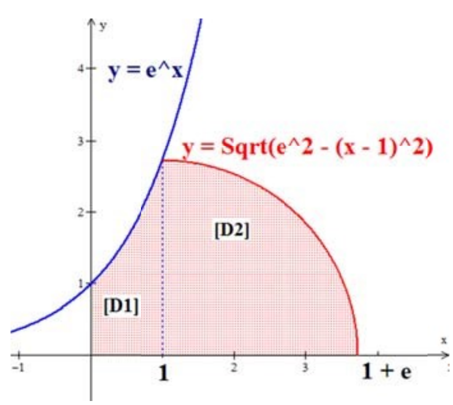
Domeinua lehenengo koadrantean dago. $y = e^x$ funtzioa eta $(x-1)^2 + y^2 = e^2$ zirkunferentzia (zentroa: (1,0); erradioa: e) domeinuaren mugak dira.

Ebakidura puntua:

$$\begin{cases} y = e^x \\ (x-1)^2 + y^2 = e^2 \end{cases} \quad \rightarrow \quad (x-1)^2 + e^2 = e^2 \quad \rightarrow \quad x = 1 \quad P(1,e)$$

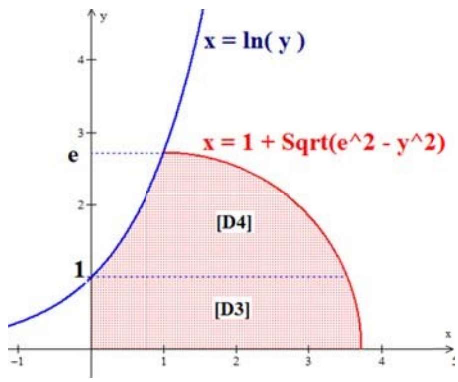
- (y) lehenengo integrazio-aldagaitzat hartuz:

$$I = \int_0^1 dx \int_0^{e^x} f(x,y)\,dy + \int_1^{1+e} dx \int_0^{\sqrt{e^2-(x-1)^2}} f(x,y)\,dy$$



- (x) lehenengo integrazio-aldagaitzat hartuz:

$$I=\int_0^1 dy\int_0^{1+\sqrt{e^2-y^2}} f(x,y)\,dx+\int_1^e dy\int_{\ln y}^{1+\sqrt{e^2-y^2}} f(x,y)\,dx$$



$[D]$ domeinuaren azalera $[D_1]$ domeinuaren azalera gehi $[D_2]$ domeinuaren azalera da:

$$A_T=A_1+A_2=\int_0^1 e^x dx+\frac{1}{4}\pi e^2=\left[e^x\right]_0^1+\frac{\pi e^2}{4}=e-1+\frac{\pi e^2}{4}=\frac{\pi e^2+4e-4}{4}\quad u^2$$

---

2. ORRIA (20 puntu)

A) Klasifikatu eta ebatzi hurrengo EDA: $\left(e^x+\ln y+\dfrac{y}{x}\right)dx+\left(\dfrac{x}{y}+\ln x+\operatorname{sen} y\right)dy=0$

(4 p)

Ebazpena

$$\begin{cases} X(x,y)=e^x+\ln y+\dfrac{y}{x} \\ Y(x,y)=\dfrac{x}{y}+\ln x+\operatorname{sen} y \end{cases}$$

$$\frac{\partial X}{\partial y}=\frac{1}{y}+\frac{1}{x}=\frac{\partial Y}{\partial x}$$

Beraz, EDA **zehatza** da.

Soluzio orokorra hurrengoa da: $\displaystyle\int_a^x\left(e^x+\ln y+\frac{y}{x}\right)dx+\int_b^y\left(\frac{a}{y}+\ln a+\operatorname{sen} y\right)dy=C$

Kalkulua sinplifikatzeko $a=1$; $b=1$ aukeratzen da:

$$\int_1^x \left(e^x + \ln y + \frac{y}{x} \right) dx + \int_1^y \left(\frac{1}{y} + \operatorname{sen} y \right) dy = C$$

$$\left[e^x + x \ln y + y \ln x \right]_1^x + \left[\ln y - \cos y \right]_1^y = C$$

$$e^x + x \ln y + y \ln x - (e + \ln y) + \ln y - \cos y - (-\cos 1) =$$

$$= e^x + x \ln y + y \ln x - e - \cos y + \cos 1 = C$$

Beraz, soluzio orokorra hurrengoa da:

$$\boxed{e^x + x \ln y + y \ln x - \cos y = K}$$

non $k = C + e - \cos 1$ den.

B) Ebatzi hurrengo EDA**:** $x^2 y'' - 3x\, y' + 3y = x + x^2 \cdot \ln x$

(5 p)

Ebazpena:

Euler-en EDA da.

$$y = x^r; \quad y' = r\, x^{r-1}; \quad y'' = r(r-1) x^{r-2}$$

$$x^2 y'' - 3xy' + 3y = x^2 r(r-1)\, x^{r-2} - 3x\, r\, x^{r-1} + 3x^r = x^r [r(r-1) - 3r + 3] = 0$$

$$r^2 - 4r + 3 = 0 \quad \rightarrow \quad r = \frac{4 \pm \sqrt{16-12}}{2} = \begin{cases} 3 \\ 1 \end{cases} \quad \Rightarrow \quad y = C_1 x + C_2 x^3$$

Parametroen aldakuntzaren metodoa erabiliz, soluzio orokorra hurrengoa da:

$$y = L_1(x) \cdot x + L_2(x) \cdot x^3$$

non $L_1(x)$ y $L_2(x)$ hurrengo sistemarekin kalkulatzen diren:

$$\begin{cases} L_1' \cdot x + L_2' \cdot x^3 = 0 \\ L_1' \cdot 1 + L_2' \cdot 3x^2 = \dfrac{x + x^2 \ln x}{x^2} = \dfrac{1}{x} + \ln x \end{cases}$$

$$L_1'(x) = \frac{\begin{vmatrix} 0 & x^3 \\ \frac{1}{x}+\ln x & 3x^2 \end{vmatrix}}{\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}} = \frac{-x^2 - x^3\ln x}{2x^3} = -\frac{1}{2x} - \frac{1}{2}\ln x$$

$$L_2'(x) = \frac{\begin{vmatrix} x & 0 \\ 1 & \frac{1}{x}+\ln x \end{vmatrix}}{\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}} = \frac{1+x\ln x}{2x^3} = \frac{1}{2x^3} + \frac{1}{2x^2}\ln x$$

Integratuz

$$L_1(x) = -\frac{1}{2}\int\frac{dx}{x} - \frac{1}{2}\int \ln x\,dx = -\frac{1}{2}\ln|x| - \frac{1}{2}\left(x\ln|x| - x\right) + A = -\frac{1}{2}\ln|x| - \frac{x}{2}\ln|x| + \frac{x}{2} + A$$

$$L_2(x) = \frac{1}{2}\int\frac{dx}{x^3} + \int\frac{1}{2x^2}\ln x\,dx = I + J$$

$$I = \frac{1}{2}\int\frac{dx}{x^3} = \frac{1}{2}\int x^{-3}dx = \frac{1}{2}\cdot\frac{x^{-2}}{(-2)} = -\frac{1}{4x^2} + cte$$

$$J = \int\frac{1}{2x^2}\ln x\,dx = \left\{\begin{matrix} \ln x = u & \Rightarrow & du = dx/x \\ \frac{1}{2x^2}dx = dv & \Rightarrow & v = -\frac{1}{2x} \end{matrix}\right\} = -\frac{1}{2x}\cdot\ln x + \int\frac{1}{2x^2}dx = -\frac{1}{2x}\cdot\ln x - \frac{1}{2x} + cte$$

$$L_2(x) = I + J = -\frac{1}{4x^2} - \frac{1}{2x}\cdot\ln x - \frac{1}{2x} + B$$

Orduan, soluzio orokorra hurrengoa da:

$$\boxed{y} = L_1(x)\cdot x + L_2(x)\cdot x^3 = \left(-\frac{1}{2}\ln|x| - \frac{x}{2}\ln|x| + \frac{x}{2} + A\right)\cdot x + \left(-\frac{1}{4x^2} - \frac{1}{2x}\cdot\ln x - \frac{1}{2x} + B\right)\cdot x^3 =$$

$$= -\frac{x}{2}\ln|x| - \frac{x^2}{2}\ln|x| + \frac{x^2}{2} + Ax - \frac{x}{4} - \frac{x^2}{2}\cdot\ln x - \frac{x^2}{2} + Bx^3 =$$

$$= \boxed{Ax + Bx^3 - \left(x^2 + \frac{x}{2}\right)\ln|x| - \frac{x}{4}}$$

2. ORRIA (240 puntu)

A) Kalkulatu: $\displaystyle\int \frac{dx}{(x-1)\,\sqrt{x^2+x+1}}$

(40 p)

Ebazpena

$$\int \frac{dx}{(x-1)\sqrt{x^2+x+1}} = \left\{ \begin{array}{l} x-1=\frac{1}{t} \quad \Rightarrow \quad dx=-\frac{dt}{t^2} \\ x^2+x+1=\frac{(1+t)^2}{t^2}+\frac{1+t}{t}+1=\frac{3t^2+3t+1}{t^2} \end{array} \right\} = \int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\cdot\frac{\sqrt{3t^2+3t+1}}{t}} =$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{t^2+t+\frac{1}{3}}} = -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{12}}} = \left\{ \begin{array}{l} t+\frac{1}{2}=z \\ dt=dz \end{array} \right\} =$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dz}{\sqrt{z^2+\frac{1}{12}}} = -\frac{1}{\sqrt{3}}\ln\left|z+\sqrt{z^2+\frac{1}{12}}\right|+C = -\frac{1}{\sqrt{3}}\ln\left|t+\frac{1}{2}+\sqrt{t^2+t+\frac{1}{3}}\right|+C =$$

$$= -\frac{1}{\sqrt{3}}\ln\left|\frac{1}{x-1}+\frac{1}{2}+\sqrt{\frac{1}{(x-1)^2}+\frac{1}{(x-1)}+\frac{1}{3}}\right|+C = -\frac{1}{\sqrt{3}}\ln\left|\frac{1}{x-1}+\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{\sqrt{3}(x-1)}\right|+C$$

B) Izan bedi hurrengo $[D]$ domeinua:

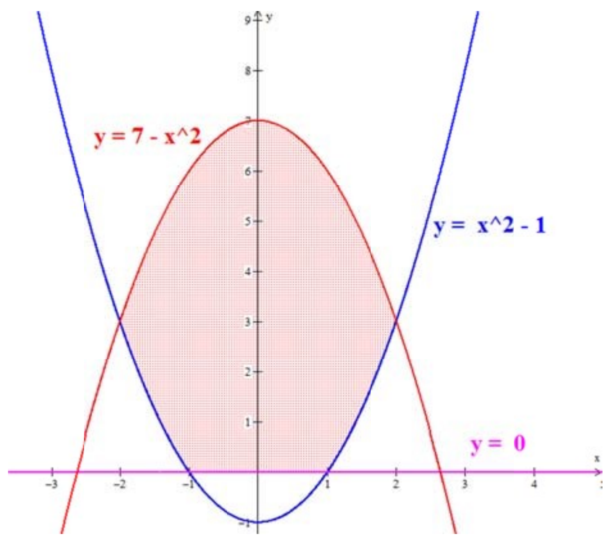
$$D=\left\{(x,y)\in\mathbb{R}^2 \,/\, (y\le 7-x^2)\wedge(y\ge x^2-1)\wedge(y\ge 0)\right\}$$

Kalkulatu:

1.-$[D]$ domeinu lauaren perimetroa. **(50 p)**

2.- $[D]$ abzisa ardatzaren inguruan biratzean sortutako bolumena **(50 p)**

Ebazpena



1.- $[D]$ domeinu lauaren perimetroa:

$$P=2\left[1+L_1+L_2\right]$$

Non:

$$L_1=\int_1^2\sqrt{1+4x^2}\,dx=\frac{1}{4}\left[2x\sqrt{1+4x^2}+\ln\left|2x+\sqrt{1+4x^2}\right|\right]_1^2=$$

$$=\frac{1}{4}\left[\left(4\sqrt{17}+\ln\left(4+\sqrt{17}\right)\right)-\left(2\sqrt{5}+\ln\left(2+\sqrt{5}\right)\right)\right]=\sqrt{17}-\frac{\sqrt{5}}{2}+\frac{1}{4}\ln\frac{4+\sqrt{17}}{2+\sqrt{5}}$$

$$L_2=\int_0^2\sqrt{1+4x^2}\,dx=\frac{1}{4}\left[2x\sqrt{1+4x^2}+\ln\left|2x+\sqrt{1+4x^2}\right|\right]_0^2=$$

$$=\frac{1}{4}\left[\left(4\sqrt{17}+\ln\left(4+\sqrt{17}\right)\right)-\ln(1)\right]=\sqrt{17}+\frac{1}{4}\ln\left(4+\sqrt{17}\right)$$

Beraz, perimetroa honako hau da:

$$P = 2\left[1 + L_1 + L_2\right] = 2\left[1 + \sqrt{17} - \frac{\sqrt{5}}{2} + \frac{1}{4}\ln\frac{4+\sqrt{17}}{2+\sqrt{5}} + \sqrt{17} + \frac{1}{4}\ln\left(4+\sqrt{17}\right)\right] =$$

$$= 2 + 4\sqrt{17} - \sqrt{5} + \frac{1}{4}\ln\frac{\left(4+\sqrt{17}\right)^2}{2+\sqrt{5}} \quad u$$

2.- $[D]$ abzisa ardatzaren inguruan biratzean sortutako bolumena.

$$V = 2V_1$$

$$V_1 = \pi\left[\int_0^2 \left(7 - x^2\right)^2 dx - \int_1^2 \left(x^2 - 1\right)^2 dx\right] = \pi\left[\left(49x - \frac{14}{3}x^3 + \frac{x^5}{5}\Big|_0^2\right) - \left(\frac{x^5}{5} - \frac{2}{3}x^3 + x\right)\Big|_1^2\right] =$$

$$= \pi\left[\left(98 - \frac{112}{3} + \frac{32}{5}\right) - \left(\frac{32}{5} - \frac{16}{3} + 2 - \frac{1}{5} + \frac{2}{3} - 1\right)\right] = \pi\left[\frac{1006}{15} - \frac{38}{15}\right] = \frac{968}{15}\pi$$

Beraz, bolumena honako hau da:

$$V = 2V_1 = \frac{1936}{15}\pi \quad u^3$$

C) Izan bedi hurrengo $[D]$ domeinua:

$$D=\left\{(x,y)\in\mathbb{R}^2/(9x^2+25y^2-225\le 0)\wedge(3x-5y+15\le 0)\right\}$$

1.- Bi era desberdinetan, $I=\iint_{[D]} f(x,y)\,dx\,dy$ integralean, integrazio-limiteak zehaztu.

(30 p)

2.- $[D]$ domeinu lauaren grabitate-zentro geometrikoaren abzisa koordenatua kalkulatu, integral bikoitzak erabiliz.

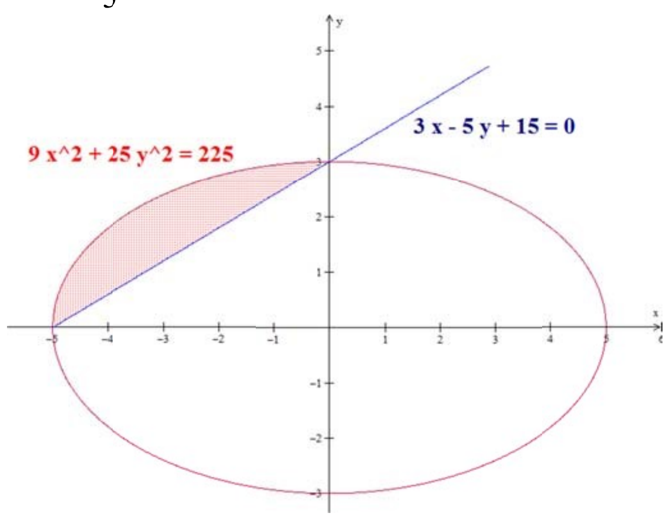
(70 p)

Ebazpena

1.- Bi era desberdinetan, $I=\iint_{[D]} f(x,y)\,dx\,dy$ integralean, integrazio-limiteak zehaztu.

$$9x^2+25y^2-225=0 \quad\rightarrow\quad \frac{x^2}{25}+\frac{y^2}{9}=1 \quad\rightarrow\quad a=5 \quad ; \quad b=3 \text{ erdi-ardatzetako elipsea}$$

$$3x-5y+15=0 \quad\rightarrow\quad y=\frac{3}{5}x+3 \quad\rightarrow\quad (-5,0) \text{ eta } (0,3) \text{ puntuetatik pasatzen den zuzena}$$



Lehenengo integrazio aldagaitzat (x) hartuz

$$I=\iint_{[D]} f(x,y)\,dx\,dy=\int_0^3 dy\int_{-(5/3)\sqrt{9-y^2}}^{(5/3)(y-3)} f(x,y)\,dx$$

Lehenengo integrazio aldagaitzat (y) hartuz

$$I=\iint_{[D]} f(x,y)\,dx\,dy=\int_{-5}^{0} dx\int_{(3/5)x+3}^{(3/5)\sqrt{25-x^2}} f(x,y)\,dy$$

2.- $[D]$ domeinu lauaren grabitate-zentro geometrikoaren abzisa koordenatua kalkulatu, integral bikoitzak erabiliz.

$$A=\iint_D dx\,dy=\int_{-5}^{0}dx\int_{(3/5)x+3}^{(3/5)\sqrt{25-x^2}}dy \;=\int_{-5}^{0}\left[\frac{3}{5}\sqrt{25-x^2}-\frac{3}{5}x-3\right]dx \;=$$

$$=\frac{3}{5}\int_{-5}^{0}\sqrt{25-x^2}\,dx\;-\left[\frac{3x^2}{10}+3x\right]_{-5}^{0}\;=\frac{3}{5}J+\frac{75}{10}-15=\frac{3}{5}J-\frac{15}{2}$$

$$J=\int_{-5}^{0}\sqrt{25-x^2}\,dx\;=\begin{bmatrix}x=5\operatorname{sen}t & x=-5 & \Rightarrow t=-\pi/2\\ dx=5\cos t\,dt & x=0 & \Rightarrow t=0\end{bmatrix}=25\int_{-\pi/2}^{0}\cos^2 t\,dt=$$

$$=25\left[\frac{t}{2}+\frac{\operatorname{sen}2t}{4}\right]_{-\pi/2}^{0}=\frac{25\pi}{4}\quad\Rightarrow\quad\boxed{A}=\frac{3}{5}J-\frac{15}{2}=\frac{15\pi}{4}-\frac{15}{2}=\boxed{\frac{15}{4}(\pi-2)}$$

$$I=\iint_D x\,dx\,dy=\int_{-5}^{0}x\,dx\int_{(3/5)x+3}^{(3/5)\sqrt{25-x^2}}dy\;=\int_{-5}^{0}x\left[\frac{3}{5}\sqrt{25-x^2}-\frac{3}{5}x-3\right]dx=$$

$$=\frac{3}{5}\int_{-5}^{0}x\sqrt{25-x^2}\;dx-\left[\frac{x^3}{5}+\frac{3x^2}{2}\right]_{-5}^{0}=\frac{3}{5}H-25+\frac{75}{2}=\frac{3}{5}H+\frac{25}{2}$$

$$H=\int_{-5}^{0}x\sqrt{25-x^2}\;dx=\begin{bmatrix}25-x^2=t^2 & x=-5 & \Rightarrow t=0\\ -xdx=tdt & x=0 & \Rightarrow t=5\end{bmatrix}=-\int_0^5 t^2dt=\left[-\frac{t^3}{3}\right]_0^5=-\frac{125}{3}$$

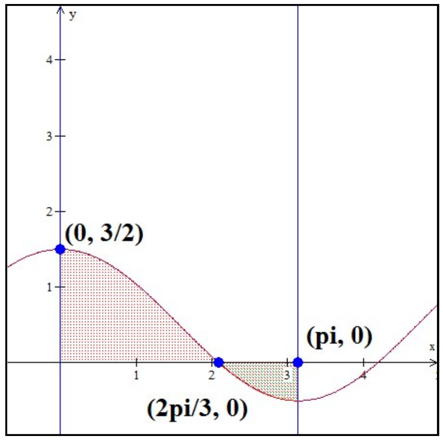
$$I=\frac{3}{5}H+\frac{25}{2}=-25+\frac{25}{2}=-\frac{25}{2}$$

$$\boxed{x_c}=\frac{1}{A}\iint_D x\,dx\,dy=\frac{-25/2}{15(\pi-2)/4}=\boxed{\frac{-10}{3(\pi-2)}}$$

D) $y=\dfrac{1}{2}+\cos x$, absiza-ardatzak eta hurrengo zuzenek $x=0$ eta $x=\pi$ mugatzen duten azalera kalkulatu.

(6 puntu)

Ebazpena



Bere azalera integralen bidez kalkulatuko dugu:

$$A=\int_0^{\frac{2\pi}{3}}\left(\frac{1}{2}+\cos x\right)dx-\int_{\frac{2\pi}{3}}^{\pi}\left(\frac{1}{2}+\cos x\right)dx=\left(\frac{1}{2}x+\operatorname{sen}x\right)\Bigg]_0^{\frac{2\pi}{3}}-\left(\frac{1}{2}x+\operatorname{sen}x\right)\Bigg]_{\frac{2\pi}{3}}^{\pi}=$$

$$=\left(\frac{\pi}{3}+\operatorname{sen}\frac{2\pi}{3}\right)-\left(\frac{\pi}{2}-\frac{\pi}{3}-\operatorname{sen}\frac{2\pi}{3}\right)=\frac{2\pi}{3}-\frac{\pi}{2}+2\frac{\sqrt{3}}{2}=\frac{\pi}{6}+\sqrt{3}\quad u^2$$

---

2. ORRIA (20 puntu)

A) Izan bitez $O=(0,0)$, $A=(0,2)$, $B=(2,2)$, $C=(4,2)$ eta $E=(6,0)$ puntuak. $[D]$ domeinua hurrengo eran mugatuta dago:

- $\overline{OA}$ zuzenaren segmentua zeinak *O* eta *A* puntuak lotzen dituen.
- $\overline{AB}$ zuzenaren segmentua zeinak *A* eta *B* puntuak lotzen dituen.
- $(3,2)$ zentrodun eta 1 erradiodun zirkunferentziaren beheko erdi-zirkulua.
- $\overline{CE}$ zuzenaren segmentua zeinak *C* eta *E* puntuak lotzen dituen.
- $\overline{EO}$ zuzenaren segmentua zeinak *E* eta *O* puntuak lotzen dituen.

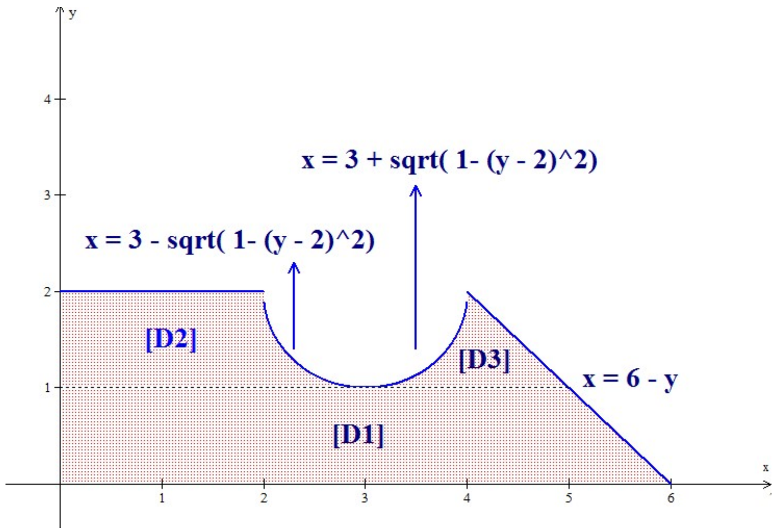
1.- $I=\iint_{[D]}f(x,y)\,dx\,dy$ integralean integrazio-limiteak bi era desberdinetan planteatu.

2.- Integral bikoitzak erabiliz, $[D]$ domeinu lauaren azalera kalkulatu, eta emaitza egiaztatu oinarrizko geometria erabiliz.

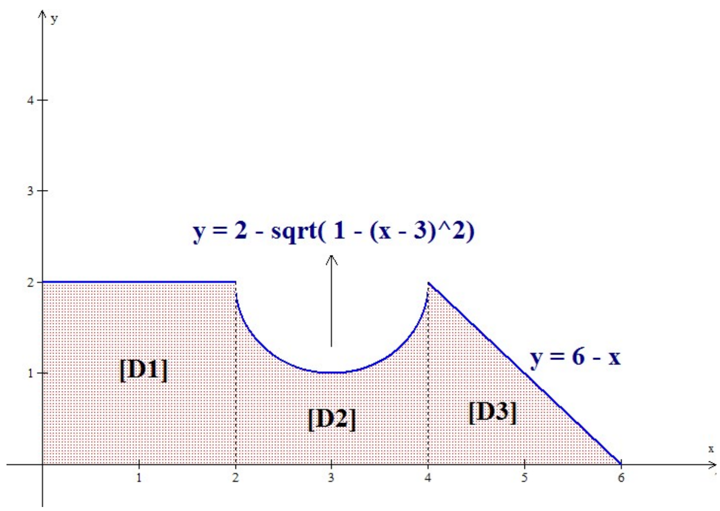
(6 puntu)

Ebazpena

Integralaren limiteak



$$\iint_D f(x,y)\,dx\,dy = \int_0^1 dy \int_0^{6-y} f(x,y)\,dx + \int_1^2 dy \int_0^{3-\sqrt{1-(y-2)^2}} f(x,y)\,dx \quad + \int_1^2 dy \int_{3+\sqrt{1-(y-2)^2}}^{6-y} f(x,y)\,dx$$



$$\iint_D f(x,y)\,dx\,dy = \int_0^2 dx \int_0^2 f(x,y)\,dy + \int_2^4 dx \int_0^{2-\sqrt{1-(x-3)^2}} f(x,y)\,dy \quad + \int_4^6 dx \int_0^{-x+6} f(x,y)\,dy$$

Azaleraren kalkulua

$$A = \iint_D dx\,dy = \int_0^2 dx \int_0^2 dy + \int_2^4 dx \int_0^{2-\sqrt{1-(x-3)^2}} dy \quad + \int_4^6 dx \int_0^{-x+6} dy \quad =$$

$$= \int_0^2 2\,dx + \int_2^4 \left(2-\sqrt{1-(x-3)^2}\right) dx + \int_4^6 (-x+6)\,dx =$$

$$\underset{*}{=} 2x\Big]_0^2 + \left(2x - \frac{1}{2}\left((x-3)\sqrt{1-(x-3)^2} + \operatorname{arcsen}(x-3)\right)\right)\Bigg]_2^4 + \left(-\frac{x^2}{2} + 6x\right)\Bigg]_4^6 =$$

$$= 4 + \left(\left(8 - \frac{\pi}{4}\right) - \left(4 + \frac{\pi}{4}\right)\right) + \left((-18+36) - (-8+24)\right) = 10 - \frac{\pi}{2} \quad u^2$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2 \operatorname{arcsen}\frac{x}{a}\right) + C \qquad (*)$$

Geometrikoki

$$A_D = A_{\text{cuadrado}} + \left(A_{\text{cuadrado}} - \frac{1}{2} A_{\text{círculo}}\right) + A_{\text{triángulo}} = 2\cdot 2 + \left(2\cdot 2 - \frac{\pi \cdot 1^2}{2}\right) + \frac{2\cdot 2}{2} = 10 - \frac{\pi}{2} \quad u^2$$

B) Hurrengo EDA sailkatu eta ebatzi: $\left(\dfrac{2x}{x^2+y^2+1} - 2y\right)dx + \left(\dfrac{2y}{x^2+y^2+1} - 2x\right)dy = 0$

(4 puntu)

Ebazpena

$$\begin{cases} X(x,y) = \dfrac{2x}{x^2+y^2+1} - 2y \\ Y(x,y) = \dfrac{2y}{x^2+y^2+1} - 2x \end{cases}$$

$$\frac{\partial X}{\partial y} = \frac{-4xy}{(x^2+y^2+1)^2} - 2 = \frac{\partial Y}{\partial x}$$

Beraz, EDA **zehatza** da.

Ebazpen orokorra hurrengoa da:

$$\int_a^x \left(\frac{2x}{x^2+y^2+1} - 2y\right)dx + \int_b^y \left(\frac{2y}{a^2+y^2+1} - 2a\right)dy = C$$

Kalkuluak sinplifikatzeko hurrengo erabiliko dugu: $a = 0$; $b = 0$:

$$\int_0^x \left(\frac{2x}{x^2+y^2+1} - 2y\right)dx + \int_0^y \frac{2y}{y^2+1}\,dy = C$$

$$\left[\ln\left|x^2+y^2+1\right| - 2xy\right]_0^x + \left[\ln\left|y^2+1\right|\right]_0^y = C$$

$$\ln\left|x^2+y^2+1\right|-2xy-\ln\left|y^2+1\right|+\ln\left|y^2+1\right|=C \quad\rightarrow\quad \boxed{\ln\left|x^2+y^2+1\right|-2xy=C}$$

C) Hurrengo EDA ebatzi**:** $y''-y=xe^x$

(5 puntu)

Ebazpena

1.- Koefiziente indeterminatuen metodoa

Elkartutako ekuazio homogeneoaren soluzio orokorra:

$$r^2-1=0 \quad\rightarrow\quad r=\pm 1 \quad\Rightarrow\quad y_h=C_1e^{-x}+C_2e^x$$

Ekuazio osoaren soluzio partikularra:

$$f(x)=xe^x \quad\rightarrow\quad Y=x(Ax+B)\,e^x$$

$x(Ax+b)\,e^x$ erabiliko dugu $(Ax+B)\,e^x$ erabili beharrean. Horrela, ekuazio homogeneoaren soluzioetako batekin sortuko litzatekeen bikoiztasuna ekiditen da. (Y)-ren koefizienteak identifikatzeko, (Y) eta bere deribatuak ekuazio osoan ordezkatzen dira.:

$$Y=(Ax^2+Bx)\,e^x \quad\rightarrow\quad Y'=(2Ax+B)\,e^x+(Ax^2+Bx)\,e^x=[Ax^2+(2A+B)\,x+B]e^x \quad\rightarrow$$

$$Y''=(2Ax+2A+B)\,e^x+[Ax^2+(2A+B)x+B]e^x=[Ax^2+(4A+B)x+(2A+2B)]e^x$$

$$Y''-Y=[Ax^2+(4A+B)x+(2A+2B)]e^x-(Ax^2+Bx)\,e^x=(4Ax+2A+2B)\,e^x\equiv x\,e^x$$

$$\rightarrow\quad \begin{cases} 4A=1 \\ 2A+2B=0 \end{cases} \quad\rightarrow\quad \begin{cases} A=1/4 \\ B=-1/4 \end{cases} \quad\rightarrow\quad Y(x)=\left(\frac{x^2}{4}-\frac{x}{4}\right)e^x$$

Ekuazio osoaren soluzio orokorra:

$$y=y_h+Y \quad\Rightarrow\quad \boxed{y(x)=C_1e^{-x}+C_2e^x+\frac{1}{4}(x^2-x)\,e^x}$$

2.- Parametroen aldakuntzaren metodoa

Parametroen aldakuntzaren metodoa erabiliz, ekuazio osoaren soluzio orokorra planteatzen da:

$$y=L_1(x)e^{-x}+L_2(x)e^x$$

(L_1') eta (L_2') hurrengo sisteman egonda:

$$\begin{cases} e^{-x}\, L_1'(x) + e^{x}\, L_2'(x) = 0 \\ -e^{-x}\, L_1'(x) + e^{x}\, L_2'(x) = x\,e^{x} \end{cases} \qquad \Rightarrow$$

$$L_1'(x) = \frac{\begin{vmatrix} 0 & e^{x} \\ xe^{x} & e^{x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix}} = \frac{-x\,e^{2x}}{1+1} = -\frac{x}{2}e^{2x}$$

$$L_2'(x) = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & xe^{x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix}} = \frac{x}{2}$$

$\int$ eragilea aplikatuz:

$$\boxed{L_1(x)} = \int L_1'(x)\,dx = -\frac{1}{2}\int xe^{2x}dx = \begin{Bmatrix} x = u & \Rightarrow & du = dx \\ e^{2x}\,dx = dv & \Rightarrow & v = e^{2x}/2 \end{Bmatrix} =$$

$$= -\frac{1}{2}\left[\frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}dx\right] = \boxed{-\frac{x}{4}e^{2x} + \frac{e^{2x}}{8} + A}$$

$$\boxed{L_2(x)} = \int L_2'(x)\,dx = \int \frac{x}{2}dx = \boxed{\frac{x^2}{4} + B}$$

Soluzio orokorra lortzen dugu:

$$\boxed{y} = L_1(x)e^{-x} + L_2(x)e^{-x} = \left[\left(-\frac{x}{4}+\frac{1}{8}\right)e^{2x} + A\right]\cdot e^{-x} + \left(\frac{x^2}{4}+B\right)\cdot e^{x} =$$

$$= Ae^{-x} + Be^{x} + e^{x}\left(-\frac{x}{4}+\frac{1}{8}+\frac{x^2}{4}\right) = \boxed{Ae^{-x} + Ke^{x} + \frac{1}{4}(x^2 - x)\,e^{x}}$$

Minimo ala maximo bat den jakiteko, matrize hessiarraren determinantea kalkulatzen da:

$$HL(\lambda,(x,y))=\begin{pmatrix}0 & \varphi'_x & \varphi'_y\\ \varphi'_x & L''_{xx} & L''_{xy}\\ \varphi'_y & L''_{yx} & L''_{yy}\end{pmatrix}=\begin{pmatrix}0 & 2x & 2y\\ 2x & 2+2\lambda & 0\\ 2y & 0 & 2+2\lambda\end{pmatrix}$$

$$\left|HL\left(-\frac{9}{4},\left(\frac{12}{5},-\frac{16}{5}\right)\right)\right|=\begin{vmatrix}0 & \frac{24}{5} & -\frac{32}{5}\\ \frac{24}{5} & -\frac{5}{2} & 0\\ -\frac{32}{5} & 0 & -\frac{5}{2}\end{vmatrix}>0\rightarrow\ P_1\left(\frac{12}{5},-\frac{16}{5}\right)$$ maximo lokal bat da.

$$\left|HL\left(\frac{1}{4},\left(-\frac{12}{5},\frac{16}{5}\right)\right)\right|=\begin{vmatrix}0 & -\frac{24}{5} & \frac{32}{5}\\ -\frac{24}{5} & \frac{5}{2} & 0\\ \frac{32}{5} & 0 & \frac{5}{2}\end{vmatrix}<0\rightarrow\ P_2\left(-\frac{12}{5},\frac{16}{5}\right)$$ maximo lokal bat da.

D) Kalkulatu, integralak erabiliz, $y=\sqrt{(3-x)(1+x)}$ kurbaren luzera

(6 puntu)

<u>*Ebazpena*</u>

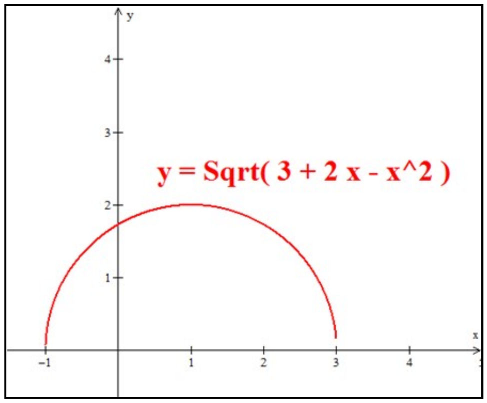
Zirkunferentzierdi bat da. Hain zuzen ere:

$$y=\sqrt{(3-x)(1+x)}\quad\rightarrow\quad y^2=(3-x)(1+x)=3-x+3x-x^2=3+2x-x^2$$

$x^2+y^2-2x=3\quad\rightarrow\quad(x-1)^2+y^2=4\quad\rightarrow\quad(1,0)$ zentroko eta 2 erradioko zirkunferentzia.

$$y=\pm\sqrt{4-(x-1)^2}=\pm\sqrt{3+2x-x^2}$$

Orduan, emandako kurba $(1,0)$ zentroko eta 2 erradioko goiko zirkunferentzierdia da.



Luzeraren kalkulua:

$$y' = \frac{-2x+2}{2\sqrt{3+2x-x^2}} = \frac{1-x}{\sqrt{3+2x-x^2}} \quad \rightarrow \quad y'^2 = \frac{(1-x)^2}{3+2x-x^2}$$

$$\boxed{L} = \int_{-1}^{3} \sqrt{1+y'^2}\,dx = \int_{-1}^{3} \sqrt{1+\frac{(1-x)^2}{3+2x-x^2}}\,dx = \int_{-1}^{3} \sqrt{\frac{3+2x-x^2+1-2x+x^2}{3+2x-x^2}}\,dx =$$

$$= \int_{-1}^{3} \sqrt{\frac{4}{3+2x-x^2}}\,dx = 2\int_{-1}^{3} \frac{dx}{\sqrt{3+2x-x^2}} = 2\int_{-1}^{3} \frac{dx}{\sqrt{4-(x-1)^2}} =$$

$$= 2\left[\arcsin\left(\frac{x-1}{2}\right)\right]_{-1}^{3} = 2\left[\arcsin 1 - \arcsin(-1)\right] = 2\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \boxed{2\pi}$$

Oinarrizko geometria erabiliz (emaitza konprobatzeko): $L = \dfrac{2\pi r}{2} = \pi r = 2\pi$

<u>2. ORRIA</u> (20 puntu)

A) Izan bedi hurrengo [D] domeinu laua:

$$D = \left\{ (x,y) \in \mathbb{R}^2 \,/\, y \geq |2-x| \;;\; (x-2)^2 + y^2 \leq 4 \right\}$$

1.- $I = \iint_{[D]} f(x,y)\,dx\,dy$ integralean integrazio-limiteak bi era desberdinetan planteatu.

2.- Integral bikoitzak erabiliz, $[D]$ domeinu lauaren azalera kalkulatu.

(6 puntu)

Ebazpena

Ebakidura puntuak kalkulatu:

Si $x \le 2$:

$$\begin{cases}(x-2)^2+y^2=4\\ \qquad y=2-x\end{cases} \Rightarrow \begin{cases}y=\sqrt{4-(x-2)^2}\\ y=2-x\end{cases} \Rightarrow 2-x=\sqrt{4-(x-2)^2} \Rightarrow$$

$$(2-x)^2=4-(x-2)^2 \Rightarrow 2(2-x)^2=4 \Rightarrow x=2\pm\sqrt{2} \Rightarrow$$

$$\Rightarrow \left(2-\sqrt{2},\sqrt{2}\right)$$

Si $x > 2$:

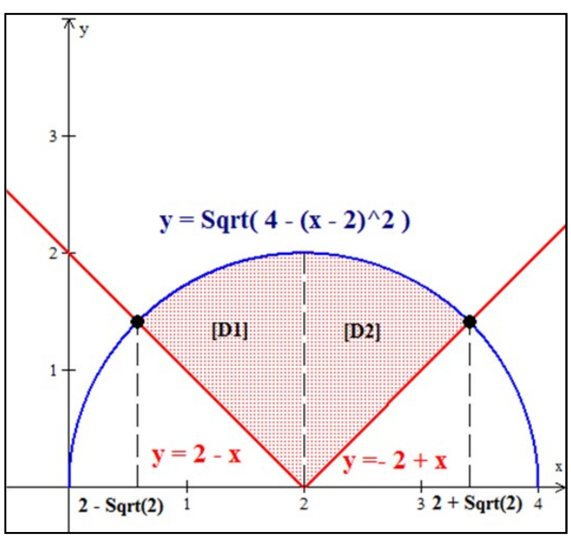
$$\begin{cases}(x-2)^2+y^2=4\\ \qquad y=x-2\end{cases} \Rightarrow \begin{cases}y=\sqrt{4-(x-2)^2}\\ y=x-2\end{cases} \Rightarrow x-2=\sqrt{4-(x-2)^2} \Rightarrow$$

$$(x-2)^2=4-(x-2)^2 \Rightarrow 2(x-2)^2=4 \Rightarrow x=2\pm\sqrt{2} \Rightarrow$$

$$\Rightarrow \left(2+\sqrt{2},\sqrt{2}\right)$$

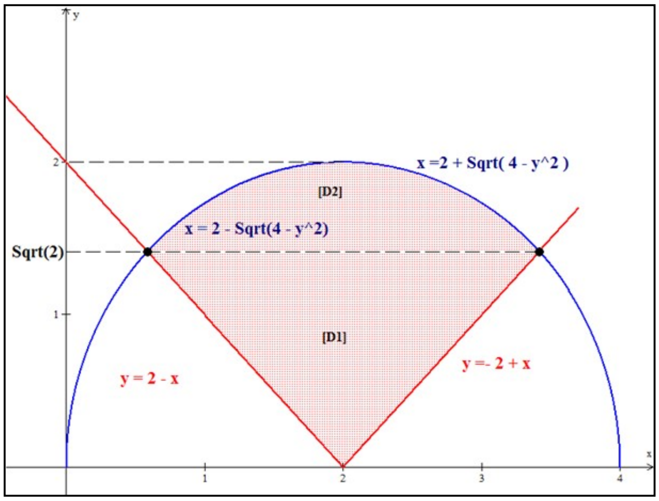
Integralaren limiteak

(y) lehenengo integrazio-aldagaitzat hartuz:



$$\iint_D f(x,y)\,dx\,dy=\int_{2-\sqrt{2}}^{2}dx\int_{2-x}^{\sqrt{4-(x-2)^2}} f(x,y)\,dy+\int_{2}^{2+\sqrt{2}}dx\int_{-2+x}^{\sqrt{4-(x-2)^2}} f(x,y)\,dy$$

(x) lehenengo integrazio-aldagaitzat hartuz:



$$\iint_D f(x,y)\,dx\,dy = \int_0^{\sqrt{2}} dy \int_{2-y}^{y+2} f(x,y)\,dx + \int_{\sqrt{2}}^{2} dy \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} f(x,y)\,dx$$

<u>*Azaleraren kalkulua:*</u>

$$A = \int_{2-\sqrt{2}}^{2} dx \int_{2-x}^{\sqrt{4-(x-2)^2}} dy + \int_{2}^{2+\sqrt{2}} dx \int_{x-2}^{\sqrt{4-(x-2)^2}} dy$$

Dagoen simetria kontuan hartuta:

$$A = 2\int_{2}^{2+\sqrt{2}} dx \int_{x-2}^{\sqrt{4-(x-2)^2}} dy = 2\int_{2}^{2+\sqrt{2}} \left(\sqrt{4-(x-2)^2} - (x-2)\right) dx =$$

$$= 2\left[\int_{2}^{2+\sqrt{2}} \left(\sqrt{4-(x-2)^2}\right) dx - \left[\frac{x^2}{2} - 2x\right]_2^{2+\sqrt{2}}\right] = 2\left[I - \frac{1}{2}\left(\left(2+\sqrt{2}\right)^2 - 4\left(2+\sqrt{2}\right) - 4 + 8\right)\right] = 2[I-1]$$

$$I = \int_{2}^{2+\sqrt{2}} \left(\sqrt{4-(x-2)^2}\right) dx = \begin{bmatrix} x-2 = 2\sin t & \to & dx = 2\cos t\,dt \\ x = 2 & \to & t = 0 \\ x = 2+\sqrt{2} & \to & t = \arcsin\left(\sqrt{2}/2\right) = \pi/4 \end{bmatrix} = \int_0^{\pi/4} \sqrt{4-4\sin^2 t}\; 2\cos t\,dt =$$

$$I = \int_0^{\pi/4} 4\cos^2 t\,dt = 2\int_0^{\pi/4} (1+\cos 2t)\,dt = 2\left[t + \frac{\sin 2t}{2}\right]_0^{\pi/4} = 2\left[\frac{\pi}{4} + \frac{1}{2}\right] = \frac{\pi}{2} + 1$$

Beraz, azalera hurrengoa da:

$$A = 2\left[I-1\right] = 2\left[\frac{\pi}{2}+1-1\right] = \pi \quad u^2$$

B) Ebatzi hurrengo EDA faktore integratzaile bat erabiliz:

$$\left(x^2+y^2\right)\cos x\,dx - \frac{2}{y}\left(2x\cos x + \left(x^2-2\right)\sin x\right)dy = 0$$

(5 puntu)

<u>*Ebazpena*</u>

$$X(x,y) = \left(x^2+y^2\right)\cos x \quad \Rightarrow \quad \frac{\partial X}{\partial y} = 2y\cos x$$

$$Y(x,y) = -\frac{2}{y}\left(2x\cos x + \left(x^2-2\right)\operatorname{sen} x\right) \quad \Rightarrow \quad \frac{\partial Y}{\partial x} = \frac{-2x^2\cos x}{y}$$

EDA ez da zehatza: $\dfrac{\partial X}{\partial y} \neq \dfrac{\partial Y}{\partial x}$

Ezin da $z(x)$ faktore integratzaile bat lortu:

$$\frac{\dfrac{\partial X}{\partial y} - \dfrac{\partial Y}{\partial x}}{Y} = \frac{2y\cos x + \dfrac{2x^2\cos x}{y}}{-\dfrac{2}{y}\left(2x\cos x + \left(x^2-2\right)\operatorname{sen} x\right)} \neq \varphi(x)$$

$z(y)$ faktore integratzaile bat lortu ahal da:

$$\frac{\dfrac{\partial Y}{\partial x} - \dfrac{\partial X}{\partial y}}{X} = \frac{-\dfrac{2x^2\cos x}{y} - 2y\cos x}{\left(x^2+y^2\right)\cos x} = \frac{\dfrac{-2x^2-2y^2}{y}}{\left(x^2+y^2\right)} = \frac{-2}{y} = \varphi(y)$$

$$z(y) = e^{\int \frac{-2}{y}dx} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = \frac{1}{y^2}$$

EDA bider $z(y)$ biderkatu ondoren zehatza da.

$$\left(\frac{x^2+y^2}{y^2}\right)\cos x\,dx - \frac{2}{y^3}\left(2x\cos x + \left(x^2-2\right)\sin x\right)dy = 0$$

$$X'(x,y)=\left(\frac{x^2+y^2}{y^2}\right)\cos x \quad \Rightarrow \quad \frac{\partial X'}{\partial y}=-\frac{2x^2}{y^3}\cos x$$

$$Y'(x,y)=-\frac{2}{y^3}\left(2x\cos x+\left(x^2-2\right)\sin x\right) \quad \Rightarrow \quad \frac{\partial Y'}{\partial x}=-\frac{2x^2\cos x}{y^3}$$

$$\Rightarrow \quad \frac{\partial X'}{\partial y}=\frac{\partial Y'}{\partial x}$$

Soluzio orokorra hurrengoa da:

$$\int_a^x X'(x,y)\,dx+\int_b^y Y'(a,y)\,dx=C$$

(a,b)=(0,1) hartuz:

$$\int_0^x\left(\frac{x^2+y^2}{y^2}\right)\cos x\,dx+\int_1^y 0\,dx=C \quad \to \quad \int_0^x\left(\frac{x^2+y^2}{y^2}\right)\cos x\,dx=C$$

$$\frac{1}{y^2}\int_0^x\left(x^2+y^2\right)\cos x\,dx=\begin{bmatrix}u=x^2+y^2 & du=2x\,dx\\ dv=\cos x\,dx & v=\sin x\end{bmatrix}=\frac{1}{y^2}\left[\left[\left(x^2+y^2\right)\sin x\right]_0^x-\int_0^x 2x\sin x\,dx\right]=$$

$$=\begin{bmatrix}u=2x & du=2\,dx\\ dv=\sin x\,dx & v=-\cos x\end{bmatrix}=\frac{1}{y^2}\left[\left(x^2+y^2\right)\sin x-\left(\left[-2x\cos x\right]_0^x+\int_0^x 2\cos x\,dx\right)\right]=$$

$$=\frac{1}{y^2}\left[\left(x^2+y^2\right)\sin x+2x\cos x-2\sin x\right]$$

Beraz, soluzio orokorra hurrengoa da:

$$\frac{1}{y^2}\left[\left(x^2+y^2\right)\operatorname{sen}x+2x\cos x-2\operatorname{sen}x\right]=C \quad \Rightarrow \quad Cy^2=\left(x^2+y^2-2\right)\operatorname{sen}x+2x\cos x$$

C) $x(2x+3)y''-6(x+1)y'+6y=0$ ekuazio diferentzialaren soluzio orokorra lortu, $y=x^3$ ekuazioren soluzio partikular bat dela jakinda.

(4 puntu)

Ebazpena

Beste soluzioa hurrengoa formula erabiliz lortzen da:

$$y_2 = y_1 \int \frac{\exp\left(-\int P(x)dx\right)}{y_1^2} dx \text{ , non } P(x) = \frac{-6(x+1)}{x(2x+3)}$$

Zatiki sinpleetan deskonposatuz:

$$\frac{-6(x+1)}{x(2x+3)} = \frac{A}{x} + \frac{B}{2x+3}$$

$$-6(x+1) = A(2x+3) + Bx$$

$$\begin{cases} -6 = 2A + B \\ -6 = 3A \end{cases} \quad \Rightarrow \quad A = -2\,;\, B = -2$$

Integratuz:

$$\exp\left[-\int P(x)dx\right] = \exp\left[\int\left(\frac{2}{x} + \frac{2}{2x+3}\right)dx\right] = \exp\left[\ln x^2\left(2x+3\right)\right] = x^2\left(2x+3\right)$$

eta ordezkatuz:

$$y_2 = x^3\left[\int \frac{(2x+3)}{x^4}\, dx\right] = x^3\left[\int\left(\frac{2}{x^3} + \frac{3}{x^4}\right)dx\right] = x^3\left[-x^{-2} - x^{-3}\right] = -\left(x+1\right)$$

Soluzio orokorra lortzen da:

$$y = C_1 x^3 + C_2\left(x+1\right)$$

D) Ebatzi hurrengo ekuazio diferentziala Laplace-ren transformatua erabiliz.

$$y'' - 2y' + 2y = 6\, e^{-t} \cos t \qquad y\left(0\right) = y'\left(0\right) = 0$$

(5 puntu)

<u>*Ebazpena*</u>

$$y'' - 2y' + 2y = 6\, e^{-t} \cos t \qquad y\left(0\right) = y'\left(0\right) = 0$$

$$L\left[y'' - 2y' + 2y\right] = p^2 Y(p) - p\; y(0) - y'(0) - 2\left[pY(p) - y(0)\right] + 2\, Y(p)$$

$$\wedge \quad L\left[6\, e^{-t} \cos t\right] = \frac{6\left(p+1\right)}{\left(p+1\right)^2 + 1}, \quad \text{non } y(0) = y'(0) = 0$$

$$\left(p^2-2p+2\right)Y(p)=\frac{6\left(p+1\right)}{\left(p+1\right)^2+1} \quad \Rightarrow \quad Y(p)=\frac{6\left(p+1\right)}{\left(\left(p-1\right)^2+1\right)\left(\left(p+1\right)^2+1\right)}$$

Deskonposatuz:

$$\frac{6(p+1)}{\left((p-1)^2+1\right)\left((p+1)^2+1\right)}=\frac{ap+b}{(p-1)^2+1}+\frac{cp+d}{(p+1)^2+1}$$

$$6(p+1)=\left(ap+b\right)\left((p+1)^2+1\right)+\left(cp+d\right)\left((p-1)^2+1\right)$$

$$6p+6=\left(a+c\right)p^3+\left(2a+b-2c+d\right)p^2+\left(2a+2b+2c-2d\right)p+\left(2b+2d\right)$$

$$\begin{cases} 0=a+c \\ 0=2a+b-2c+d \\ 6=2a+2b+2c-2d \\ 6=2b+2d \end{cases} \Rightarrow \begin{cases} c=-a \\ 0=4a+b+d \\ 3=b-d \\ 3=b+d \end{cases} \Rightarrow \begin{cases} a=-\dfrac{3}{4} \\ b=3 \\ c=\dfrac{3}{4} \\ d=0 \end{cases}$$

Beraz:

$$Y(p)=\frac{-\frac{3}{4}p+3}{(p-1)^2+1}+\frac{\frac{3}{4}p}{(p+1)^2+1}=-\frac{3}{4}\left(\frac{p-4}{(p-1)^2+1}\right)+\frac{3}{4}\left(\frac{p}{(p+1)^2+1}\right)$$

$$Y(p)=-\frac{3}{4}\left(\frac{p-1-3}{(p-1)^2+1}\right)+\frac{3}{4}\left(\frac{p+1-1}{(p+1)^2+1}\right)$$

$$Y(p)=-\frac{3}{4}\left(\frac{p-1}{(p-1)^2+1}\right)+\frac{9}{4}\left(\frac{1}{(p-1)^2+1}\right)+\frac{3}{4}\left(\frac{p+1}{(p+1)^2+1}\right)-\frac{3}{4}\left(\frac{1}{(p+1)^2+1}\right)$$

Alderantziko transformatua kalkulatuz:

$$y(t)=-\frac{3}{4}\left(e^t\cos t\right)+\frac{9}{4}\left(e^t\sin t\right)+\frac{3}{4}\left(e^{-t}\cos t\right)-\frac{3}{4}\left(e^{-t}\sin t\right)$$

$$y(t)=-\frac{3}{2}\left(\frac{e^t-e^{-t}}{2}\right)\cos t+\frac{3}{2}\left(\frac{e^t-e^{-t}}{2}\right)\sin t+\frac{3}{2}\left(e^t\sin t\right)$$

$$y(t)=-\frac{3}{2}\operatorname{sh}t\cdot\cos t+\frac{3}{2}\operatorname{sh}t\cdot\sin t+\frac{3}{2}\left(e^t\sin t\right)$$

OHAR OROKORRAK:

2. ORRIA (200 puntu)

A) Kalkulatu: $\int \ln(\operatorname{sen} x)\cdot \operatorname{sen} x \, dx$ $\qquad \int \frac{1}{x^5\cdot\sqrt{1+\frac{1}{x^2}}}\,dx$

(60 p)

Ebazpena

$$I=\int \ln(\operatorname{sen} x)\cdot \operatorname{sen} x\, dx = \left\| \begin{matrix} u=\ln(\operatorname{sen} x) & du=\frac{\cos x}{\operatorname{sen} x}dx \\ dv=\operatorname{sen} x\, dx & v=-\cos x \end{matrix}\right\| =$$

$$=-\ln(\operatorname{sen} x)\cdot\cos x+\int\frac{\cos^2 x}{\operatorname{sen} x}dx=-\ln(\operatorname{sen} x)\cdot\cos x+J$$

$$J=\int\frac{\cos^2 x}{\operatorname{sen} x}dx=\int\frac{1-\operatorname{sen}^2 x}{\operatorname{sen} x}dx=\int\left(\frac{1}{\operatorname{sen} x}-\operatorname{sen} x\right)dx=\int\frac{1}{\operatorname{sen} x}dx-\int \operatorname{sen} x\, dx=H+\cos x$$

$$H=\int\frac{1}{\operatorname{sen} x}dx=\left\| \begin{matrix} t=\operatorname{tg}\frac{x}{2} & dx=\frac{2}{1+t^2}dt \\ \operatorname{sen} x=\frac{2t}{1+t^2} & \end{matrix}\right\| =\int\frac{1+t^2}{2t}\cdot\frac{2}{1+t^2}dt=\int\frac{1}{t}dt=$$

$$=\ln|t|+K=\ln\left|\operatorname{tg}\frac{x}{2}\right|+K_1$$

$$\boxed{I}=-\ln(\operatorname{sen} x)\cdot\cos x+J=-\ln(\operatorname{sen} x)\cdot\cos x+\cos x+H=$$

$$=\boxed{-\ln(\operatorname{sen} x)\cdot\cos x+\cos x+\ln\left|\operatorname{tg}\frac{x}{2}\right|+K}$$

$$I=\int\frac{1}{x^5\cdot\sqrt{1+x^{-2}}}dx=\int x^{-5}\left(1+x^{-2}\right)^{-1/2}dx=\left[\begin{matrix} m=-5 & n=-2 \\ p=-\frac{1}{2}\notin\mathbb{Z} & \frac{m+1}{n}=2\in\mathbb{Z}\end{matrix}\right]=\left(\begin{matrix}\text{binomia}\\ 2^{\text{o}}\text{ caso}\end{matrix}\right)=$$

$$=\left\| \begin{array}{l} x^{-2}=t \rightarrow x=t^{-1/2} \\ dx=-\dfrac{1}{2}t^{-3/2}dt \end{array} \right\| = \int t^{5/2}\left(1+t\right)^{-1/2}\left(-\frac{1}{2}\right)t^{-3/2}dt=-\frac{1}{2}\int t\left(1+t\right)^{-1/2}\,dt=$$

$$=\left\| \begin{array}{l} 1+t=z^2 \\ dt=2z\,dz \end{array} \right\| =-\frac{1}{2}\int\left(z^2-1\right)z^{-1}\,2z\,dz=-\int\left(z^2-1\right)dz=$$

$$=z-\frac{z^3}{3}+K=\left(1+t\right)^{1/2}-\frac{\left(1+t\right)^{3/2}}{3}+K=\boxed{\left(1+x^{-2}\right)^{1/2}-\frac{\left(1+x^{-2}\right)^{3/2}}{3}+K}$$

B) Zehaztu a konstante erreal positiboaren balioa, hurrengo domeinuak

$$D=\left\{(x,y)\in\mathbb{R}^2\;/\;x^2+y-4\le 0\;\wedge\;y\ge a\cdot x^2\;\wedge\;y\ge 0\right\}$$

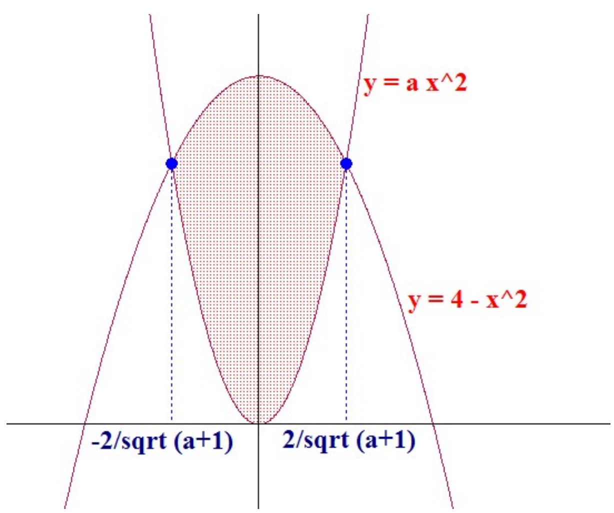
definitutako azalera $A=\dfrac{16}{3}\quad u^2$ izan dadin.

(80 p)

Ebazpena

Bi parabolak ditugu ardatz bertikala dutenak. Ebakidura puntuak hurrengoak dira:

$$\left.\begin{array}{l} y=a\,x^2 \\ y=4-x^2 \end{array}\right\} \Rightarrow \quad x=\pm\frac{2}{\sqrt{a+1}}$$



Simetria kontuan izanda, azalera horrela kalkulatuko dugu:

$$A=2\left[\int_0^{\frac{2}{\sqrt{a+1}}}\left(4-x^2\right)dx-\int_0^{\frac{2}{\sqrt{a+1}}}\left(ax^2\right)dx\right]=2\left[\left.\left(4x-\frac{a+1}{3}\,x^3\right)\right|_0^{\frac{2}{\sqrt{a+1}}}\right]=\frac{16}{3}\quad\rightarrow$$

$$\frac{8}{\sqrt{a+1}}-\frac{8}{3\sqrt{a+1}}=\frac{8}{3} \;\rightarrow\; \frac{2}{\sqrt{a+1}}=1 \;\rightarrow\; 2=\sqrt{a+1} \;\rightarrow\; \boxed{a=3}$$

C) Hurrengo integral inpropioak kalkulatu:

$$\int_4^{\infty}\frac{1}{x(\ln x)^2}\,dx \qquad\qquad \int_0^{6}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx$$

(60 p)

Ebazpena

$$\int_4^{\infty}\frac{1}{x(\ln x)^2}\,dx=\lim_{b\to\infty}\int_4^{b}\frac{1}{x(\ln x)^2}\,dx \underset{*}{=} \lim_{b\to\infty}\left[-\frac{1}{\ln b}+\frac{1}{\ln 4}\right]=\frac{1}{\ln 4}$$

$$\int_4^{b}\frac{1}{x(\ln x)^2}\,dx=\left[\ln x=t \quad \frac{1}{x}dx=dt\right]=\int_{\ln 4}^{\ln b}\frac{1}{t^2}\,dt=-\frac{1}{t}\bigg|_{\ln 4}^{\ln b}=-\frac{1}{\ln b}+\frac{1}{\ln 4} \quad (*)$$

$$\int_0^{6}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx=\int_0^{2}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx+\int_2^{6}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx=$$

$$=\lim_{\varepsilon\to 0}\left[\int_0^{2-\varepsilon}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx+\int_{2+\varepsilon}^{6}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx\right]\underset{*}{=}$$

$$=\lim_{\varepsilon\to 0}\left[3\left(\sqrt[3]{(2-\varepsilon)^2-4}-\sqrt[3]{-4}\right)+3\left(\sqrt[3]{32}-\sqrt[3]{(2+\varepsilon)^2-4}\right)\right]=3\left(\sqrt[3]{4}+\sqrt[3]{32}\right)=3\left(\sqrt[3]{4}+2\sqrt[3]{4}\right)=9\sqrt[3]{4}$$

$$\int_0^{2-\varepsilon}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx=3\left(x^2-4\right)^{1/3}\bigg|_0^{2-\varepsilon}=3\left[\sqrt[3]{(2-\varepsilon)^2-4}-\sqrt[3]{-4}\right] \quad (*)$$

$$\int_{2+\varepsilon}^{6} \frac{2x}{\left(x^2-4\right)^{2/3}}\,dx = 3\left(x^2-4\right)^{1/3}\Big|_{2+\varepsilon}^{6} = 3\left[\sqrt[3]{32} - \sqrt[3]{\left(2+\varepsilon\right)^2-4}\right] \quad (*)$$