# KALKULUA

## AZTERKETA PARTZIALA. 2018ko Apirilaren 13an

Kudeaketaren eta Informazio Sistemen Informatikaren Ingeniaritzako Gradua

**1. Ariketa**

Kalkulatu hurrengo integralak:

a) $\displaystyle\int \frac{x\sqrt{1+x^2}}{2+x^2}\,dx$

b) $\displaystyle\int \frac{\sin x \cos x}{1+\sin^4 x}\,dx$

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a) ataleren ebazpena

$$\int \frac{x\sqrt{1+x^2}}{2+x^2}\,dx = \left\{ \begin{array}{l} \sqrt{1+x^2} = t \\ 1+x^2 = t^2 \quad \Rightarrow \quad x\,dx = t\,dt \end{array} \right\} = \int \frac{t}{1+t^2}\,t\,dt = \int \frac{t^2}{1+t^2}\,dt =$$

$$\int \left(1 - \frac{1}{1+t^2}\right) dt = t - \operatorname{arctg} t + C = \sqrt{1+x^2} - \operatorname{arctg}\sqrt{1+x^2} + C$$

b) ataleren ebazpena

$$\int \frac{\operatorname{sen} x \cos x}{1+\operatorname{sen}^4 x}\,dx = \left\{ \begin{array}{l} \operatorname{sen} x = t \\ \cos x\,dx = dt \end{array} \right\} = \int \frac{t\,dt}{1+t^4} = \left\{ \begin{array}{l} t^2 = z \\ 2t\,dt = dz \end{array} \right\} = \frac{1}{2}\int \frac{dz}{1+z^2} =$$

$$= \frac{1}{2}\operatorname{arctg} z + C = \frac{1}{2}\operatorname{arctg}\left(t^2\right) + C = \frac{1}{2}\operatorname{arctg}\left(\operatorname{sen}^2 x\right) + C$$

## 2. Ariketa

Izan bedi $[D]$ hurrengo eran definitutako domeinu laua:

$$D = \left\{ (x, y) \in \mathbb{R}^2 \,/\, x^2 + y^2 - 2y \le 0, \quad y \ge x^2 \right\}$$

**Integral mugatuaren kontzeptua erabiliz,** kalkulatu:

1.- $[D]$ domeinu lauaren azalera

2.- $[D]$ absiza ardatzen inguruan biratzerakoan sortutako bolumena.

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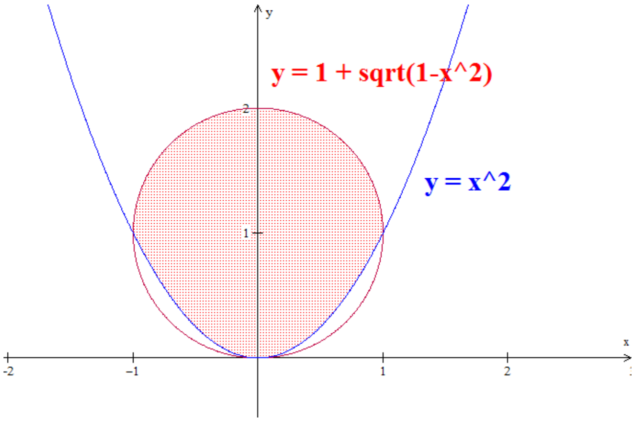
*Ebazpena:*




Ebakidura puntuak kalkulatu egiten dira:

$$\left.\begin{array}{l} x^2 + y^2 - 2y = 0 \\ y = x^2 \end{array}\right\} \Rightarrow \quad (x = 0;\, y = 0) \vee (x = \pm 1;\, y = 1)$$

Irudiari begira esan daiteke kalkulatu beharreko azalera hurrengoa dela:

$$A = 2\left[\int_0^1 1 + \sqrt{1-x^2}\, dx - \int_0^1 x^2\, dx\right] = 2\left[x - \frac{x^3}{3}\right]_0^1 + 2\int_0^1 \sqrt{1-x^2}\, dx = \frac{4}{3} + J = \frac{4}{3} + \frac{\pi}{2} = \frac{8+3\pi}{6} \quad u^2$$

$$J = 2\int_0^1 \sqrt{1-x^2}\, dx = \left\| \begin{array}{l} x = \sin(t) \\ dx = \cos(t)\, dt \\ x = 1 \to t = \frac{\pi}{2} \\ x = 0 \to t = 0 \end{array} \right\| = 2\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2(t)} \cdot \cos(t)\, dt = 2\int_0^{\frac{\pi}{2}} \cos^2(t)\, dt =$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1 + \cos(2t)}{2}\, dt = \left[t + \frac{\sin(2t)}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Sortutako bolumena $[D]$ x ardatzaren inguruan biratzerakoan hurrengoa da:

$$V = 2\pi\int_0^1 \left(1+\sqrt{1-x^2}\right)^2 dx - 2\pi\int_0^1 (x^2)^2\,dx = 2\pi\int_0^1 \left(2-x^2-x^4+2\sqrt{1-x^2}\right)dx =$$

$$= 2\pi\left[2x-\frac{x^3}{3}-\frac{x^5}{5}+x\sqrt{1-x^2}+\operatorname{arcsen} x\right]_0^1 = 2\pi\left[\frac{22}{15}+\frac{\pi}{2}\right] \quad u^3$$

## 3. Ariketa

Alderantzikatu integrazio ordena honako integral honetan:

$$I = \int_0^2 dx \int_0^x f(x,y)\,dy + \int_2^4 dx \int_0^{2+\sqrt{1-(x-3)^2}} f(x,y)\,dy + \int_4^6 dx \int_0^{6-x} f(x,y)\,dy$$

eta kalkulatu integrazio domeinuaren azalera

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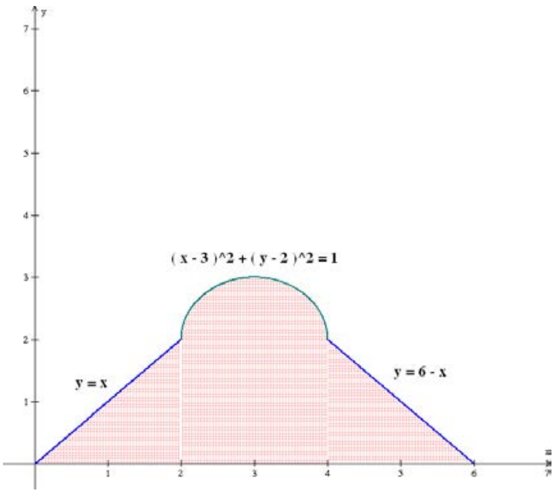
*Ebazpena:*




Integrazio ordena alderantzikatuko dugu. Domeinua bi zatitan deskonposatuko dugu:

$(x-3)^2 + (y-2)^2 = 1 \quad \to \quad (x-3)^2 = 1-(y-2)^2 \quad \to \quad x = 3 \pm \sqrt{1-(y-2)^2}$

$$I = \int_0^2 dy \int_y^{6-y} f(x,y)\,dx + \int_2^3 dy \int_{3-\sqrt{1-(y-2)^2}}^{3+\sqrt{1-(y-2)^2}} f(x,y)\,dx$$

$$I = \int_0^2 dy \int_y^{6-y} dx + \int_2^3 dy \int_{3-\sqrt{1-(y-2)^2}}^{3+\sqrt{1-(y-2)^2}} dx = \int_0^2 (6-2y)\,dy + \int_2^3 2\sqrt{1-(y-2)^2}\,dy =$$

$$= \left[ 6y - \frac{2y^2}{2} \right]_0^2 + J = 8 + \frac{\pi}{2} \quad u^2$$

non J hurrengo eran ebazten dugun:

$$J=\int_2^3 2\sqrt{1-(y-2)^2}\,dy=\left\|\begin{array}{l} y-2=\sin(t) \\ dy=\cos(t)\,dt \\ y=3\to t=\dfrac{\pi}{2} \\ y=2\to t=0 \end{array}\right\|=2\int_0^{\frac{\pi}{2}}\sqrt{1-\sin^2(t)}\cdot\cos(t)\,dt=2\int_0^{\frac{\pi}{2}}\cos^2(t)\,dt=$$

$$=2\int_0^{\frac{\pi}{2}}\frac{1+\cos(2t)}{2}\,dt=\left[t+\frac{\sin(2t)}{2}\right]_0^{\frac{\pi}{2}}=\frac{\pi}{2}$$

## 4. Ariketa

Integral hirukoitzak erabiliz, hurrengo gainazalek mugatutako $[C]$ gorputz homogeneoaren bolumena kalkulatu:

$$x^2+y^2-4z^2=0 \;\; (z\geq 0), \quad x^2+y^2-z^2+18z-81=0 \;\; (z\leq 9)$$

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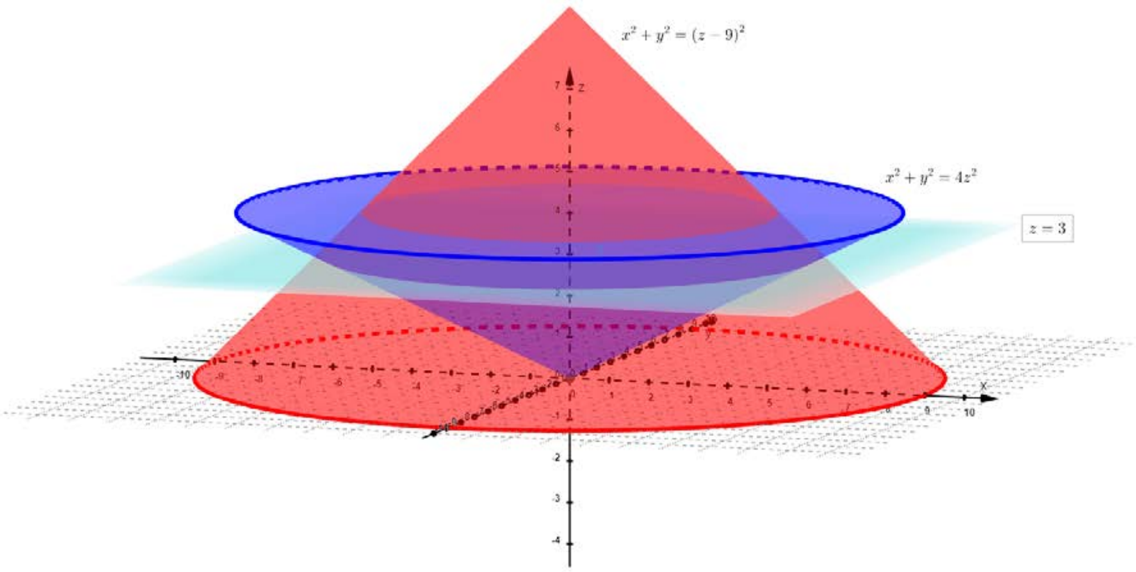
Ebazpena:

Irudikapen grafikoan ikus daitekeenez bi kono ditugu.




Bi konoek mugatutako [C] gorputzaren bolumena, kono urdinetik ( $x^2+y^2-4z^2=0$ ) kono gorrirakoa ( $x^2+y^2=(z-9)^2$ ) da. Bolumen hori kalkulatzeko lehendabizi ebakidura planoa kalkulatu behar da.

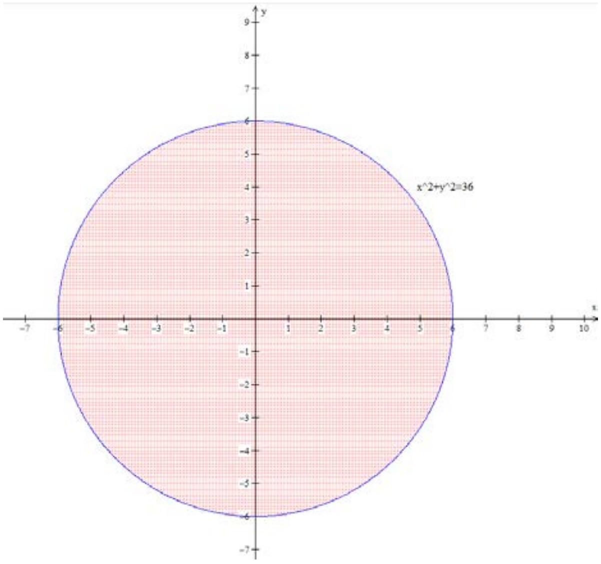
$$\begin{cases} x^2+y^2=4z^2 \\ x^2+y^2=(z-9)^2 \end{cases} \Bigg\{ 4z^2=(z-9)^2 \;\;\Rightarrow\;\; 4z^2=z^2+18z-81 \;\;\Rightarrow\;\; 3z^2-18z+81=0 \;\;\Rightarrow\;\; z^2-6z+27=0$$

$$z^2-6z+27=0 \;\;\Rightarrow\;\; \begin{cases} z=-9 \\ \boxed{z=3} \end{cases}$$

Koordenatu zilindrikoetan ebatziko da ariketa. Beraz, hurrengo aldagai aldaketa aplikatzen da:

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \\ z=z \\ J(\rho,\theta,z)=\rho \end{cases} \quad \begin{cases} x^2+y^2=4z^2 \quad \Rightarrow \quad \rho^2=4z^2 \quad \Rightarrow \quad z=\rho/2 \\ x^2+y^2=(z-9)^2 \quad \Rightarrow \quad \rho^2=(z-9)^2 \quad \Rightarrow \quad z=9-\rho \end{cases}$$

Behin *z*-ren mugak zehaztuta daudela, *XOY* planoaren gaineko proiekzioa egiten dugu eta hurrengoa ikusten da, $x^2+y^2=36$ zirkunferentzia, zentroa *C*(0,0) eta *R*=6.




Ditugun hiru aldagaien mugak orduan hauexek izango dira:

$\theta=[0,2\pi]; \quad \rho=[0,6]; \quad z=[\rho/2, 9-\rho]$

Orduan, bolumena kalkulatzeko hurrengo integral hirukooitza planteatzen dugu:

$$V=\int_0^{2\pi}d\theta\int_0^6\rho\,d\rho\int_{\rho/2}^{9-\rho}dz=\int_0^{2\pi}d\theta\int_0^6\rho(9-\rho-\frac{\rho}{2})d\rho=\int_0^{2\pi}d\theta\int_0^6(9\rho-\frac{3\rho^2}{2})d\rho=$$

$$=\int_0^{2\pi}\left[\frac{9\rho^2}{2}-\frac{\rho^3}{2}\right]_0^6 d\theta=\pi\left[9\cdot6^2-\ 6^3\right]=36\pi\left[9-\ 6\right]=108\pi$$

$$\boxed{V=108\pi \quad u^3}$$