KALKULUA (INDUSTRIALAK)
EZ-OHIKO DEIALDIA. 2016KO EKAINAREN 24A

1. ORRIA (20 puntu)

A) Lortu $z \in \mathbb{C}$ barneko zenbakiak jakinda $z^2 + 2\overline{z}^2 + z - \overline{z} + 9 = 0$

(4 puntu)

Ebazpena

Izan bedi $z = a + bi$. Hurrengo ekuazioa $z^2 + 2\overline{z}^2 + z - \overline{z} + 9 = 0$ transformatzen da:

$$(a+bi)^2 + 2(a-bi)^2 + (a+bi) - (a-bi) + 9 = 0 \quad \to$$

$$\to \quad a^2 - b^2 + 2abi + 2a^2 - 2b^2 - 4abi + 2bi + 9 = 0 \quad \to \quad (3a^2 - 3b^2 + 9) + (-2ab + 2b)i = 0$$

$$\begin{cases} 3a^2 - 3b^2 + 9 = 0 \\ -2ab + 2b = 0 \end{cases} \to \begin{cases} a^2 - b^2 = -3 \\ b(1-a) = 0 \end{cases}$$

- 1. kasua: $b = 0 \quad \to \quad a^2 = -3 \quad (a \notin \mathbb{R})$ zentzugabea
- 2. kasua: $b \neq 0 \quad \to \quad a = 1 \quad \to \quad 1 - b^2 = -3 \quad \to \quad b^2 = 4 \quad \to \quad b = \pm 2$

Beraz, eskatzen ziren zenbaki konplexuak hurrengoak dira: $z_1 = 1 + 2i \quad \wedge \quad z_2 = 1 - 2i$

B) Zehaztu eta adierazi grafikoki hurrengo funtzioaren definizio-eremua:

$$f(x,y) = \frac{\operatorname{arg\,ch}\left(x^2 + y^2 - 3\right)}{\ln\left(x^2 - y^2\right)}$$

(6 p)

Ebazpena

- Kosinu hiperbolikoaren argumentua existitu dadin:

$$x^2 + y^2 - 3 \geq 1 \quad \Rightarrow \quad x^2 + y^2 \geq 4$$

Beraz, jatorrian zentraturik dagoen eta 2 erradiodun zirkunferentziaren kanpoaldea eta zirkunferentziaren puntuak.

- Nepertarra existitu dadin:

$$x^2 - y^2 > 0 \quad \to \quad (x-y)(x+y) > 0 \quad \to \quad \begin{cases} (x - y > 0) \wedge (x + y > 0) \\ (x - y < 0) \wedge (x + y < 0) \end{cases} \to$$

$$\rightarrow \begin{cases} (y < x) \wedge (y > -x) \\ (y > x) \wedge (y < -x) \end{cases}$$

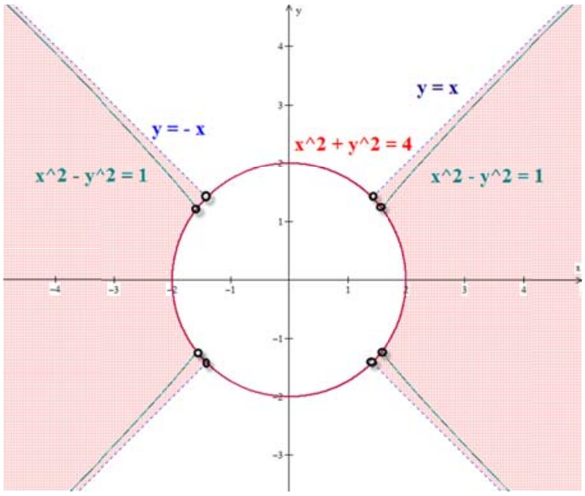
- Izendatzailea zero izan ez dadin:

$$x^2 - y^2 \neq 1$$

Beraz, $x^2 - y^2 = 1$ hiperbolaren puntuak domeinutik kanpo geratzen dira.

Beraz:

$$D = \left\{ (x,y) \in \mathbb{R}^2 \,/\, (x^2 + y^2 \geq 4) \wedge ([(y < x) \wedge (y > -x)] \vee [(y > x) \wedge (y < -x)]) \wedge (x^2 - y^2 \neq 1) \right\}$$



C) Hurrengo funtzioaren mutur erlatiboak zehaztu $f(x,y) = xy - \dfrac{2}{x} - \dfrac{4}{y} + 8$

(4 p)

Ebazpena

Puntu kritikoak:

$$f(x,y) = xy - \frac{2}{x} - \frac{4}{y} + 8 \;\rightarrow\; \begin{cases} \dfrac{\partial f}{\partial x} = 0 \rightarrow y + \dfrac{2}{x^2} = 0 \;\rightarrow\; x^2 y + 2 = 0 \\ \dfrac{\partial f}{\partial y} = 0 \rightarrow x + \dfrac{4}{y^2} = 0 \;\rightarrow\; xy^2 + 4 = 0 \end{cases} \;\rightarrow\; x = -1;\, y = -2$$

Hessiarraren irizpidearen arabera:

$$H[f(x,y)] = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} -\dfrac{4}{x^3} & 1 \\ 1 & -\dfrac{8}{y^3} \end{vmatrix} = \frac{32}{x^3 y^3} - 1$$

$$H\left[f(-1,-2)\right]=3>0; \quad \frac{\partial^2 f}{\partial x^2}(-1,-2)=4>0 \quad \wedge \quad f(-1,-2)=14$$

$M(-1,-2,14)$ puntua **minimo erlatiboa** da.

D) Integrazio-limiteak bi era desberdinetan jarri integral honetan $I=\iint_D f(x,y)\,dx\,dy$, hurrengo $[D]$ domeinuarentzat:

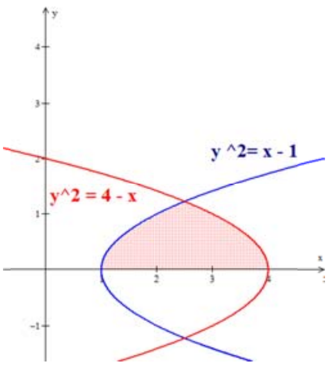
$$D=\left\{(x,y)\in\mathbb{R}^2 \,/\, x\geq 0 \;;\; y\geq 0 \;;\; y^2\leq x-1 \;;\; y^2\leq 4-x\right\}$$

Kalkulatu $[D]$ domeinua x ardatzaren inguruan biratzekoan sorrarazten den bolumena.

(6 p)

Ebazpena

Domeinua bat dator lehenengo koadrantean 0X ardatzeko $y^2=x-1$; $y^2=4-x$ parabolek mugatutako eskualdearekin.

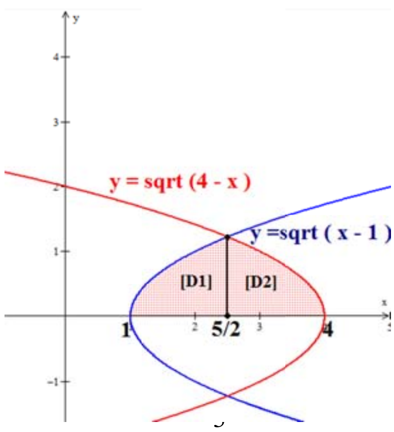


Bi kurba horien mozketa puntua lehenengo koadrantean:

$$\begin{cases} y^2=x-1 \\ y^2=4-x \end{cases} \rightarrow \quad x-1=4-x \quad \rightarrow \quad x=\frac{5}{2} \quad \rightarrow \quad P\left(\frac{5}{2},\sqrt{\frac{3}{2}}\right)$$

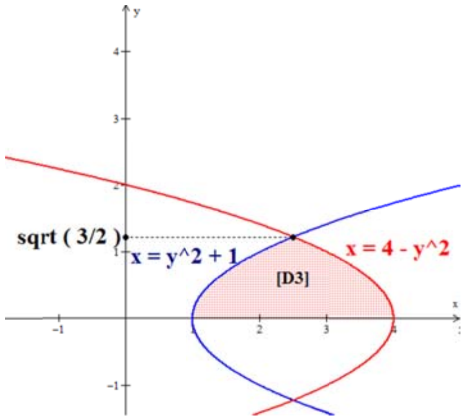
- Lehenengo integrazio aldagaitzat (y) hartuz:

$$I=\int_1^{5/2}dx\int_0^{\sqrt{x-1}}f(x,y)\,dy+\int_{5/2}^{4}dx\int_0^{\sqrt{4-x}}f(x,y)\,dy$$



- Lehenengo integrazio aldagaitzat (x) hartuz:

$$I=\int_0^{\sqrt{\frac{3}{2}}} dy \int_{y^2+1}^{4-y^2} f(x,y)\,dx$$



$[D]$ domeinua x ardatzaren inguruan biratzekoan sorrarazten den bolumena:

$$V=\pi\left[\int_1^{5/2}\left(\sqrt{x-1}\right)^2 dx+\int_{5/2}^{4}\left(\sqrt{4-x}\right)^2 dx\right]=\pi\left[\left.\frac{x^2}{2}-x\right|_1^{5/2}+\left.4x-\frac{x^2}{2}\right|_{5/2}^{4}\right]=\frac{9}{4}\pi \quad u^3$$

Existitzen den simetria kontuan hartuz, bolumena honela geratzen da:

$$V=2\pi\left[\int_1^{5/2}\left(\sqrt{x-1}\right)^2 dx\right]=2\pi\left[\left.\frac{x^2}{2}-x\right|_1^{5/2}\right]=\frac{9}{4}\pi \quad u^3$$

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2. ORRIA (20 puntu)

A) Klasifikatu eta ebatzi hurrengo EDA: $\left(xy-2y^2\right)dx-\left(x^2-3xy\right)dy=0$

(4 p)

Ebazpena

EDA **homogeneoa**:

$$y'=\frac{xy-2y^2}{x^2-3xy} \quad\rightarrow\quad y'=\frac{\dfrac{xy-2y^2}{x^2}}{\dfrac{x^2-3xy}{x^2}}=\frac{\dfrac{y}{x}-2\left(\dfrac{y}{x}\right)^2}{1-3\left(\dfrac{y}{x}\right)}$$

Hurrengo aldaketarekin: $\dfrac{y}{x}=u \quad\Rightarrow\quad y=xu \quad\Rightarrow\quad y'=u+xu'$

$$u+xu'=\frac{u-2u^2}{1-3u} \quad\rightarrow\quad x\frac{du}{dx}=\frac{u-2u^2}{1-3u}-u=\frac{u-2u^2-u+3u^2}{1-3u}=\frac{u^2}{1-3u}$$

Aldagai banangarrien EDA lortzen dugu: $\frac{1-3u}{u^2}du = \frac{1}{x}dx$

Integratuz:

$$-\frac{1}{u} - 3\ln|u| = \ln|x| + C \quad \rightarrow \quad -\frac{x}{y} = \ln|x| + 3\ln\left|\frac{y}{x}\right| + C \quad \rightarrow \quad \boxed{-\frac{x}{y} = \ln\left|\frac{y^3}{x^2}\right| + C}$$

B) Ebatzi hurrengo koefiziente aldakorreko ekuazioa

$$\left(x^2-1\right)y'' - 2x\,y' + 2xy = \left(x^2-1\right)^2$$

jakinda $y_1(x) = x$ ekuazio homogeneoaren soluzio partikularra dela. ***(6 p)***

Ebazpena

$$\left(x^2-1\right)y'' - 2x\,y' + 2xy = \left(x^2-1\right)^2 \quad \rightarrow \quad y'' - \frac{2x}{x^2-1}y' + \frac{2x}{x^2-1}y = x^2-1$$

Elkartutako ekuazio homogeneoaren beste soluzio partikularra (y_2), (y_1)-ekin linealki independentea dena, hurrengo formularen bidez lor daiteke:

$$y_2 = y_1 \int \frac{\exp\left(-\int P(x)dx\right)}{y_1^2}dx$$

$$P(x) = \frac{-2x}{x^2-1} \quad \Rightarrow \quad \int P(x)\,dx = -\int \frac{2x}{x^2-1}\,dx = -\ln\left|x^2-1\right| + cte$$

$$\exp\left(-\int P(x)\,dx\right) = e^{\ln\left|x^2-1\right|} = x^2-1 \quad \Rightarrow$$

$$\int \frac{\exp\left(-\int P(x)dx\right)}{y_1^2}dx = \int \frac{(x^2-1)}{x^2}dx = \int\left(1-\frac{1}{x^2}\right)dx = x + \frac{1}{x} + cte$$

$$y_2 = y_1 \int \frac{\exp\left(-\int P(x)dx\right)}{y_1^2}dx = x\left[x + \frac{1}{x}\right] = x^2+1$$

Beraz, elkartutako ekuazio homogeneoaren soluzio orokorra hurrengoa da:

$$y_h = C_1 \cdot x + C_2(x^2+1)$$

Parametroen aldakuntzaren metodoa aplikatuko dugu:

$$y = L_1(x)\cdot x + L_2(x)\cdot(x^2+1) \qquad [*]$$

$L'_1(x)$ y $L'_2(x)$ hurrengo sistema ebatziz lortuko dira:

$$\begin{cases} L_1' \cdot x + L_2' \cdot (x^2+1) = 0 \\ L_1' \cdot 1 + L_2' \cdot 2x = x^2 - 1 \end{cases}$$

$$L_1'(x) = \frac{\begin{vmatrix} 0 & x^2+1 \\ x^2-1 & 2x \end{vmatrix}}{\begin{vmatrix} x & x^2+1 \\ 1 & 2x \end{vmatrix}} = \frac{-(x^2-1)(x^2+1)}{2x^2-x^2-1} = \frac{-(x^2-1)(x^2+1)}{x^2-1} = -(x^2+1)$$

$$L_2'(x) = \frac{\begin{vmatrix} x & 0 \\ 1 & x^2-1 \end{vmatrix}}{\begin{vmatrix} x & x^2+1 \\ 1 & 2x \end{vmatrix}} = \frac{x(x^2-1)}{x^2-1} = x$$

$$L_1(x) = -\int (x^2+1)\,dx = -\frac{x^3}{3} - x + A \qquad ; \qquad L_2(x) = \int x\,dx = \frac{x^2}{2} + B$$

[*] adierazpenean ordezkatuz, soluzio orokorra lortzen da:

$$\boxed{y} = \left[-\frac{x^3}{3} - x + A\right]\cdot x + \left[\frac{x^2}{2} + B\right]\cdot(x^2+1) = Ax + B(x^2+1) - \frac{x^4}{3} - x^2 + \frac{x^4}{2} + \frac{x^2}{2} =$$

$$= \boxed{Ax + B(x^2+1) + \frac{x^4}{6} - \frac{x^2}{2}}$$

C) Hurrengo EDA ebatzi**:** $y'' + 4y = (t-1)^2 u_1 \quad ; \quad y(0) = y'(0) = 0$

(6 p)

<u>*Ebazpena*</u>

Laplace transformatua aplikatuko da.

Hurrengo propietatea kontuan hartuz $\mathfrak{L}\,[f(t)\cdot u_a] = e^{-pa}\mathfrak{L}\,[f(t+a)]$

$$\mathfrak{L}\,[(t-1)^2 \cdot u_1] = e^{-p}\mathfrak{L}\,[(t+1-1)^2] = e^{-p}\mathfrak{L}\,[t^2] = e^{-p}\cdot\frac{2}{p^3}$$

$$\left[p^2 Y(p) - p y(0) - y'(0)\right] + 4Y(p) = \frac{2e^{-p}}{p^3}$$

$$\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases} \quad \rightarrow \quad (p^2+4)Y(p)=\frac{2e^{-p}}{p^3} \quad \Rightarrow \quad Y(p)=2e^{-p}\cdot\frac{1}{p^3(p^2+4)}$$

$$\frac{1}{p^3(p^2+4)}=\frac{A}{p}+\frac{B}{p^2}+\frac{C}{p^3}+\frac{Dp+E}{p^2+4}$$

$$1\equiv Ap^2(p^2+4)+Bp(p^2+4)+C(p^2+4)+(Dp+E)p^3$$

$$1\equiv (A+D)\,p^4+(B+E)\,p^3+(4A+C)\,p^2+4Bp+4C$$

$$\begin{cases} A+D=0 \\ B+E=0 \\ 4A+C=0 \\ 4B=0 \\ 4C=1 \end{cases} \quad \Rightarrow \quad \begin{cases} A=-1/16 \\ B=0 \\ C=1/4 \\ D=1/16 \\ E=0 \end{cases}$$

$$Y(p)=2e^{-p}\cdot\frac{1}{p^3(p^2+4)}=2e^{-p}\left[\frac{-1/16}{p}+\frac{1/4}{p^3}+\frac{p/16}{p^2+4}\right]$$

Honako propietatea $\mathfrak{L}^{-1}\left[e^{-pa}F(p)\right]=f(t-a)\cdot u_a$ aplikatu eta beraz:

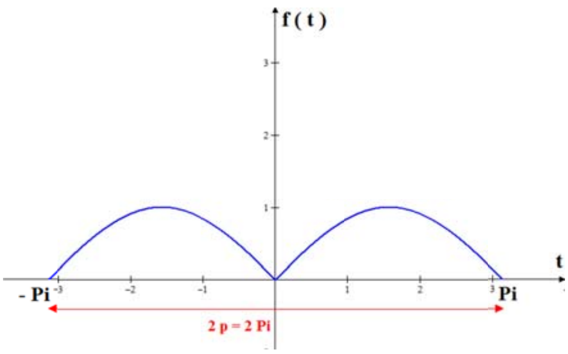
$$y(t)=\mathfrak{L}^{-1}\left[Y(p)\right]=\frac{1}{2}\cdot\mathfrak{L}^{-1}\left(e^{-p}\left[\frac{-\frac{1}{4}}{p}+\frac{1}{p^3}+\frac{\frac{1}{4}p}{p^2+4}\right]\right)=\frac{1}{2}\left(-\frac{1}{4}+\frac{(t-1)^2}{2}+\frac{\cos 2(t-1)}{4}\right)\cdot u_1$$

D) Fourieren serieak erabiliz, garatu hurrengo funtzio periodikoa $f(t)=\left|\operatorname{sen} t\right|$ zeina $-\pi\le t\le\pi$ periodoan deskribaturik dagoen.

(4 p)

Ebazpena

$$f(t)=\left|\operatorname{sen} t\right|=\begin{cases} -\operatorname{sen} t & -\pi<t<0 \\ \operatorname{sen} t & 0<t<\pi \end{cases}$$



Funtzioa bikoitia da**,** eta periodoa $2p = 2\pi \quad \to \quad p = \pi$, horregatik Fourieren bidezko serie garapenak soilik kosinu motako harmonikoak ditu.

$$f(t) = \frac{a_0}{2} + \sum_{1}^{\infty} a_k \cos(kt)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt = \frac{2}{\pi}\int_0^{\pi} \operatorname{sen} t\,dt = \frac{2}{\pi}\left[-\cos t\right]_0^{\pi} = \frac{2}{\pi}\left(-\cos\pi + \cos 0\right) = \frac{4}{\pi}$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(kt)\,dt = \frac{2}{\pi}\int_0^{\pi} \operatorname{sen} t\cdot\cos(kt)\,dt = \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi}\left[\operatorname{sen}(t+kt) + \operatorname{sen}(t-kt)\right]dt =$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\operatorname{sen}(1+k)t + \operatorname{sen}(1-k)t\right]dt = \frac{1}{\pi}\left[-\frac{\cos(1+k)t}{1+k} - \frac{\cos(1-k)t}{1-k}\right]_0^{\pi} =$$

$$= -\frac{1}{\pi}\left[\frac{\cos(1+k)t}{1+k} + \frac{\cos(1-k)t}{1-k}\right]_0^{\pi} = -\frac{1}{\pi}\left[\frac{\cos(1+k)\pi}{1+k} + \frac{\cos(1-k)\pi}{1-k} - \frac{1}{1+k} - \frac{1}{1-k}\right] =$$

$$= \frac{1}{\pi}\left[\frac{\cos k\pi}{1+k} + \frac{\cos k\pi}{1-k} + \frac{1}{1+k} + \frac{1}{1-k}\right] \Rightarrow a_k = \frac{2\cos k\pi + 2}{\pi\left(1-k^2\right)} \qquad k \neq 1$$

a_1 lortzeko integrala bereizi egingo da:

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \operatorname{sen} t\cdot\cos t\,dt = \frac{1}{\pi}\int_0^{\pi} \operatorname{sen} 2t\,dt = \frac{1}{\pi}\left[\frac{-\cos(2t)}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(-\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = 0$$

$$f(t) = \frac{2}{\pi} + \sum_{k=2}^{\infty}\frac{2\cos k\pi + 2}{\pi\left(1-k^2\right)}\cdot\cos(kt) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{k=2}^{\infty}\frac{\cos k\pi + 1}{\left(1-k^2\right)}\cdot\cos(kt)$$

Oharra: Seriea beste era batean adieraz daiteke:

k bakoitia bada:

$$-\frac{1}{\pi}\left[\frac{1}{1+k} + \frac{1}{1-k} - \frac{1}{1+k} - \frac{1}{1-k}\right] = -\frac{1}{\pi}[0] = 0$$

k bikoitia bada:

$$-\frac{1}{\pi}\left[\frac{-1}{1+k} + \frac{-1}{1-k} - \frac{1}{1+k} - \frac{1}{1-k}\right] = -\frac{1}{\pi}\left[-\frac{2}{1+k} - \frac{2}{1-k}\right] = \frac{2}{\pi}\left[\frac{1}{1+k} + \frac{1}{1-k}\right] =$$

$$= \frac{2}{\pi}\left[\frac{1-k+1+k}{1-k^2}\right] = \frac{4}{\pi(1-k^2)}$$

$$a_k = \begin{cases} 0 & si \quad k\ es \text{ impar} \\ \dfrac{4}{\pi(1-k^2)} & si \quad k\ es \text{ par} \end{cases}$$

Beraz, (k) erabili beharrean ($2n$) erabili daiteke eta:

$$f(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{(1-4n^2)\pi} \cdot \cos\left(2nt\right) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(1-4n^2)} \cdot \cos\left(2nt\right)$$