15. ARIKETA

$$\int \frac{dx}{x\sqrt{x^2+x+1}} = \int \frac{dx}{x\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}} = \left\| \begin{matrix} t = x+\frac{1}{2} \\ dt = dx \end{matrix} \right\| = \int \frac{dt}{\left(t-\frac{1}{2}\right)\sqrt{t^2+\frac{3}{4}}} = \left\| \begin{matrix} t=\frac{\sqrt{3}}{2}\tan z \\ dt = \frac{\sqrt{3}}{2\cos^2 z}\,dz \end{matrix} \right\| =$$

$$= \int \frac{\sqrt{3}\;dz}{2\cos^2 z\cdot\left(\frac{\sqrt{3}}{2}\tan z-\frac{1}{2}\right)\cdot\sqrt{\frac{3}{4}\tan^2 z+\frac{3}{4}}} = \cancel{\frac{\sqrt{3}}{2}} \int \frac{dz}{\frac{1}{2}\cos^2 z\left(\sqrt{3}\tan z-1\right)\cdot\cancel{\frac{\sqrt{3}}{2}}\sqrt{\tan^2 z+1}} =$$

$$= 2\int \frac{dz}{\left(\sqrt{3}\tan z-1\right)\cos^2 z\cdot\cancel{\sqrt{\frac{1}{\cos^2 z}}}} = 2\int \frac{dz}{\left(\sqrt{3}\sin z-\cos z\right)} = \left\| \begin{matrix} \tan\frac{z}{2} = w \Rightarrow z = 2\arctan w \Rightarrow dz = \frac{2}{1+w^2}\,dw \\ \sin z = \frac{2w}{1+w^2} \qquad \cos z = \frac{1-w^2}{1+w^2} \end{matrix} \right\|$$

$$= \cancel{2}\int \frac{\cancel{2}\cdot dw}{\cancel{(1+w^2)}\left(\sqrt{3}\cdot\frac{2w}{\cancel{1+w^2}}-\frac{(1-w^2)}{\cancel{1+w^2}}\right)} = \int \frac{4\cdot dw}{w^2+2\sqrt{3}w-1} = 4\int \frac{dw}{(w+\sqrt{3})^2-4} = -4\int \frac{dw}{4-(w+\sqrt{3})^2} =$$

$$= -4\int \frac{dw}{4-(w+\sqrt{3})^2} = -\frac{4}{4}\ln\left|\frac{2+w+\sqrt{3}}{2-w-\sqrt{3}}\right|+C = -\ln\left|\frac{2+w+\sqrt{3}}{2-w-\sqrt{3}}\right|+C = -\ln\left|\frac{2+\sqrt{3}+\tan\frac{z}{2}}{2-\sqrt{3}-\tan\frac{z}{2}}\right|+C$$

$$= -\ln\left|\frac{2+\sqrt{3}+\tan\left(\frac{\arctan\left(\frac{2t}{\sqrt{3}}\right)}{2}\right)}{2-\sqrt{3}-\tan\left(\frac{\arctan\left(\frac{2t}{\sqrt{3}}\right)}{2}\right)}\right|+C = -\ln\left|\frac{2+\sqrt{3}+\tan\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{2}\right)}{2-\sqrt{3}-\tan\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{2}\right)}\right|+C$$