

**KALKULUA (INDUSTRIALAK)
OHIKO DEIALDIA. 2017KO MAIATZAREN 15A**

1. ORRIA (20 puntu)

A) Zein kasutan $z + \frac{1}{z}$ ($z \in \mathbb{C}, z \neq 0$) zenbakia erreala da? Eraitza interpretatu.

(4 puntu)

Ebazpena

Izan bedi $z = x + iy$:

$$z + \frac{1}{z} = x + iy + \frac{1}{x + iy} = x + iy + \frac{x - iy}{(x + iy)(x - iy)} = x + iy + \frac{x - iy}{x^2 + y^2} = x + \frac{x}{x^2 + y^2} + i \left(y - \frac{y}{x^2 + y^2} \right)$$

$z + \frac{1}{z}$ erreala izateko:

$$\operatorname{Im}(z) = 0 \Rightarrow \left(y - \frac{y}{x^2 + y^2} \right) = 0 \Rightarrow y \left(1 - \frac{1}{x^2 + y^2} \right) = 0 \Rightarrow \begin{cases} a) & y = 0 \\ b) & \left(1 - \frac{1}{x^2 + y^2} \right) = 0 \end{cases}$$

a) $y = 0 \Rightarrow z = x + 0i$ (zenbaki erreala) \rightarrow Ardatz erreala, jatorria izan ezik.

b) $\left(1 - \frac{1}{x^2 + y^2} \right) = 0 \Rightarrow x^2 + y^2 = 1 \rightarrow (0,0)$ zentroko eta 1 erradioko zirkunferentzia.

B) $e^{xy-1} + 2x + 4\sqrt{y} = 7$ ekuazioak $y = f(x)$ funtzioa inplizituki definitzen du. $y(1) = 1$ dela jakinda, $y''(1)$ kalkulatu.

(5 puntu)

Ebazpena

(x) -ekiko inplizituki deribatuz: $e^{xy-1}(y + xy') + 2 + \frac{2y'}{\sqrt{y}} = 0$ [1]

$$x = 1 \rightarrow 1(1 + 1 \cdot y'(1)) + 2 + 2y'(1) = 0 \rightarrow 3 + 3y'(1) = 0 \rightarrow \boxed{y'(1) = -1}$$

[1] (x) -ekiko berriro deribatuz:

$$e^{xy-1} \left[(y + xy')^2 + y' + y' + xy'' \right] + 2 \cdot \frac{\sqrt{y} \cdot y'' - \frac{y'^2}{2\sqrt{y}}}{y} = 0$$

$$x=1 \rightarrow 1[0-2+y''(1)] + 2 \left[y''(1) - \frac{1}{2} \right] = 0 \rightarrow -2 + 3y''(1) - 1 = 0 \rightarrow \boxed{y''(1)=1}$$

C) Izan bedi $f(x, y) = x^2 + y^2 + 6x - 8y + 25$ funtzioa, kalkulatu mutur erlatiboak $x^2 + y^2 = 16$ kurban.

(5 puntu)

Ebazpena

Funtzio Lagrangiarra hurrengoa da:

$$L(x, y, \lambda) = x^2 + y^2 + 6x - 8y + 25 + \lambda(x^2 + y^2 - 16)$$

Baldintza hurrengoa da:

$$\begin{cases} \frac{\partial L}{\partial x} = 2x + 6 + \lambda 2x = 0 \\ \frac{\partial L}{\partial y} = 2y - 8 + \lambda 2y = 0 \\ x^2 + y^2 - 16 = 0 \end{cases} \Rightarrow \begin{cases} -\frac{2x+6}{2x} = \lambda \\ -\frac{2y-8}{2y} = \lambda \\ x^2 + y^2 - 16 = 0 \end{cases} \Rightarrow \begin{cases} \frac{x+3}{x} = \frac{y-4}{y} \\ x^2 + y^2 - 16 = 0 \end{cases} \Rightarrow \begin{cases} y = -\frac{4}{3}x \\ x^2 + y^2 - 16 = 0 \end{cases} \Rightarrow$$

$$x^2 + \left(-\frac{4}{3}x\right)^2 - 16 = 0 \Rightarrow \frac{25}{9}x^2 = 16 \Rightarrow x = \pm \frac{12}{5}$$

$$x = \pm \frac{12}{5} \rightarrow y = \mp \frac{16}{5}$$

Orduan, mutur erlatiboak hurrengoak izan daitezke:

- $P_1\left(\frac{12}{5}, -\frac{16}{5}\right)$ con $\lambda = -\frac{9}{4}$
- $P_2\left(-\frac{12}{5}, \frac{16}{5}\right)$ con $\lambda = \frac{1}{4}$

Minimo ala maximo bat den jakiteko, matrize hessiarraren determinantea kalkulatzeko da:

$$HL(\lambda, (x, y)) = \begin{pmatrix} 0 & \phi'_x & \phi'_y \\ \phi'_x & L''_{xx} & L''_{xy} \\ \phi'_y & L''_{yx} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & 2x & 2y \\ 2x & 2+2\lambda & 0 \\ 2y & 0 & 2+2\lambda \end{pmatrix}$$

$$\left| HL\left(-\frac{9}{4}, \left(\frac{12}{5}, -\frac{16}{5}\right)\right) \right| = \begin{vmatrix} 0 & \frac{24}{5} & -\frac{32}{5} \\ \frac{24}{5} & -\frac{5}{2} & 0 \\ -\frac{32}{5} & 0 & -\frac{5}{2} \end{vmatrix} > 0 \rightarrow P_1\left(\frac{12}{5}, -\frac{16}{5}\right) \text{ maximo lokal bat da.}$$

$$\left| HL\left(\frac{1}{4}, \left(-\frac{12}{5}, \frac{16}{5}\right)\right) \right| = \begin{vmatrix} 0 & -\frac{24}{5} & \frac{32}{5} \\ -\frac{24}{5} & \frac{5}{2} & 0 \\ \frac{32}{5} & 0 & \frac{5}{2} \end{vmatrix} < 0 \rightarrow P_2\left(-\frac{12}{5}, \frac{16}{5}\right) \text{ maximo lokal bat da.}$$

D) Kalkulatu, integralak erabiliz, $y = \sqrt{(3-x)(1+x)}$ kurbaren luzera

(6 puntu)

Ebazpena

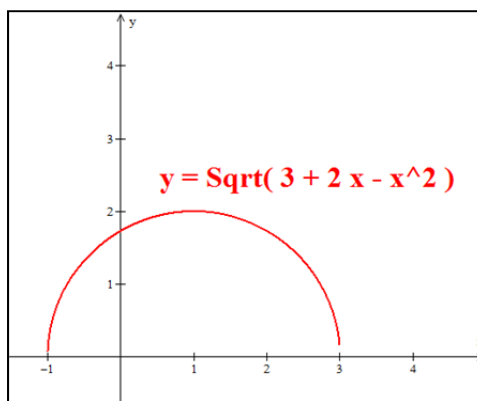
Zirkunferentzierdi bat da. Hain zuzen ere:

$$y = \sqrt{(3-x)(1+x)} \rightarrow y^2 = (3-x)(1+x) = 3 - x + 3x - x^2 = 3 + 2x - x^2$$

$$x^2 + y^2 - 2x = 3 \rightarrow (x-1)^2 + y^2 = 4 \rightarrow (1,0) \text{ zentroko eta 2 erradioko zirkunferentzia.}$$

$$y = \pm \sqrt{4 - (x-1)^2} = \pm \sqrt{3 + 2x - x^2}$$

Orduan, emandako kurba $(1,0)$ zentroko eta 2 erradioko goiko zirkunferentzierdia da.



Luzeraren kalkulua:

$$y' = \frac{-2x+2}{2\sqrt{3+2x-x^2}} = \frac{1-x}{\sqrt{3+2x-x^2}} \rightarrow y'^2 = \frac{(1-x)^2}{3+2x-x^2}$$

$$\boxed{L} = \int_{-1}^3 \sqrt{1+y'^2} dx = \int_{-1}^3 \sqrt{1 + \frac{(1-x)^2}{3+2x-x^2}} dx = \int_{-1}^3 \sqrt{\frac{3+2x-x^2+1-2x+x^2}{3+2x-x^2}} dx =$$

$$= \int_{-1}^3 \sqrt{\frac{4}{3+2x-x^2}} dx = 2 \int_{-1}^3 \frac{dx}{\sqrt{3+2x-x^2}} = 2 \int_{-1}^3 \frac{dx}{\sqrt{4-(x-1)^2}} =$$

$$= 2 \left[\arcsin\left(\frac{x-1}{2}\right) \right]_{-1}^3 = 2 [\arcsin 1 - \arcsin(-1)] = 2 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \boxed{2\pi}$$

Oinarrizko geometria erabiliz (emaitza konprobatzeko): $L = \frac{2\pi r}{2} = \pi r = 2\pi$

2. ORRIA (20 puntu)

A) Izan bedi hurrengo $[D]$ domeinu laua:

$$D = \left\{ (x, y) \in \mathbb{R}^2 / y \geq |2-x| ; (x-2)^2 + y^2 \leq 4 \right\}$$

1.- $I = \iint_{[D]} f(x, y) dx dy$ integralean integrazio-limiteak bi era desberdinetan planteatu.

2.- Integral bikoitzak erabiliz, $[D]$ domeinu lauaren azalera kalkulatu.

(6 puntu)

Ebazpena

Ebakidura puntuak kalkulatu:

Si $x \leq 2$:

$$\begin{cases} (x-2)^2 + y^2 = 4 \\ y = 2-x \end{cases} \Rightarrow \begin{cases} y = \sqrt{4-(x-2)^2} \\ y = 2-x \end{cases} \Rightarrow 2-x = \sqrt{4-(x-2)^2} \Rightarrow$$

$$(2-x)^2 = 4-(x-2)^2 \Rightarrow 2(2-x)^2 = 4 \Rightarrow x = 2 \pm \sqrt{2} \Rightarrow$$

$$\Rightarrow (2-\sqrt{2}, \sqrt{2})$$

Si $x > 2$:

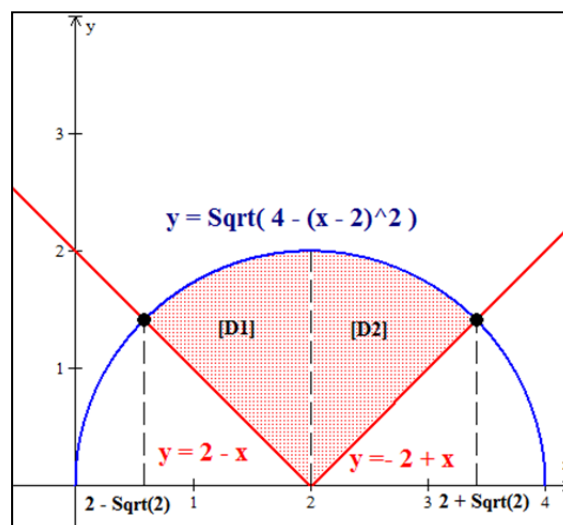
$$\begin{cases} (x-2)^2 + y^2 = 4 \\ y = x-2 \end{cases} \Rightarrow \begin{cases} y = \sqrt{4-(x-2)^2} \\ y = x-2 \end{cases} \Rightarrow x-2 = \sqrt{4-(x-2)^2} \Rightarrow$$

$$(x-2)^2 = 4-(x-2)^2 \Rightarrow 2(x-2)^2 = 4 \Rightarrow x = 2 \pm \sqrt{2} \Rightarrow$$

$$\Rightarrow (2+\sqrt{2}, \sqrt{2})$$

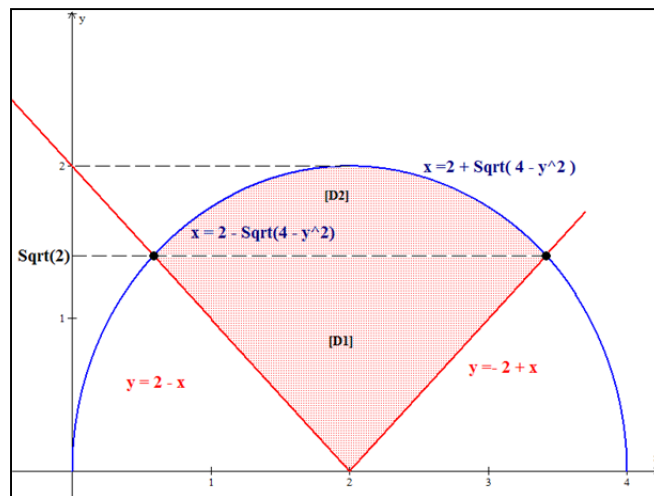
Integralaren limiteak

(y) lehenengo integrazio-aldagaitzat hartuz:



$$\iint_D f(x, y) dx dy = \int_{2-\sqrt{2}}^2 dx \int_{2-x}^{\sqrt{4-(x-2)^2}} f(x, y) dy + \int_2^{2+\sqrt{2}} dx \int_{-2+x}^{\sqrt{4-(x-2)^2}} f(x, y) dy$$

(x) lehenengo integrazio-aldagaitzat hartuz:



$$\iint_D f(x, y) dx dy = \int_0^{\sqrt{2}} dy \int_{2-y}^{y+2} f(x, y) dx + \int_{\sqrt{2}}^2 dy \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} f(x, y) dx$$

Azaleraren kalkulua:

$$A = \int_{2-\sqrt{2}}^2 dx \int_{2-x}^{\sqrt{4-(x-2)^2}} dy + \int_2^{2+\sqrt{2}} dx \int_{x-2}^{\sqrt{4-(x-2)^2}} dy$$

Dagoen simetria kontuan hartuta:

$$\begin{aligned} A &= 2 \int_2^{2+\sqrt{2}} dx \int_{x-2}^{\sqrt{4-(x-2)^2}} dy = 2 \int_2^{2+\sqrt{2}} \left(\sqrt{4-(x-2)^2} - (x-2) \right) dx = \\ &= 2 \left[\int_2^{2+\sqrt{2}} \left(\sqrt{4-(x-2)^2} \right) dx - \left[\frac{x^2}{2} - 2x \right]_2^{2+\sqrt{2}} \right] = 2 \left[I - \frac{1}{2} \left((2+\sqrt{2})^2 - 4(2+\sqrt{2}) - 4 + 8 \right) \right] = 2[I - 1] \end{aligned}$$

$$I = \int_2^{2+\sqrt{2}} \left(\sqrt{4-(x-2)^2} \right) dx = \left[\begin{array}{l} x-2 = 2 \sin t \rightarrow dx = 2 \cos t dt \\ x = 2 \rightarrow t = 0 \\ x = 2 + \sqrt{2} \rightarrow t = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \pi/4 \end{array} \right] = \int_0^{\pi/4} \sqrt{4-4 \sin^2 t} 2 \cos t dt =$$

$$I = \int_0^{\pi/4} 4 \cos^2 t dt = 2 \int_0^{\pi/4} (1 + \cos 2t) dt = 2 \left[t + \frac{\sin 2t}{2} \right]_0^{\pi/4} = 2 \left[\frac{\pi}{4} + \frac{1}{2} \right] = \frac{\pi}{2} + 1$$

Beraz, azalera hurrengoa da:

$$A = 2[I - 1] = 2\left[\frac{\pi}{2} + 1 - 1\right] = \pi \quad u^2$$

B) Ebatzi hurrengo EDA faktore integratzaile bat erabiliz:

$$(x^2 + y^2)\cos x \, dx - \frac{2}{y}(2x \cos x + (x^2 - 2)\sin x) \, dy = 0$$

(5 puntu)

Ebazpena

$$X(x, y) = (x^2 + y^2)\cos x \Rightarrow \frac{\partial X}{\partial y} = 2y \cos x$$

$$Y(x, y) = -\frac{2}{y}(2x \cos x + (x^2 - 2)\sin x) \Rightarrow \frac{\partial Y}{\partial x} = \frac{-2x^2 \cos x}{y}$$

EDA ez da zehatza: $\frac{\partial X}{\partial y} \neq \frac{\partial Y}{\partial x}$

Ezin da $z(x)$ faktore integratzaile bat lortu:

$$\frac{\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}}{Y} = \frac{2y \cos x + \frac{2x^2 \cos x}{y}}{-\frac{2}{y}(2x \cos x + (x^2 - 2)\sin x)} \neq \varphi(x)$$

$z(y)$ faktore integratzaile bat lortu ahal da:

$$\frac{\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}}{X} = \frac{-\frac{2x^2 \cos x}{y} - 2y \cos x}{(x^2 + y^2)\cos x} = \frac{\frac{-2x^2 - 2y^2}{y}}{(x^2 + y^2)} = \frac{-2}{y} = \varphi(y)$$

$$z(y) = e^{\int \frac{-2}{y} dy} = e^{-2 \ln y} = e^{\ln y^{-2}} = y^{-2} = \frac{1}{y^2}$$

EDA bider $z(y)$ biderkatu ondoren zehatza da.

$$\left(\frac{x^2 + y^2}{y^2}\right)\cos x \, dx - \frac{2}{y^3}(2x \cos x + (x^2 - 2)\sin x) \, dy = 0$$

$$X'(x, y) = \left(\frac{x^2 + y^2}{y^2} \right) \cos x \Rightarrow \frac{\partial X'}{\partial y} = -\frac{2x^2}{y^3} \cos x$$

$$Y'(x, y) = -\frac{2}{y^3} (2x \cos x + (x^2 - 2) \sin x) \Rightarrow \frac{\partial Y'}{\partial x} = -\frac{2x^2 \cos x}{y^3} \Rightarrow \frac{\partial X'}{\partial y} = \frac{\partial Y'}{\partial x}$$

Soluzio orokorra hurrengoa da:

$$\int_a^x X'(x, y) dx + \int_b^y Y'(a, y) dy = C$$

(a,b)=(0,1) hartuz:

$$\int_0^x \left(\frac{x^2 + y^2}{y^2} \right) \cos x dx + \int_1^y 0 dy = C \rightarrow \int_0^x \left(\frac{x^2 + y^2}{y^2} \right) \cos x dx = C$$

$$\frac{1}{y^2} \int_0^x (x^2 + y^2) \cos x dx = \left[\begin{array}{ll} u = x^2 + y^2 & du = 2x dx \\ dv = \cos x dx & v = \sin x \end{array} \right] = \frac{1}{y^2} \left[\left[(x^2 + y^2) \sin x \right]_0^x - \int_0^x 2x \sin x dx \right] =$$

$$= \left[\begin{array}{ll} u = 2x & du = 2 dx \\ dv = \sin x dx & v = -\cos x \end{array} \right] = \frac{1}{y^2} \left[(x^2 + y^2) \sin x - \left([-2x \cos x]_0^x + \int_0^x 2 \cos x dx \right) \right] =$$

$$= \frac{1}{y^2} \left[(x^2 + y^2) \sin x + 2x \cos x - 2 \sin x \right]$$

Beraz, soluzio orokorra hurrengoa da:

$$\frac{1}{y^2} \left[(x^2 + y^2) \sin x + 2x \cos x - 2 \sin x \right] = C \Rightarrow Cy^2 = (x^2 + y^2 - 2) \sin x + 2x \cos x$$

C) $x(2x+3)y'' - 6(x+1)y' + 6y = 0$ ekuazio diferentzialaren soluzio orokorra lortu, $y = x^3$ ekuazioaren soluzio partikular bat dela jakinda.

(4 puntu)

Ebazpena

Beste soluzioa hurrengoa formula erabiliz lortzen da:

$$y_2 = y_1 \int \frac{\exp\left(-\int P(x)dx\right)}{y_1^2} dx, \text{ non } P(x) = \frac{-6(x+1)}{x(2x+3)}$$

Zatiki sinpleetan deskonposatuz:

$$\frac{-6(x+1)}{x(2x+3)} = \frac{A}{x} + \frac{B}{2x+3}$$

$$-6(x+1) = A(2x+3) + Bx$$

$$\begin{cases} -6 = 2A + B \\ -6 = 3A \end{cases} \Rightarrow A = -2; B = -2$$

Integratuz:

$$\exp\left[-\int P(x)dx\right] = \exp\left[\int\left(\frac{2}{x} + \frac{2}{2x+3}\right)dx\right] = \exp\left[\ln x^2(2x+3)\right] = x^2(2x+3)$$

eta ordezkatzuz:

$$y_2 = x^3 \left[\int \frac{(2x+3)}{x^4} dx \right] = x^3 \left[\int \left(\frac{2}{x^3} + \frac{3}{x^4} \right) dx \right] = x^3 \left[-x^{-2} - x^{-3} \right] = -(x+1)$$

Soluzio orokorra lortzen da:

$$y = C_1 x^3 + C_2 (x+1)$$

D) Ebatzi hurrengo ekuazio diferentziala Laplace-ren transformatua erabiliz.

$$y'' - 2y' + 2y = 6e^{-t} \cos t \quad y(0) = y'(0) = 0$$

(5 puntu)

Ebazpena

$$y'' - 2y' + 2y = 6e^{-t} \cos t \quad y(0) = y'(0) = 0$$

$$L[y'' - 2y' + 2y] = p^2 Y(p) - p y(0) - y'(0) - 2[pY(p) - y(0)] + 2Y(p)$$

$$\wedge \quad L[6e^{-t} \cos t] = \frac{6(p+1)}{(p+1)^2 + 1}, \quad \text{non } y(0) = y'(0) = 0$$

$$(p^2 - 2p + 2)Y(p) = \frac{6(p+1)}{(p+1)^2 + 1} \Rightarrow Y(p) = \frac{6(p+1)}{((p-1)^2 + 1)((p+1)^2 + 1)}$$

Deskonposatuz:

$$\frac{6(p+1)}{((p-1)^2 + 1)((p+1)^2 + 1)} = \frac{ap+b}{(p-1)^2 + 1} + \frac{cp+d}{(p+1)^2 + 1}$$

$$6(p+1) = (ap+b)((p+1)^2 + 1) + (cp+d)((p-1)^2 + 1)$$

$$6p + 6 = (a+c)p^3 + (2a+b-2c+d)p^2 + (2a+2b+2c-2d)p + (2b+2d)$$

$$\begin{cases} 0 = a+c \\ 0 = 2a+b-2c+d \\ 6 = 2a+2b+2c-2d \\ 6 = 2b+2d \end{cases} \Rightarrow \begin{cases} c = -a \\ 0 = 4a+b+d \\ 3 = b-d \\ 3 = b+d \end{cases} \Rightarrow \begin{cases} a = -\frac{3}{4} \\ b = 3 \\ c = \frac{3}{4} \\ d = 0 \end{cases}$$

Beraz:

$$Y(p) = \frac{-\frac{3}{4}p+3}{(p-1)^2 + 1} + \frac{\frac{3}{4}p}{(p+1)^2 + 1} = -\frac{3}{4} \left(\frac{p-4}{(p-1)^2 + 1} \right) + \frac{3}{4} \left(\frac{p}{(p+1)^2 + 1} \right)$$

$$Y(p) = -\frac{3}{4} \left(\frac{p-1-3}{(p-1)^2 + 1} \right) + \frac{3}{4} \left(\frac{p+1-1}{(p+1)^2 + 1} \right)$$

$$Y(p) = -\frac{3}{4} \left(\frac{p-1}{(p-1)^2 + 1} \right) + \frac{9}{4} \left(\frac{1}{(p-1)^2 + 1} \right) + \frac{3}{4} \left(\frac{p+1}{(p+1)^2 + 1} \right) - \frac{3}{4} \left(\frac{1}{(p+1)^2 + 1} \right)$$

Alderantziko transformatua kalkulatzuz:

$$y(t) = -\frac{3}{4}(e^t \cos t) + \frac{9}{4}(e^t \sin t) + \frac{3}{4}(e^{-t} \cos t) - \frac{3}{4}(e^{-t} \sin t)$$

$$y(t) = -\frac{3}{2} \left(\frac{e^t - e^{-t}}{2} \right) \cos t + \frac{3}{2} \left(\frac{e^t - e^{-t}}{2} \right) \sin t + \frac{3}{2}(e^t \sin t)$$

$$y(t) = -\frac{3}{2} \operatorname{sh} t \cdot \cos t + \frac{3}{2} \operatorname{sh} t \cdot \sin t + \frac{3}{2}(e^t \sin t)$$

OHAR OROKORRAK:

MATEMATIKA APLIKATUA

- Iraupena: HIRU ORDU ETA ERDI.
- Azterketako zati bakoitza (ORRIA) banan-banan entregatuko da.
- Apunteetako BOST ORRI (ariketarik gabe) erabil daitezke.
- Azterketa egiteko nortasun agiria aurkeztu behar da.
- Notak GAUR plataforman maiatzaren 25an argitaratuko dira eta bertan berrikusteko data eta lekua adieraziko dira.
- Esandako berrikustera joateko, azterketa ebatzia eramatea beharrezkoa da. Ebazpena egela plataforman egongo da.