1. Proba / 1. ariketa

Izan bedi A matrizea sistema homogeneo baten koefiziente matrizea

$$A=\begin{pmatrix} 2 & 0 & 2 \\ 2 & 1+p & 2 \\ 0 & p & 3 \end{pmatrix}$$

a) Sailkatu sistema $p\in\mathbb{R}$ parametroaren balioen arabera. (2 puntu)

Sistema homogeneoa denez, bateragarria izango da $h(A)=h(A^*)$

$$|A|=\begin{vmatrix} 2 & 0 & 2 \\ 2 & 1+p & 2 \\ 0 & p & 3 \end{vmatrix} \underset{\sim}{E_2-E_1} \begin{vmatrix} 2 & 0 & 2 \\ 0 & 1+p & 0 \\ 0 & p & 3 \end{vmatrix} = 2(-1)^{1+1}\begin{vmatrix} 1+p & 0 \\ p & 3 \end{vmatrix} =$$

$$= 2(1+p)\cdot 3 = 6(1+p)$$

$\boxed{p=-1} \to h(A)=2=h(A^*) \to \boxed{S.B.I}$

$\boxed{p\neq -1} \to h(A)=h(A^*)=3 \to \boxed{S.B.D.}$

b) Kalkulatu A^{-1} Gauss-en metodoa erabiliz, $p=1$ denean (2 puntu)

$$A=\left(\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \underset{\sim}{E_2-E_1} \left(\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\underset{\sim}{E_3-E_2/2} \left(\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1/2 & -1/2 & 1 \end{array}\right) \underset{\substack{E_2/2 \\ E_3/3}}{\overset{E_1/2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/6 & -1/6 & 1/3 \end{array}\right) \sim$$

$$\underset{\sim}{E_1-E_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/6 & -1/3 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/6 & -1/6 & 1/3 \end{array}\right)$$

$$\underbrace{\begin{pmatrix} 1/3 & 1/6 & -1/3 \\ -1/2 & 1/2 & 0 \\ 1/6 & -1/6 & 1/3 \end{pmatrix}}_{A^{-1}}$$

c) Kalkulatu $\mathrm{Det}(4A^{-2})$ $p=0$ denean (puntu 1)

$$|A|\Big|_{p=0}=6\;;\quad |4A^{-2}|=4^3|A^{-2}|=4^3|A^{-1}|^2=4^3\cdot\frac{1}{|A|^2}=4^3\cdot\frac{1}{36}=\frac{64}{36}=\boxed{\frac{16}{9}}$$

$\boxed{\text{1. Proba / 2. ariketa}}$

Izan bedi $\mathbb{P}_2(x)$-ren ondoko azpiespazioa:

$$V = \{ p(x) = ax^2 + bx + c \in \mathbb{P}_2(x) \ / \ 2a - 3b + c = 0 \}$$

a) Frogatu azpiespazio bat dela (1'5 puntu)

$$p_1(x) \in V \rightarrow p_1(x) = a_1 x^2 + b_1 x + (3b_1 - 2a_1)$$

$$p_2(x) \in V \rightarrow p_2(x) = a_2 x^2 + b_2 x + (3b_2 - 2a_2)$$

$$\alpha p_1(x) + \beta p_2(x) \in V ?$$

$$\alpha p_1(x) + \beta p_2(x) = \alpha(a_1 x^2 + b_1 x + 3b_1 - 2a_1) + \beta(a_2 x^2 + b_2 x + 3b_2 - 2a_2) =$$

$$= \underbrace{\underbrace{(\alpha a_1 + \beta a_2)}_{A} x^2 + \underbrace{(\alpha b_1 + \beta b_2)}_{B} x + \underbrace{(3\alpha b_1 - 2\alpha a_1 + 3\beta b_2 - 2\beta a_2)}_{3B - 2A}}_{\in V \ \checkmark}$$

$\rightarrow$ Bai, azpiespazioa da.

b) Lortu V-ren dimentsioa eta oinarri bat (1'5 puntu)

$$p(x) = ax^2 + bx + 3b - 2a = a(x^2 - 2) + b(x + 3)$$

$$\boxed{B_V = \{ x^2 - 2, \ x + 3 \}} \ \boxed{\dim(V) = 2}$$

l.i. dira.

c) $r(x) = 3x^2 - x - 9 \in V$? Egia bada, lortu $r(x)$ polinomioaren koordenatuak aurreko atalean lortutako oinarrian (puntu 1)

$$3x^2 - x - 9 = \alpha_1 (x^2 - 2) + \alpha_2 (x + 3)$$

$$3x^2 - x - 9 = \alpha_1 x^2 - 2\alpha_1 + \alpha_2 x + 3\alpha_2$$

x^2 koef. $\rightarrow$ $\boxed{3 = \alpha_1}$ koordenatuak

x koef. $\rightarrow$ $\boxed{-1 = \alpha_2}$

g.c. $\rightarrow$ $-9 = -2\alpha_1 + 3\alpha_2 \rightarrow -9 = -2(3) + 3(-1) \ \checkmark$ $\boxed{\text{Bai, } \in V}$

c) Osatu b) atalean lortutako oinarria $\mathbb{P}_2(x)$-ren oinarri bat lortu arte (puntu 1)

$$B_V = \{x^2-2,\ x+3\}$$

B_V-ri x^2 polinomioa gehituko diogu:

$$B = \underbrace{\{x^2-2,\ x+3,\ x^2\}}_{\text{l.i. dira?}}$$

$$\Downarrow$$

$$\begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{vmatrix} = 2 \neq 0 \Rightarrow \text{Bai, l.i. dira}$$

Beraz:

$$\boxed{B_{\mathbb{P}_2(x)} = \{x^2-2,\ x+3,\ x^2\}} \quad (\dim = 3)$$

2. Zatia / 1. ariketa

Kalkulatu, Karratu txikien metodoa erabiliz, ondoko sistemaren soluzio hurbildua eta egindako errorea lortu: (5 puntu)

$$\begin{cases} 2x+y=1 \\ -x+y=2 \\ x+y=0 \end{cases}$$

$$AM=\left(\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{array}\right) \Rightarrow |AM| = 0+4-1-1-4=-2\neq 0 \rightarrow \underbrace{h(AM)\neq h(A)}_{\text{sistema bateraezina da.}}$$

$\bar{a}_1=\{(2,-1,1)\}$; $\bar{a}_2=\{(1,1,1)\}$; $\bar{b}=\{(1,2,0)\}$

$S=L\{\bar{a}_1,\bar{a}_2\} \rightarrow \bar{b}\notin S$

l.i. dira, beraz S-ren oinarria bat $\rightarrow B_S=\{\bar{a}_1,\bar{a}_2\}$

B_S oinarria ortogonala da?

$$\langle \bar{a}_1,\bar{a}_2\rangle = 2-1+1=2\neq 0 \Longrightarrow \text{Ez}$$

Gram-Schmidt-en metodoa aplikatuko dugu oinarri ortogonal bat lortzeko:

$$\bar{u}_1=\bar{a}_1=(2,-1,1)$$

$$\bar{u}_2=\bar{a}_2-\frac{\langle \bar{a}_2,\bar{u}_1\rangle}{\|\bar{u}_1\|^2}\cdot\bar{u}_1=(1,1,1)-\frac{2}{4+1+1}(2,-1,1)=$$

$$=(1,1,1)-\frac{1}{3}(2,-1,1)=(1/3,4/3,2/3)\xrightarrow{\text{bider } 3}(1,4,2)$$

oinarri ortogonala:

$$\boxed{B_O=\{\underbrace{(2,-1,1)}_{\bar{u}_1},\underbrace{(1,4,2)}_{\bar{u}_2}\}}$$

$\bar{b}$ bektorearen hurbilketarik onena S azpiespazioan:

$$\boxed{\bar{b}'} = \frac{\langle \bar{b}, \bar{u}_1 \rangle}{\|\bar{u}_1\|^2} \cdot \bar{u}_1 + \frac{\langle \bar{b}, \bar{u}_2 \rangle}{\|\bar{u}_2\|^2} \cdot \bar{u}_2 = 0 + \frac{1+8}{1+16+4}(1,4,2) =$$

$$= \frac{9}{21}(1,4,2) = \boxed{(3/7,\ 12/7,\ 6/7)}$$

Sistemaren soluzio hurbildua:

$$\left\{\begin{array}{l} 2x+y = 3/7 \quad ① \\ \left.\begin{array}{l} -x+y = 12/7 \\ x+y = 6/7 \end{array}\right\} \longrightarrow 2y = \frac{18}{7} \Rightarrow \boxed{y = 9/7} \end{array}\right.$$

$$\boxed{x = \frac{6}{7} - y = \frac{6}{7} - \frac{9}{7} = -\frac{3}{7}}$$

① $2(-3/7) + 9/7 = 3/7$ ✓

Egindako errorea:

$$\|\bar{b} - \bar{b}'\| = \sqrt{(1-3/7)^2 + (2-12/7)^2 + (0-6/7)^2} = \boxed{\frac{\sqrt{56}}{7}}$$

$\boxed{\text{2. Zatia / 2. ariketa}}$

Lortu $A \in M_{3\times 3}(\mathbb{R})$ matrizea, ondokoa jakinda:

- $\lambda_1 = 2$ balio propioari elkartutako azpiespazioa ondokoa da:

$$V(2) = L\{(1,1,1)\}$$

- $\lambda_2 = -1$ balio propioari elkartutako azpiespazioa ondokoa da:

$$V(-1) = \{(x,y,z) \in \mathbb{R}^3 / z = -x-y\}$$

(5 puntu)

$\lambda_1 = 2$ balio propioari elkartutako azpiespazioaren dimentsioa $\boxed{d_1 = 1}$ da.

$\lambda_2 = -1$ balio propioari elkartutako azpiespazioa:

$$(x,y,-x-y) = x(1,0,-1) + y(0,1,-1)$$

$$V(-1) = L\{(1,0,-1),(0,1,-1)\} \rightarrow \boxed{d_2 = 2}, \text{ beraz}$$

$(d_i \leq k_i)$ $\boxed{k_2 = 2}$ eta $\boxed{k_1 = 1}$

$$\left.\begin{array}{l} k_1 + k_2 = n \Rightarrow 1+2 = 3 \checkmark \\ k_1 = d_1 \rightarrow 1 = 1 \checkmark \\ k_2 = d_2 \rightarrow 2 = 2 \checkmark \end{array}\right\} \boxed{\text{A matrizea diagonalizagarria da.}}$$

$$\Downarrow$$

$$A = P \cdot D \cdot P^{-1}$$

$$\boxed{D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; \quad P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}}$$

P^{-1} Kalkulatuko dugu:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \overset{E_2-E_1}{\underset{E_3-E_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{E_3+2E_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -3 & 1 & -2 & 1 \end{array}\right) \overset{E_3(-1/3)}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1/3 & +2/3 & -1/3 \end{array}\right) \sim$$

$$\overset{E_2-E_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & -1/3 & 2/3 & -1/3 \end{array}\right) \overset{E_2(-1)}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & +2/3 & -1/3 & -1/3 \\ 0 & 0 & 1 & -1/3 & 2/3 & -1/3 \end{array}\right) \sim$$

$$\overset{E_1-E_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & 0 & 1 & -1/3 & 2/3 & -1/3 \end{array}\right)$$

$$\underbrace{\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \end{pmatrix}}_{= P^{-1}}$$

Beraz:

$$\boxed{A = P.D.P^{-1}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \end{pmatrix} =$$

$$= \boxed{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}$$