KALKULUA (INDUSTRIALAK)
OHIKO DEIALDIA. 2016KO MAIATZAREN 31

1. ORRIA (20 puntu)

A) Adierazi era binomikoan hurrengo zenbaki konplexua: $z = \dfrac{i^{1121}}{1+\sqrt{3}\,i}$. Kalkulatu $\ln z$

(4 p)

Ebazpena:

$$z = \frac{i^{1121}}{1+\sqrt{3}\,i} = \frac{i^{(4\bullet 280+1)}}{1+\sqrt{3}\,i} = \frac{i}{1+\sqrt{3}\,i} = \frac{i(1-\sqrt{3}\,i)}{(1+\sqrt{3}\,i)\ (1-\sqrt{3}\,i)} = \frac{\sqrt{3}+i}{1+3} = \frac{\sqrt{3}}{4} + \frac{1}{4}i$$

$$\begin{cases} |z| = \sqrt{\left(\dfrac{\sqrt{3}}{4}\right)^2 + \left(\dfrac{1}{4}\right)^2} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2} \\ \varphi = \operatorname{arctg}\left(\dfrac{1/4}{\sqrt{3}/4}\right) = \operatorname{arctg}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6} \end{cases} \quad \to \quad z = \frac{1}{2}e^{i\,[(\pi/6)+2k\pi]} = \left(\frac{1}{2}\right)_{\frac{\pi}{6}}$$

$$\ln z = \ln\left(\frac{i^{1121}}{1+\sqrt{3}\,i}\right) = \ln\left[\frac{1}{2}e^{i\,[(\pi/6)+2k\pi]}\right] = \ln\frac{1}{2} + i\left[\frac{\pi}{6}+2k\pi\right] = -\ln 2 + i\left[\frac{\pi}{6}+2k\pi\right]$$

B) Hurrengo funtzioaren definizio eremua kalkulatu eta grafikoki adierazi:

$$f\left(x,y\right) = \frac{\arccos\left(x^2+y^2-5\right)}{\ln\left(e^x-y\right)}$$

(6 p)

Ebazpena:

- Arku-kosinuaren baldintza:

$$-1 \le x^2+y^2-5 \le 1 \quad \Rightarrow \quad 4 \le x^2+y^2 \le 6$$

Jatorrian zentratuak eta 2 eta $\sqrt{6}$ erradioko zirkunferentzien artean dagoen koroa zirkularra da.

- Logaritmo nepertarra existitzeko: $e^x - y > 0 \quad \rightarrow \quad y < e^x$

Beraz, $y = e^x$ funtzioaren azpian dagoen eskualdea.

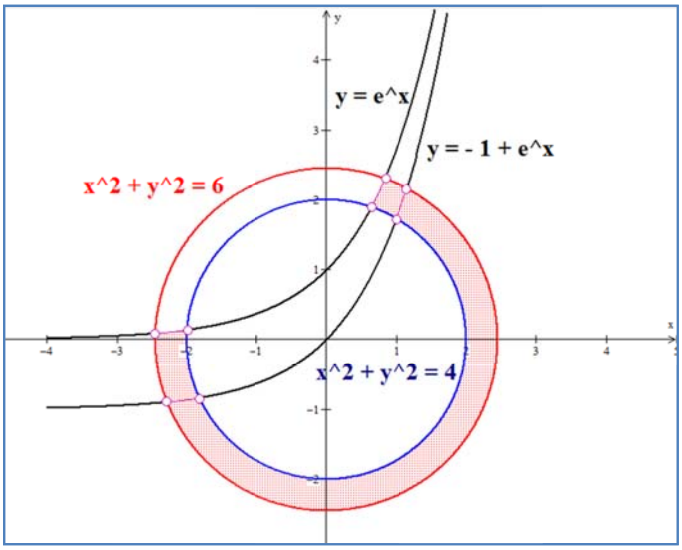
- Izendatzailea ez-nulua izateko:

$$e^x - y \neq 1 \quad \Rightarrow \quad y \neq e^x - 1$$

$y = e^x - 1$ domeinuaren kanpoan dago

Beraz:

$$D = \left\{ (x,y) \in \mathbb{R}^2 \, / \, (4 \leq x^2 + y^2 \leq 6) \wedge (y < e^x) \wedge (y \neq e^x - 1) \right\}$$



C) Izan bedi $z = e^{u \cdot v + u - v}$ funtzioa, non $\begin{cases} u = x^2 - y^2 \\ v = \dfrac{1}{e^{x-y}} \end{cases}$ diren, kalkulatu: $\dfrac{\partial z}{\partial x}(0,0)$, $\dfrac{\partial z}{\partial y}(0,0)$

(4 p)

<u>*Ebazpena*</u>

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = (v+1) \cdot e^{u \cdot v + u - v} \cdot 2x + (u-1) \cdot e^{u \cdot v + u - v} \cdot (-e^{y-x})$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = (v+1) \cdot e^{u \cdot v + u - v} \cdot (-2y) + (u-1) \cdot e^{u \cdot v + u - v} \cdot e^{y-x}$$

Para $\begin{cases} x = 0 \\ y = 0 \end{cases} \rightarrow \begin{cases} u = 0 \\ v = 1 \end{cases}$ orduan:

$$\frac{\partial z}{\partial x}(0,0) = 2 \cdot e^{-1} \cdot 0 + (-1) \cdot e^{-1} \cdot (-1) = \frac{1}{e}$$

$$\frac{\partial z}{\partial y}(0,0) = 2\cdot e^{-1}\cdot 0 + (-1)\cdot e^{-1}\cdot(1) = -\frac{1}{e}$$

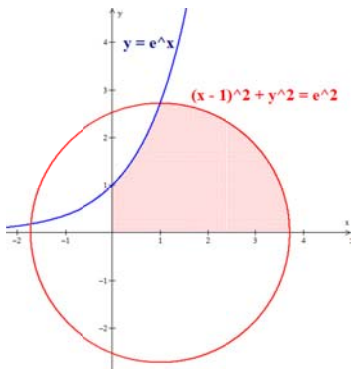
D) Jarri integrazio-limiteak bi era desberdinetan $I = \iint_D f(x,y)\,dx\,dy$ integralean, hurrengo $[D]$ eremurako:

$$D = \left\{(x,y)\in\mathbb{R}^2 \,/\, x\geq 0 \;;\; y\geq 0 \;;\; y\leq e^x \;;\; (x-1)^2+y^2\leq e^2\right\}$$

$[D]$ eremu lauaren azalera kalkulatu.

(6 p)

<u>*Ebazpena*</u>



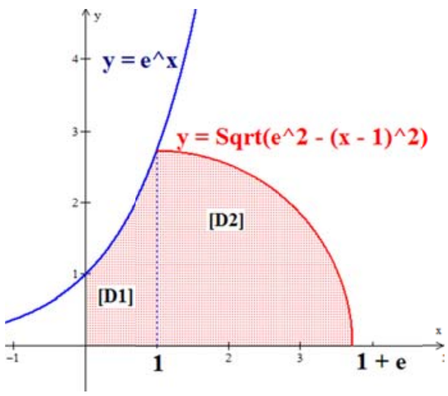
Domeinua lehenengo koadrantean dago. $y = e^x$ funtzioa eta $(x-1)^2 + y^2 = e^2$ zirkunferentzia (zentroa: (1,0); erradioa: e) domeinuaren mugak dira.

Ebakidura puntua:

$$\begin{cases} y = e^x \\ (x-1)^2 + y^2 = e^2 \end{cases} \rightarrow \quad (x-1)^2 + e^2 = e^2 \quad \rightarrow \quad x = 1 \quad P(1,e)$$

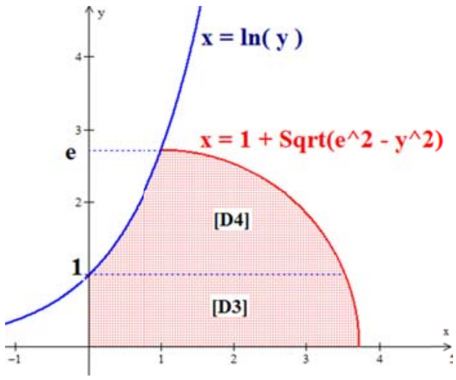
- (y) lehenengo integrazio-aldagaitzat hartuz:

$$I = \int_0^1 dx \int_0^{e^x} f(x,y)\,dy + \int_1^{1+e} dx \int_0^{\sqrt{e^2-(x-1)^2}} f(x,y)\,dy$$



- (x) lehenengo integrazio-aldagaitzat hartuz:

$$I=\int_0^1 dy\int_0^{1+\sqrt{e^2-y^2}} f(x,y)\,dx+\int_1^e dy\int_{\ln y}^{1+\sqrt{e^2-y^2}} f(x,y)\,dx$$



$[D]$ domeinuaren azalera $[D_1]$ domeinuaren azalera gehi $[D_2]$ domeinuaren azalera da:

$$A_T=A_1+A_2=\int_0^1 e^x dx+\frac{1}{4}\pi e^2=\left[e^x\right]_0^1+\frac{\pi e^2}{4}=e-1+\frac{\pi e^2}{4}=\frac{\pi e^2+4e-4}{4}\quad u^2$$

2. ORRIA (20 puntu)

A) Klasifikatu eta ebatzi hurrengo EDA: $\left(e^x+\ln y+\dfrac{y}{x}\right)dx+\left(\dfrac{x}{y}+\ln x+\operatorname{sen} y\right)dy=0$

(4 p)

Ebazpena

$$\begin{cases} X(x,y)=e^x+\ln y+\dfrac{y}{x} \\ Y(x,y)=\dfrac{x}{y}+\ln x+\operatorname{sen} y \end{cases}$$

$$\frac{\partial X}{\partial y}=\frac{1}{y}+\frac{1}{x}=\frac{\partial Y}{\partial x}$$

Beraz, EDA **zehatza** da.

Soluzio orokorra hurrengoa da: $\displaystyle\int_a^x\left(e^x+\ln y+\frac{y}{x}\right)dx+\int_b^y\left(\frac{a}{y}+\ln a+\operatorname{sen} y\right)dy=C$

Kalkulua sinplifikatzeko $a=1$; $b=1$ aukeratzen da:

$$\int_1^x \left(e^x + \ln y + \frac{y}{x} \right) dx + \int_1^y \left(\frac{1}{y} + \operatorname{sen} y \right) dy = C$$

$$\left[e^x + x\ln y + y\ln x \right]_1^x + \left[\ln y - \cos y \right]_1^y = C$$

$$e^x + x\ln y + y\ln x - \left(e + \ln y \right) + \ln y - \cos y - \left(-\cos 1 \right) =$$

$$= e^x + x\ln y + y\ln x - e - \cos y + \cos 1 = C$$

Beraz, soluzio orokorra hurrengoa da:

$$\boxed{e^x + x\ln y + y\ln x - \cos y = K}$$

non $k = C + e - \cos 1$ den.

B) Ebatzi hurrengo EDA**:** $x^2 y'' - 3x\,y' + 3y = x + x^2 \cdot \ln x$

(5 p)

Ebazpena:

Euler-en EDA da.

$$y = x^r; \quad y' = r\,x^{r-1}; \quad y'' = r(r-1)x^{r-2}$$

$$x^2 y'' - 3xy' + 3y = x^2 r(r-1)\,x^{r-2} - 3x\,r\,x^{r-1} + 3x^r = x^r [r(r-1) - 3r + 3] = 0$$

$$r^2 - 4r + 3 = 0 \quad \rightarrow \quad r = \frac{4 \pm \sqrt{16-12}}{2} = \begin{cases} 3 \\ 1 \end{cases} \quad \Rightarrow \quad y = C_1 x + C_2 x^3$$

Parametroen aldakuntzaren metodoa erabiliz, soluzio orokorra hurrengoa da:

$$y = L_1(x) \cdot x + L_2(x) \cdot x^3$$

non $L_1(x)$ y $L_2(x)$ hurrengo sistemarekin kalkulatzen diren:

$$\begin{cases} L_1' \cdot x + L_2' \cdot x^3 = 0 \\ L_1' \cdot 1 + L_2' \cdot 3x^2 = \dfrac{x + x^2 \ln x}{x^2} = \dfrac{1}{x} + \ln x \end{cases}$$

$$L_1'(x)=\frac{\begin{vmatrix} 0 & x^3 \\ \frac{1}{x}+\ln x & 3x^2 \end{vmatrix}}{\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}}=\frac{-x^2-x^3\ln x}{2x^3}=-\frac{1}{2x}-\frac{1}{2}\ln x$$

$$L_2'(x)=\frac{\begin{vmatrix} x & 0 \\ 1 & \frac{1}{x}+\ln x \end{vmatrix}}{\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}}=\frac{1+x\ln x}{2x^3}=\frac{1}{2x^3}+\frac{1}{2x^2}\ln x$$

Integratuz

$$L_1(x)=-\frac{1}{2}\int\frac{dx}{x}-\frac{1}{2}\int\ln x\,dx=-\frac{1}{2}\ln|x|-\frac{1}{2}\left(x\ln|x|-x\right)+A=-\frac{1}{2}\ln|x|-\frac{x}{2}\ln|x|+\frac{x}{2}+A$$

$$L_2(x)=\frac{1}{2}\int\frac{dx}{x^3}+\int\frac{1}{2x^2}\ln x\,dx=I+J$$

$$I=\frac{1}{2}\int\frac{dx}{x^3}=\frac{1}{2}\int x^{-3}dx=\frac{1}{2}\cdot\frac{x^{-2}}{(-2)}=-\frac{1}{4x^2}+cte$$

$$J=\int\frac{1}{2x^2}\ln x\,dx=\left\{\begin{array}{lcl} \ln x=u & \Rightarrow & du=dx/x \\ \frac{1}{2x^2}dx=dv & \Rightarrow & v=-\frac{1}{2x} \end{array}\right\}=-\frac{1}{2x}\cdot\ln x+\int\frac{1}{2x^2}dx=-\frac{1}{2x}\cdot\ln x-\frac{1}{2x}+cte$$

$$L_2(x)=I+J=-\frac{1}{4x^2}-\frac{1}{2x}\cdot\ln x-\frac{1}{2x}+B$$

Orduan, soluzio orokorra hurrengoa da:

$$\boxed{y}=L_1(x)\cdot x+L_2(x)\cdot x^3=\left(-\frac{1}{2}\ln|x|-\frac{x}{2}\ln|x|+\frac{x}{2}+A\right)\cdot x+\left(-\frac{1}{4x^2}-\frac{1}{2x}\cdot\ln x-\frac{1}{2x}+B\right)\cdot x^3=$$

$$=-\frac{x}{2}\ln|x|-\frac{x^2}{2}\ln|x|+\frac{x^2}{2}+Ax-\frac{x}{4}-\frac{x^2}{2}\cdot\ln x-\frac{x^2}{2}+Bx^3=$$

$$=\boxed{Ax+Bx^3-\left(x^2+\frac{x}{2}\right)\ln|x|-\frac{x}{4}}$$

C) Ebatzi hurrengo ekuazio integrala: $f(t)+2\int_0^t f(u)\cdot\cos(t-u)\,du = 4e^{-t}+\operatorname{sen} t$

(5 p)

<u>*Ebazpena*</u>

Hurrengo propietatea erabiliko dugu:

$$\mathfrak{L}\int_0^t f(u)\,g(t-u)\,du = \mathfrak{L}[f(t)*g(t)] = F(p)\cdot G(p)$$

Ekuazioaren transformatua kalkulatuko da, $F(p)$ askatuko da eta, azkenean, alderantzizko transformatua aplikatuko da:

$$F(p)+2F(p)\cdot\frac{p}{p^2+1}=\frac{4}{p+1}+\frac{1}{p^2+1} \quad\to$$

$$F(p)\left(1+\frac{2p}{p^2+1}\right)=\frac{4}{p+1}+\frac{1}{p^2+1} \quad\to\quad F(p)\left(\frac{p^2+1+2p}{p^2+1}\right)=\frac{4}{p+1}+\frac{1}{p^2+1}$$

$$F(p)\left(\frac{(p+1)^2}{p^2+1}\right)=\frac{4}{p+1}+\frac{1}{p^2+1} \quad\to\quad F(p)=\frac{4(p^2+1)}{(p+1)(p+1)^2}+\frac{p^2+1}{(p^2+1)(p+1)^2}$$

$$F(p)=\frac{4(p^2+1)}{(p+1)^3}+\frac{1}{(p+1)^2}=4\cdot\frac{p^2+1}{(p+1)^3}+\frac{1}{(p+1)^2}$$

$$\frac{p^2+1}{(p+1)^3}=\frac{A}{p+1}+\frac{B}{(p+1)^2}+\frac{C}{(p+1)^3}$$

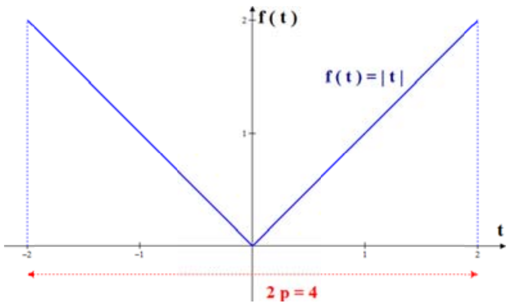
$$p^2+1\equiv A(p+1)^2+B(p+1)+C \quad\to\quad \begin{cases}A=1\\2A+B=0\\A+B+C=1\end{cases} \quad\to\quad \begin{cases}A=1\\B=-2\\C=2\end{cases}$$

$$\boxed{f(t)}=4\cdot\mathfrak{L}^{-1}\left[\frac{1}{p+1}\right]-8\cdot\mathfrak{L}^{-1}\left[\frac{1}{(p+1)^2}\right]+8\cdot\mathfrak{L}^{-1}\left[\frac{1}{(p+1)^3}\right]+\mathfrak{L}^{-1}\left[\frac{1}{(p+1)^2}\right]=$$

$$=4e^{-t}-8te^{-t}+4t^2e^{-t}+te^{-t}=4e^{-t}-7te^{-t}+4t^2e^{-t}=\boxed{\left(4t^2-7t+4\right)e^{-t}}$$

D) Garatu $f(t)=|t| \quad -2\le t\le 2$ tartean definitutako funtzio periodikoaren Fourier-en seriea.

(5 p)



Ebazpena:

Funtzioa **bikoitia** da. Funtzioaren periodoa hurrengoa da: $2p=4 \quad \to \quad p=2$, Beraz, Fourier-en seriea hurrengoa da:

$$f(t)=\frac{a_0}{2}+\sum_1^{\infty} a_k\cos\left(\frac{k\pi t}{p}\right)$$

$$a_0=\frac{2}{p}\int_0^p f(t)\,dt=\int_0^2 t\,dt=\left[\frac{t^2}{2}\right]_0^2=2$$

$$a_k=\frac{2}{p}\int_0^p f(t)\cos\left(\frac{k\pi t}{p}\right)dt=\int_0^2 t\cos\left(\frac{k\pi t}{2}\right)dt=\left\{\begin{array}{rcl} t=u & \Rightarrow & du=dt \\ \cos\left(\frac{k\pi t}{2}\right)dt=dv & \Rightarrow & v=\frac{2}{k\pi}\operatorname{sen}\left(\frac{k\pi t}{2}\right)\end{array}\right\}=$$

$$=\left[\frac{2}{k\pi}t\operatorname{sen}\left(\frac{k\pi t}{2}\right)+\frac{4}{k^2\pi^2}\cos\left(\frac{k\pi t}{2}\right)\right]_0^2=\frac{4}{k^2\pi^2}\cos(k\pi)-\frac{4}{k^2\pi^2}=\frac{4}{k^2\pi^2}\left[\cos(k\pi)-1\right]=$$

$$=\begin{cases}\dfrac{-8}{k^2\pi^2} & si \quad k \text{ es impar}\\ 0 & si \quad k \text{ es par}\end{cases}$$

$$f(t)=\frac{2}{2}+\sum_1^{\infty}\frac{4}{k^2\pi^2}\left[\cos(k\pi)-1\right]\cdot\cos\left(\frac{k\pi t}{2}\right)=1-\frac{8}{\pi^2}\sum_1^{\infty}\frac{1}{(2n-1)^2}\cdot\cos\left(\frac{(2n-1)\pi t}{2}\right)$$