KALKULUA (INDUSTRIALAK)
AZTERKETA PARTZIALA 2016KO URTARRILAREN 22A

1. ORRIA **(160 puntu)**

A) Hurrengo baldintza betetzen duten $z \in \mathbb{C}$ puntuek definitutako leku geometrikoa zehaztu

$$|z-1|+|z+3|=10$$

(60 p)

Ebazpena

Izan bedi $z = x + iy$ beraz:

$$|z-1| = |(x-1)+iy| = \sqrt{(x-1)^2+y^2}$$

$$|z+3| = |(x+3)+iy| = \sqrt{(x+3)^2+y^2}$$

Ekuazioan ordezkatuz:

$$|z-1|+|z+3|=10 \Rightarrow \sqrt{(x-1)^2+y^2}+\sqrt{(x+3)^2+y^2}=10$$

$$\sqrt{(x-1)^2+y^2}=10-\sqrt{(x+3)^2+y^2}$$

Karratura jasota:

$$-2x+1=10^2-20\sqrt{(x+3)^2+y^2}+6x+9$$

$$5\sqrt{(x+3)^2+y^2}=2x+27$$

Berriro ere karratura jasoz:

$$25((x+3)^2+y^2)=(2x+27)^2$$

$$25(x^2+6x+9+y^2)=(4x^2+108x+729) \quad \rightarrow \quad 21x^2+42x+25y^2=504$$

Karratu perfektuen eran adieraziz:

$$21(x+1)^2+25y^2=525$$

Sinplifikatuz:

$$\frac{21}{525}(x+1)^2+\frac{25}{525}y^2=1 \quad \rightarrow \quad \frac{(x+1)^2}{25}+\frac{y^2}{21}=1$$

Beraz, 5 y $\sqrt{21}$ erdi-ardatzak eta zentroa $(-1,0)$ koordenatuan duen elipsean daude puntuak.

B) $x=0$ abzisa puntuan $y=(1-x)^{x+5}$ kurbaren zuzen ukitzailearen ekuazioa kalkulatu

(40 p)

Ebazpena

$$y=(1-x)^{x+5}\Big|_{x=0}=1 \quad \Rightarrow \quad (0,1)$$ puntua ukitzen du zuzen ukitzaileak

$$y=(1-x)^{x+5} \quad \Rightarrow \quad \ln y=(x+5)\ln(1-x)$$

$$\frac{y'}{y}=\ln(1-x)+\frac{x+5}{x-1} \quad \Rightarrow \quad y'=(1-x)^{x+5}\left[\ln(1-x)+\frac{x+5}{x-1}\right]$$

$$y'=(1-x)^{x+5}\left[\ln(1-x)+\frac{x+5}{x-1}\right]\Bigg|_{x=0}=[\ln 1-5]=-5$$

Zuzen ukitzailearen ekuazio hurrengoa da:

$$y-1=-5(x-0) \quad \Rightarrow \quad y=-5x+1$$

C) $z^2+\dfrac{2}{x}-\sqrt{y^2-z^2}=0$ ekuazioak $z(x,y)$ funtzioa inplizituki definitzen du.

Frogatu hurrengo berdinketa: $x^2\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\cdot\dfrac{\partial z}{\partial y}=\dfrac{1}{z}$

(60 p)

Ebazpena

Inplizituki x-rekiko deribatuz: $$2z\frac{\partial z}{\partial x}-\frac{2}{x^2}-\frac{(-2z)\dfrac{\partial z}{\partial x}}{2\sqrt{y^2-z^2}}=0$$

$$\frac{\partial z}{\partial x}\left[2z+\frac{z}{\sqrt{y^2-z^2}}\right]=\frac{2}{x^2}\quad\rightarrow\quad\frac{\partial z}{\partial x}=\frac{\dfrac{2}{x^2}}{2z+\dfrac{z}{\sqrt{y^2-z^2}}}$$

$$x^2\frac{\partial z}{\partial x}=\frac{2}{2z+\dfrac{z}{\sqrt{y^2-z^2}}}=\frac{2\sqrt{y^2-z^2}}{2z\sqrt{y^2-z^2}+z}\qquad[1]$$

Inplizituki y-rekiko deribatuz: $$2z\frac{\partial z}{\partial y}-\frac{2y-2z\dfrac{\partial z}{\partial y}}{2\sqrt{y^2-z^2}}=0$$

$$\frac{\partial z}{\partial y}\left[2z+\frac{z}{\sqrt{y^2-z^2}}\right]=\frac{y}{\sqrt{y^2-z^2}}\quad\rightarrow\quad\frac{\partial z}{\partial y}=\frac{\dfrac{y}{\sqrt{y^2-z^2}}}{2z+\dfrac{z}{\sqrt{y^2-z^2}}}$$

$$\frac{1}{y}\cdot\frac{\partial z}{\partial y}=\frac{\dfrac{1}{\sqrt{y^2-z^2}}}{2z+\dfrac{z}{\sqrt{y^2-z^2}}}=\frac{1}{2z\sqrt{y^2-z^2}+z}\qquad[2]$$

[1]+[2]: $$x^2\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}=\frac{2\sqrt{y^2-z^2}}{2z\sqrt{y^2-z^2}+z}+\frac{1}{2z\sqrt{y^2-z^2}+z}=\frac{2\sqrt{y^2-z^2}+1}{2z\sqrt{y^2-z^2}+z}=\frac{1}{z}$$

2. ORRIA (240 puntu)

A) Kalkulatu: $\displaystyle\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

(40 p)

Ebazpena

$$\int \frac{dx}{(x-1)\sqrt{x^2+x+1}} = \left\{ \begin{array}{l} x-1=\dfrac{1}{t} \quad \Rightarrow \quad dx=-\dfrac{dt}{t^2} \\ x^2+x+1=\dfrac{(1+t)^2}{t^2}+\dfrac{1+t}{t}+1=\dfrac{3t^2+3t+1}{t^2} \end{array} \right\} = \int \frac{-\dfrac{dt}{t^2}}{\dfrac{1}{t}\cdot\dfrac{\sqrt{3t^2+3t+1}}{t}} =$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{t^2+t+\dfrac{1}{3}}} = -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\dfrac{1}{2}\right)^2+\dfrac{1}{12}}} = \left\{ \begin{array}{l} t+\dfrac{1}{2}=z \\ dt=dz \end{array} \right\} =$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dz}{\sqrt{z^2+\dfrac{1}{12}}} = -\frac{1}{\sqrt{3}}\ln\left|z+\sqrt{z^2+\frac{1}{12}}\right|+C = -\frac{1}{\sqrt{3}}\ln\left|t+\frac{1}{2}+\sqrt{t^2+t+\frac{1}{3}}\right|+C =$$

$$= -\frac{1}{\sqrt{3}}\ln\left|\frac{1}{x-1}+\frac{1}{2}+\sqrt{\frac{1}{(x-1)^2}+\frac{1}{(x-1)}+\frac{1}{3}}\right|+C = -\frac{1}{\sqrt{3}}\ln\left|\frac{1}{x-1}+\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{\sqrt{3}(x-1)}\right|+C$$

B) Izan bedi hurrengo $[D]$ domeinua:

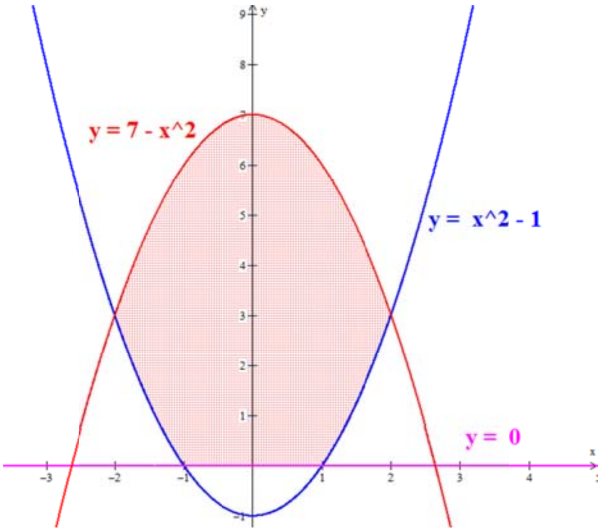
$$D=\left\{(x,y)\in\mathbb{R}^2\,/\,(y\le 7-x^2)\wedge(y\ge x^2-1)\wedge(y\ge 0)\right\}$$

Kalkulatu:

1.-$[D]$ domeinu lauaren perimetroa. **(50 p)**

2.- $[D]$ abzisa ardatzaren inguruan biratzean sortutako bolumena **(50 p)**

<u>*Ebazpena*</u>



1.- $[D]$domeinu lauaren perimetroa:

$$P=2\left[1+L_1+L_2\right]$$

Non:

$$L_1=\int_1^2\sqrt{1+4x^2}\,dx=\frac{1}{4}\left[2x\sqrt{1+4x^2}+\ln\left|2x+\sqrt{1+4x^2}\right|\right]_1^2=$$

$$=\frac{1}{4}\left[\left(4\sqrt{17}+\ln\left(4+\sqrt{17}\right)\right)-\left(2\sqrt{5}+\ln\left(2+\sqrt{5}\right)\right)\right]=\sqrt{17}-\frac{\sqrt{5}}{2}+\frac{1}{4}\ln\frac{4+\sqrt{17}}{2+\sqrt{5}}$$

$$L_2=\int_0^2\sqrt{1+4x^2}\,dx=\frac{1}{4}\left[2x\sqrt{1+4x^2}+\ln\left|2x+\sqrt{1+4x^2}\right|\right]_0^2=$$

$$=\frac{1}{4}\left[\left(4\sqrt{17}+\ln\left(4+\sqrt{17}\right)\right)-\ln(1)\right]=\sqrt{17}+\frac{1}{4}\ln\left(4+\sqrt{17}\right)$$

Beraz, perimetroa honako hau da:

$$P = 2\left[1 + L_1 + L_2\right] = 2\left[1 + \sqrt{17} - \frac{\sqrt{5}}{2} + \frac{1}{4}\ln\frac{4+\sqrt{17}}{2+\sqrt{5}} + \sqrt{17} + \frac{1}{4}\ln\left(4+\sqrt{17}\right)\right] =$$

$$= 2 + 4\sqrt{17} - \sqrt{5} + \frac{1}{4}\ln\frac{\left(4+\sqrt{17}\right)^2}{2+\sqrt{5}} \quad u$$

2.- $[D]$ abzisa ardatzaren inguruan biratzean sortutako bolumena.

$$V = 2V_1$$

$$V_1 = \pi\left[\int_0^2 \left(7-x^2\right)^2 dx - \int_1^2 \left(x^2-1\right)^2 dx\right] = \pi\left[\left(49x - \frac{14}{3}x^3 + \frac{x^5}{5}\bigg|_0^2\right) - \left(\frac{x^5}{5} - \frac{2}{3}x^3 + x\right)\bigg|_1^2\right] =$$

$$= \pi\left[\left(98 - \frac{112}{3} + \frac{32}{5}\right) - \left(\frac{32}{5} - \frac{16}{3} + 2 - \frac{1}{5} + \frac{2}{3} - 1\right)\right] = \pi\left[\frac{1006}{15} - \frac{38}{15}\right] = \frac{968}{15}\pi$$

Beraz, bolumena honako hau da:

$$V = 2V_1 = \frac{1936}{15}\pi \quad u^3$$

C) Izan bedi hurrengo $[D]$ domeinua:

$$D=\left\{(x,y)\in\mathbb{R}^2/(9x^2+25y^2-225\le 0)\wedge(3x-5y+15\le 0)\right\}$$

1.- Bi era desberdinetan, $I=\iint_{[D]} f(x,y)\,dx\,dy$ integralean, integrazio-limiteak zehaztu.

(30 p)

2.- $[D]$ domeinu lauaren grabitate-zentro geometrikoaren abzisa koordenatua kalkulatu, integral bikoitzak erabiliz.

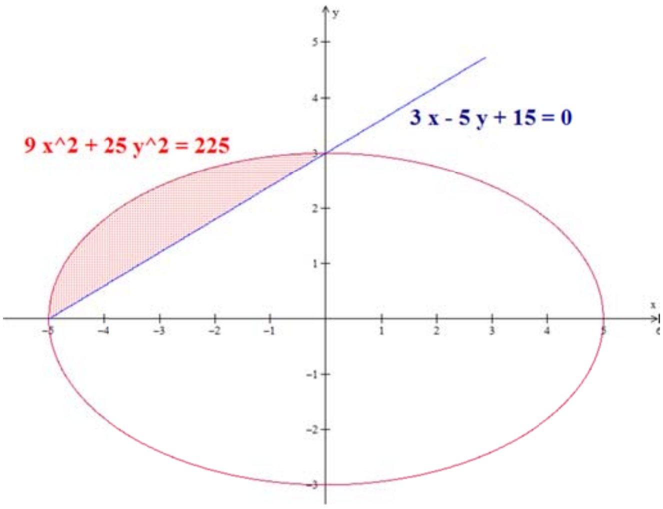
(70 p)

Ebazpena

1.- Bi era desberdinetan, $I=\iint_{[D]} f(x,y)\,dx\,dy$ integralean, integrazio-limiteak zehaztu.

$$9x^2+25y^2-225=0 \;\rightarrow\; \frac{x^2}{25}+\frac{y^2}{9}=1 \;\rightarrow\; a=5 \;\;;\;\; b=3$$ erdi-ardatzetako elipsea

$$3x-5y+15=0 \;\rightarrow\; y=\frac{3}{5}x+3 \;\rightarrow\; (-5,0) \text{ eta } (0,3)$$ puntuetatik pasatzen den zuzena



Lehenengo integrazio aldagaitzat (x) hartuz

$$I=\iint_{[D]} f(x,y)\,dx\,dy=\int_0^3 dy\int_{-(5/3)\sqrt{9-y^2}}^{(5/3)(y-3)} f(x,y)\,dx$$

Lehenengo integrazio aldagaitzat (y) hartuz

$$I=\iint_{[D]} f(x,y)\,dx\,dy=\int_{-5}^{0} dx\int_{(3/5)x+3}^{(3/5)\sqrt{25-x^2}} f(x,y)\,dy$$

2.- $[D]$ domeinu lauaren grabitate-zentro geometrikoaren abzisa koordenatua kalkulatu, integral bikoitzak erabiliz.

$$A = \iint_D dx\,dy = \int_{-5}^{0} dx \int_{(3/5)x+3}^{(3/5)\sqrt{25-x^2}} dy = \int_{-5}^{0} \left[\frac{3}{5}\sqrt{25-x^2} - \frac{3}{5}x - 3 \right] dx =$$

$$= \frac{3}{5}\int_{-5}^{0} \sqrt{25-x^2}\,dx - \left[\frac{3x^2}{10} + 3x \right]_{-5}^{0} = \frac{3}{5}J + \frac{75}{10} - 15 = \frac{3}{5}J - \frac{15}{2}$$

$$J = \int_{-5}^{0} \sqrt{25-x^2}\,dx = \begin{bmatrix} x = 5\,\text{sen}\,t & x=-5 & \Rightarrow t = -\pi/2 \\ dx = 5\cos t\,dt & x = 0 & \Rightarrow t = 0 \end{bmatrix} = 25\int_{-\pi/2}^{0} \cos^2 t\,dt =$$

$$= 25\left[\frac{t}{2} + \frac{\text{sen}\,2t}{4} \right]_{-\pi/2}^{0} = \frac{25\pi}{4} \quad \Rightarrow \quad \boxed{A} = \frac{3}{5}J - \frac{15}{2} = \frac{15\pi}{4} - \frac{15}{2} = \boxed{\frac{15}{4}(\pi - 2)}$$

$$I = \iint_D x\,dx\,dy = \int_{-5}^{0} x\,dx \int_{(3/5)x+3}^{(3/5)\sqrt{25-x^2}} dy = \int_{-5}^{0} x\left[\frac{3}{5}\sqrt{25-x^2} - \frac{3}{5}x - 3 \right] dx =$$

$$= \frac{3}{5}\int_{-5}^{0} x\sqrt{25-x^2}\,dx - \left[\frac{x^3}{5} + \frac{3x^2}{2} \right]_{-5}^{0} = \frac{3}{5}H - 25 + \frac{75}{2} = \frac{3}{5}H + \frac{25}{2}$$

$$H = \int_{-5}^{0} x\sqrt{25-x^2}\,dx = \begin{bmatrix} 25 - x^2 = t^2 & x=-5 & \Rightarrow t = 0 \\ -xdx = tdt & x = 0 & \Rightarrow t = 5 \end{bmatrix} = -\int_{0}^{5} t^2 dt = \left[-\frac{t^3}{3} \right]_{0}^{5} = -\frac{125}{3}$$

$$I = \frac{3}{5}H + \frac{25}{2} = -25 + \frac{25}{2} = -\frac{25}{2}$$

$$\boxed{x_c} = \frac{1}{A}\iint_D x\,dx\,dy = \frac{-25/2}{15(\pi-2)/4} = \boxed{\frac{-10}{3(\pi-2)}}$$