Minimo ala maximo bat den jakiteko, matrize hessiarraren determinantea kalkulatzen da:

$$HL(\lambda,(x,y))=\begin{pmatrix}0 & \varphi'_x & \varphi'_y\\ \varphi'_x & L''_{xx} & L''_{xy}\\ \varphi'_y & L''_{yx} & L''_{yy}\end{pmatrix}=\begin{pmatrix}0 & 2x & 2y\\ 2x & 2+2\lambda & 0\\ 2y & 0 & 2+2\lambda\end{pmatrix}$$

$$\left|HL\left(-\frac{9}{4},\left(\frac{12}{5},-\frac{16}{5}\right)\right)\right|=\begin{vmatrix}0 & \frac{24}{5} & -\frac{32}{5}\\ \frac{24}{5} & -\frac{5}{2} & 0\\ -\frac{32}{5} & 0 & -\frac{5}{2}\end{vmatrix}>0\rightarrow\ P_1\left(\frac{12}{5},-\frac{16}{5}\right)$$ maximo lokal bat da.

$$\left|HL\left(\frac{1}{4},\left(-\frac{12}{5},\frac{16}{5}\right)\right)\right|=\begin{vmatrix}0 & -\frac{24}{5} & \frac{32}{5}\\ -\frac{24}{5} & \frac{5}{2} & 0\\ \frac{32}{5} & 0 & \frac{5}{2}\end{vmatrix}<0\rightarrow\ P_2\left(-\frac{12}{5},\frac{16}{5}\right)$$ maximo lokal bat da.

D) Kalkulatu, integralak erabiliz, $y=\sqrt{(3-x)(1+x)}$ kurbaren luzera

(6 puntu)

<u>*Ebazpena*</u>

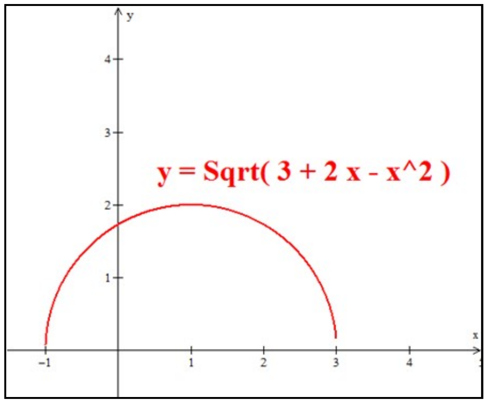
Zirkunferentzierdi bat da. Hain zuzen ere:

$$y=\sqrt{(3-x)(1+x)}\quad\rightarrow\quad y^2=(3-x)(1+x)=3-x+3x-x^2=3+2x-x^2$$

$x^2+y^2-2x=3\quad\rightarrow\quad(x-1)^2+y^2=4\quad\rightarrow\quad(1,0)$ zentroko eta 2 erradioko zirkunferentzia.

$$y=\pm\sqrt{4-(x-1)^2}=\pm\sqrt{3+2x-x^2}$$

Orduan, emandako kurba $(1,0)$ zentroko eta 2 erradioko goiko zirkunferentzierdia da.



Luzeraren kalkulua:

$$y' = \frac{-2x+2}{2\sqrt{3+2x-x^2}} = \frac{1-x}{\sqrt{3+2x-x^2}} \quad \to \quad y'^2 = \frac{(1-x)^2}{3+2x-x^2}$$

$$\boxed{L} = \int_{-1}^{3} \sqrt{1+y'^2}\, dx = \int_{-1}^{3} \sqrt{1+\frac{(1-x)^2}{3+2x-x^2}}\, dx = \int_{-1}^{3} \sqrt{\frac{3+2x-x^2+1-2x+x^2}{3+2x-x^2}}\, dx =$$

$$= \int_{-1}^{3} \sqrt{\frac{4}{3+2x-x^2}}\, dx = 2\int_{-1}^{3} \frac{dx}{\sqrt{3+2x-x^2}} = 2\int_{-1}^{3} \frac{dx}{\sqrt{4-(x-1)^2}} =$$

$$= 2\left[\arcsin\left(\frac{x-1}{2}\right)\right]_{-1}^{3} = 2\left[\arcsin 1 - \arcsin(-1)\right] = 2\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \boxed{2\pi}$$

Oinarrizko geometria erabiliz (emaitza konprobatzeko): $L = \dfrac{2\pi r}{2} = \pi r = 2\pi$

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2. ORRIA (20 puntu)

A) Izan bedi hurrengo [D] domeinu laua:

$$D = \left\{ (x,y) \in \mathbb{R}^2 \,/\, y \geq |2-x| \;;\; (x-2)^2 + y^2 \leq 4 \right\}$$

1.- $I = \iint_{[D]} f(x,y)\, dx\, dy$ integralean integrazio-limiteak bi era desberdinetan planteatu.

2.- Integral bikoitzak erabiliz, $[D]$ domeinu lauaren azalera kalkulatu.

(6 puntu)

Ebazpena

Ebakidura puntuak kalkulatu:

Si $x \le 2$:

$$\begin{cases} (x-2)^2 + y^2 = 4 \\ y = 2-x \end{cases} \Rightarrow \begin{cases} y = \sqrt{4-(x-2)^2} \\ y = 2-x \end{cases} \Rightarrow 2-x = \sqrt{4-(x-2)^2} \Rightarrow$$

$$(2-x)^2 = 4-(x-2)^2 \Rightarrow 2(2-x)^2 = 4 \Rightarrow x = 2 \pm \sqrt{2} \Rightarrow$$

$$\Rightarrow \left(2-\sqrt{2}, \sqrt{2}\right)$$

Si $x > 2$:

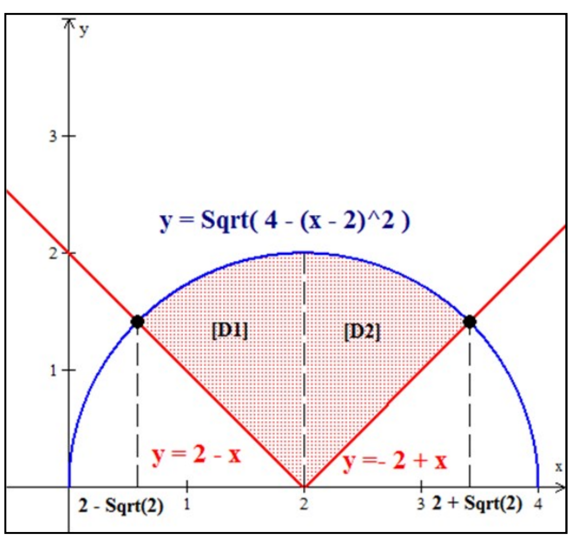
$$\begin{cases} (x-2)^2 + y^2 = 4 \\ y = x-2 \end{cases} \Rightarrow \begin{cases} y = \sqrt{4-(x-2)^2} \\ y = x-2 \end{cases} \Rightarrow x-2 = \sqrt{4-(x-2)^2} \Rightarrow$$

$$(x-2)^2 = 4-(x-2)^2 \Rightarrow 2(x-2)^2 = 4 \Rightarrow x = 2 \pm \sqrt{2} \Rightarrow$$

$$\Rightarrow \left(2+\sqrt{2}, \sqrt{2}\right)$$

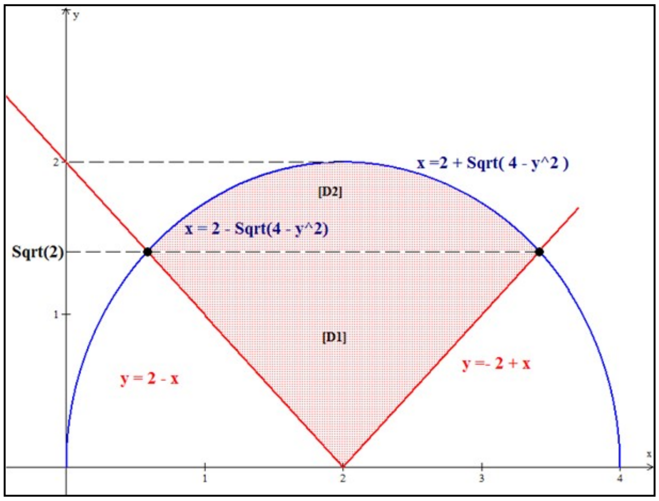
Integralaren limiteak

(y) lehenengo integrazio-aldagaitzat hartuz:



$$\iint_D f(x,y)\,dx\,dy = \int_{2-\sqrt{2}}^{2} dx \int_{2-x}^{\sqrt{4-(x-2)^2}} f(x,y)\,dy + \int_{2}^{2+\sqrt{2}} dx \int_{-2+x}^{\sqrt{4-(x-2)^2}} f(x,y)\,dy$$

(x) lehenengo integrazio-aldagaitzat hartuz:



$$\iint_D f(x,y)\,dx\,dy = \int_0^{\sqrt{2}} dy \int_{2-y}^{y+2} f(x,y)\,dx + \int_{\sqrt{2}}^{2} dy \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} f(x,y)\,dx$$

<u>*Azaleraren kalkulua:*</u>

$$A = \int_{2-\sqrt{2}}^{2} dx \int_{2-x}^{\sqrt{4-(x-2)^2}} dy + \int_{2}^{2+\sqrt{2}} dx \int_{x-2}^{\sqrt{4-(x-2)^2}} dy$$

Dagoen simetria kontuan hartuta:

$$A = 2\int_{2}^{2+\sqrt{2}} dx \int_{x-2}^{\sqrt{4-(x-2)^2}} dy = 2\int_{2}^{2+\sqrt{2}} \left(\sqrt{4-(x-2)^2} - (x-2) \right) dx =$$

$$= 2\left[\int_{2}^{2+\sqrt{2}} \left(\sqrt{4-(x-2)^2} \right) dx - \left[\frac{x^2}{2} - 2x \right]_{2}^{2+\sqrt{2}} \right] = 2\left[I - \frac{1}{2}\left(\left(2+\sqrt{2}\right)^2 - 4\left(2+\sqrt{2}\right) - 4 + 8 \right) \right] = 2[I-1]$$

$$I = \int_{2}^{2+\sqrt{2}} \left(\sqrt{4-(x-2)^2} \right) dx = \begin{bmatrix} x-2 = 2\sin t \;\; \rightarrow \;\; dx = 2\cos t\,dt \\ x = 2 \;\; \rightarrow \;\; t = 0 \\ x = 2+\sqrt{2} \;\; \rightarrow \;\; t = \arcsin\left(\sqrt{2}/2 \right) = \pi/4 \end{bmatrix} = \int_0^{\pi/4} \sqrt{4-4\sin^2 t}\; 2\cos t\,dt =$$

$$I = \int_0^{\pi/4} 4\cos^2 t\,dt = 2\int_0^{\pi/4} \left(1+\cos 2t\right) dt = 2\left[t + \frac{\sin 2t}{2} \right]_0^{\pi/4} = 2\left[\frac{\pi}{4} + \frac{1}{2} \right] = \frac{\pi}{2} + 1$$

Beraz, azalera hurrengoa da:

$$A = 2[I-1] = 2\left[\frac{\pi}{2}+1-1\right] = \pi \quad u^2$$

B) Ebatzi hurrengo EDA faktore integratzaile bat erabiliz:

$$\left(x^2+y^2\right)\cos x\,dx - \frac{2}{y}\left(2x\cos x + \left(x^2-2\right)\sin x\right)dy = 0$$

(5 puntu)

Ebazpena

$$X(x,y) = \left(x^2+y^2\right)\cos x \quad \Rightarrow \quad \frac{\partial X}{\partial y} = 2y\cos x$$

$$Y(x,y) = -\frac{2}{y}\left(2x\cos x + \left(x^2-2\right)\operatorname{sen} x\right) \quad \Rightarrow \quad \frac{\partial Y}{\partial x} = \frac{-2x^2\cos x}{y}$$

EDA ez da zehatza: $\dfrac{\partial X}{\partial y} \neq \dfrac{\partial Y}{\partial x}$

Ezin da $z(x)$ faktore integratzaile bat lortu:

$$\frac{\dfrac{\partial X}{\partial y} - \dfrac{\partial Y}{\partial x}}{Y} = \frac{2y\cos x + \dfrac{2x^2\cos x}{y}}{-\dfrac{2}{y}\left(2x\cos x + \left(x^2-2\right)\operatorname{sen} x\right)} \neq \varphi(x)$$

$z(y)$ faktore integratzaile bat lortu ahal da:

$$\frac{\dfrac{\partial Y}{\partial x} - \dfrac{\partial X}{\partial y}}{X} = \frac{-\dfrac{2x^2\cos x}{y} - 2y\cos x}{\left(x^2+y^2\right)\cos x} = \frac{\dfrac{-2x^2-2y^2}{y}}{\left(x^2+y^2\right)} = \frac{-2}{y} = \varphi(y)$$

$$z(y) = e^{\int \frac{-2}{y}dx} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = \frac{1}{y^2}$$

EDA bider $z(y)$ biderkatu ondoren zehatza da.

$$\left(\frac{x^2+y^2}{y^2}\right)\cos x\,dx - \frac{2}{y^3}\left(2x\cos x + \left(x^2-2\right)\sin x\right)dy = 0$$

$$X'(x,y)=\left(\frac{x^2+y^2}{y^2}\right)\cos x \quad\Rightarrow\quad \frac{\partial X'}{\partial y}=-\frac{2x^2}{y^3}\cos x$$

$$Y'(x,y)=-\frac{2}{y^3}\left(2x\cos x+\left(x^2-2\right)\sin x\right) \quad\Rightarrow\quad \frac{\partial Y'}{\partial x}=-\frac{2x^2\cos x}{y^3}$$

$$\Rightarrow\quad \frac{\partial X'}{\partial y}=\frac{\partial Y'}{\partial x}$$

Soluzio orokorra hurrengoa da:

$$\int_a^x X'(x,y)\,dx+\int_b^y Y'(a,y)\,dx=C$$

(a,b)=(0,1) hartuz:

$$\int_0^x\left(\frac{x^2+y^2}{y^2}\right)\cos x\,dx+\int_1^y 0\,dx=C \quad\rightarrow\quad \int_0^x\left(\frac{x^2+y^2}{y^2}\right)\cos x\,dx=C$$

$$\frac{1}{y^2}\int_0^x\left(x^2+y^2\right)\cos x\,dx=\begin{bmatrix} u=x^2+y^2 & du=2x\,dx \\ dv=\cos x\,dx & v=\sin x\end{bmatrix}=\frac{1}{y^2}\left[\left[\left(x^2+y^2\right)\sin x\right]_0^x-\int_0^x 2x\sin x\,dx\right]=$$

$$=\begin{bmatrix} u=2x & du=2\,dx \\ dv=\sin x\,dx & v=-\cos x\end{bmatrix}=\frac{1}{y^2}\left[\left(x^2+y^2\right)\sin x-\left(\left[-2x\cos x\right]_0^x+\int_0^x 2\cos x\,dx\right)\right]=$$

$$=\frac{1}{y^2}\left[\left(x^2+y^2\right)\sin x+2x\cos x-2\sin x\right]$$

Beraz, soluzio orokorra hurrengoa da:

$$\frac{1}{y^2}\left[\left(x^2+y^2\right)\operatorname{sen} x+2x\cos x-2\operatorname{sen} x\right]=C \quad\Rightarrow\quad Cy^2=\left(x^2+y^2-2\right)\operatorname{sen} x+2x\cos x$$

C) $x(2x+3)y''-6(x+1)y'+6y=0$ ekuazio diferentzialaren soluzio orokorra lortu, $y=x^3$ ekuazioren soluzio partikular bat dela jakinda.

(4 puntu)

<u>*Ebazpena*</u>

Beste soluzioa hurrengoa formula erabiliz lortzen da:

$$y_2 = y_1 \int \frac{\exp\left(-\int P(x)dx\right)}{y_1^2} dx \text{ , non } P(x) = \frac{-6(x+1)}{x(2x+3)}$$

Zatiki sinpleetan deskonposatuz:

$$\frac{-6(x+1)}{x(2x+3)} = \frac{A}{x} + \frac{B}{2x+3}$$

$$-6(x+1) = A(2x+3) + Bx$$

$$\begin{cases} -6 = 2A + B \\ -6 = 3A \end{cases} \Rightarrow \quad A = -2 \, ; \, B = -2$$

Integratuz:

$$\exp\left[-\int P(x)dx\right] = \exp\left[\int\left(\frac{2}{x} + \frac{2}{2x+3}\right)dx\right] = \exp\left[\ln x^2(2x+3)\right] = x^2(2x+3)$$

eta ordezkatuz:

$$y_2 = x^3\left[\int \frac{(2x+3)}{x^4}dx\right] = x^3\left[\int\left(\frac{2}{x^3} + \frac{3}{x^4}\right)dx\right] = x^3\left[-x^{-2} - x^{-3}\right] = -(x+1)$$

Soluzio orokorra lortzen da:

$$y = C_1 x^3 + C_2(x+1)$$

D) Ebatzi hurrengo ekuazio diferentziala Laplace-ren transformatua erabiliz.

$$y'' - 2y' + 2y = 6\, e^{-t} \cos t \qquad y(0) = y'(0) = 0$$

(5 puntu)

<u>*Ebazpena*</u>

$$y'' - 2y' + 2y = 6\, e^{-t} \cos t \qquad y(0) = y'(0) = 0$$

$$L\left[y'' - 2y' + 2y\right] = p^2 Y(p) - p\, y(0) - y'(0) - 2\left[pY(p) - y(0)\right] + 2\, Y(p)$$

$$\wedge \quad L\left[6\, e^{-t} \cos t\right] = \frac{6(p+1)}{(p+1)^2 + 1}, \quad \text{non } y(0) = y'(0) = 0$$

$$\left(p^2-2p+2\right)Y(p)=\frac{6\left(p+1\right)}{\left(p+1\right)^2+1}\quad\Rightarrow\quad Y(p)=\frac{6\left(p+1\right)}{\left(\left(p-1\right)^2+1\right)\left(\left(p+1\right)^2+1\right)}$$

Deskonposatuz:

$$\frac{6(p+1)}{\left((p-1)^2+1\right)\left((p+1)^2+1\right)}=\frac{ap+b}{(p-1)^2+1}+\frac{cp+d}{(p+1)^2+1}$$

$$6(p+1)=\left(ap+b\right)\left((p+1)^2+1\right)+\left(cp+d\right)\left((p-1)^2+1\right)$$

$$6p+6=\left(a+c\right)p^3+\left(2a+b-2c+d\right)p^2+\left(2a+2b+2c-2d\right)p+\left(2b+2d\right)$$

$$\begin{cases}0=a+c\\0=2a+b-2c+d\\6=2a+2b+2c-2d\\6=2b+2d\end{cases}\Rightarrow\begin{cases}c=-a\\0=4a+b+d\\3=b-d\\3=b+d\end{cases}\Rightarrow\begin{cases}a=-\dfrac{3}{4}\\b=3\\c=\dfrac{3}{4}\\d=0\end{cases}$$

Beraz:

$$Y(p)=\frac{-\frac{3}{4}p+3}{(p-1)^2+1}+\frac{\frac{3}{4}p}{(p+1)^2+1}=-\frac{3}{4}\left(\frac{p-4}{(p-1)^2+1}\right)+\frac{3}{4}\left(\frac{p}{(p+1)^2+1}\right)$$

$$Y(p)=-\frac{3}{4}\left(\frac{p-1-3}{(p-1)^2+1}\right)+\frac{3}{4}\left(\frac{p+1-1}{(p+1)^2+1}\right)$$

$$Y(p)=-\frac{3}{4}\left(\frac{p-1}{(p-1)^2+1}\right)+\frac{9}{4}\left(\frac{1}{(p-1)^2+1}\right)+\frac{3}{4}\left(\frac{p+1}{(p+1)^2+1}\right)-\frac{3}{4}\left(\frac{1}{(p+1)^2+1}\right)$$

Alderantziko transformatua kalkulatuz:

$$y(t)=-\frac{3}{4}\left(e^t\cos t\right)+\frac{9}{4}\left(e^t\sin t\right)+\frac{3}{4}\left(e^{-t}\cos t\right)-\frac{3}{4}\left(e^{-t}\sin t\right)$$

$$y(t)=-\frac{3}{2}\left(\frac{e^t-e^{-t}}{2}\right)\cos t+\frac{3}{2}\left(\frac{e^t-e^{-t}}{2}\right)\sin t+\frac{3}{2}\left(e^t\sin t\right)$$

$$y(t)=-\frac{3}{2}\operatorname{sh}t\cdot\cos t+\frac{3}{2}\operatorname{sh}t\cdot\sin t+\frac{3}{2}\left(e^t\sin t\right)$$

OHAR OROKORRAK: