KALKULUA (INDUSTRIALAK)
AZTERKETA FINALA 2017KO EKAINAREN 16A

1. ORRIA (20 puntu)

A) Era binomikoan adierazi hurrengo ekuazioaren soluzioak: $(3-2i)\cdot z^2-6i-4=0$
Zer nolako erlazioa dago haien artean?

(4 puntu)

Ebazpena

$$(3-2i)\cdot z^2=4+6i \quad\rightarrow\quad z^2=\frac{4+6i}{3-2i}$$

$$z^2=\frac{4+6i}{3-2i}=\frac{(4+6i)(3+2i)}{(3-2i)(3+2i)}=\frac{12+18i+8i-12}{9+4}=\frac{26i}{13}=2i$$

$$z^2=2i \quad\rightarrow\quad z=\sqrt{2i}=\sqrt{2_{\pi/2}}=\left(\sqrt{2}\right)_{(\frac{\pi}{2}+2k\pi)/2} \qquad (k=0,1)$$

$$\begin{cases} z_1=\left(\sqrt{2}\right)_{\pi/4}=\sqrt{2}\left(\cos\frac{\pi}{4}+i\,\mathrm{sen}\frac{\pi}{4}\right)=\sqrt{2}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)=1+i \\ z_2=\left(\sqrt{2}\right)_{5\pi/4}=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\,\mathrm{sen}\frac{5\pi}{4}\right)=\sqrt{2}\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=-1-i \end{cases}$$

Lortutako emaitzak bata bestearen aurkakoak dira.

B) Lortu $y^{(n)}$ hurrengoa jakinda: $y=e^{2x}+\dfrac{1}{(1-x)^2}$

(5 puntu)

Ebazpena

$y^{(n)}$ kalkulatzen da indukzio metodoa erabiliz:

$$y=e^{2x}+\frac{1}{(1-x)^2}=e^{2x}+(1-x)^{-2}$$

$$\begin{cases} y'=2e^{2x}+(-2)(-1)(1-x)^{-3}=2e^{2x}+2!(1-x)^{-3} \\ y''=2^2e^{2x}+(-3)(-2)(-1)(-1)(1-x)^{-4}=2^2e^{2x}+3!(1-x)^{-4} \\ y'''=2^3e^{2x}+(-4)(-3)(-2)(-1)(-1)(-1)(1-x)^{-5}=2^3e^{2x}+4!(1-x)^{-5} \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y^{(n)}=2^ne^{2x}+(n+1)!(1-x)^{-(n+2)} \end{cases}$$

Formula egia da $n=1$ kasurako: $y^{(1)}=2^1e^{2x}+(1+1)!(1-x)^{-(1+2)}=2e^{2x}+2!(1-x)^{-3}$.

$n=k$ kasurako formula betetzen dela suposatuz, $n=k+1$ kasurako ere egia dela frogatu beharra dago, hau da:

$$y^{(k+1)}=2^{k+1}e^{2x}+(k+2)!(1-x)^{-(k+3)}$$

$$y^{(k)}=2^k e^{2x}+(k+1)!(1-x)^{-(k+2)} \xrightarrow{\text{Deribatuz}} y^{(k+1)}=2\cdot 2^k e^{2x}-(k+2)(k+1)!(-1)(1-x)^{-(k+2)}=$$

$$2^{k+1}e^{2x}+(k+2)!(1-x)^{-(k+3)}$$

Beraz:

$$y^{(n)}=2^n e^{2x}+(n+1)!(1-x)^{-(n+2)}$$

C) izan bedi $z=(x-y)\cos\left(\dfrac{y}{x-y}\right)$. Hurrengo adierazpenaren balioa era sinplifikatuan lortu: $x\cdot\dfrac{\partial z}{\partial x}+y\cdot\dfrac{\partial z}{\partial y}$

(5 puntu)

Ebazpena

(x) -rekiko deribatuz:

$$\frac{\partial z}{\partial x}=\cos\left(\frac{y}{x-y}\right)-(x-y)\cdot\operatorname{sen}\left(\frac{y}{x-y}\right)\cdot\left(\frac{-y}{(x-y)^2}\right)=\cos\left(\frac{y}{x-y}\right)+\frac{y}{(x-y)}\cdot\operatorname{sen}\left(\frac{y}{x-y}\right)$$

$$x\cdot\frac{\partial z}{\partial x}=x\cos\left(\frac{y}{x-y}\right)+\frac{x\,y}{(x-y)}\cdot\operatorname{sen}\left(\frac{y}{x-y}\right) \qquad [1]$$

$$\frac{\partial z}{\partial y}=-\cos\left(\frac{y}{x-y}\right)-(x-y)\cdot\operatorname{sen}\left(\frac{y}{x-y}\right)\cdot\left(\frac{(x-y)-y(-1)}{(x-y)^2}\right)=-\cos\left(\frac{y}{x-y}\right)-\frac{x}{(x-y)}\cdot\operatorname{sen}\left(\frac{y}{x-y}\right)$$

$$y\cdot\frac{\partial z}{\partial y}=-y\cos\left(\frac{y}{x-y}\right)-\frac{x\,y}{(x-y)}\cdot\operatorname{sen}\left(\frac{y}{x-y}\right) \qquad [2]$$

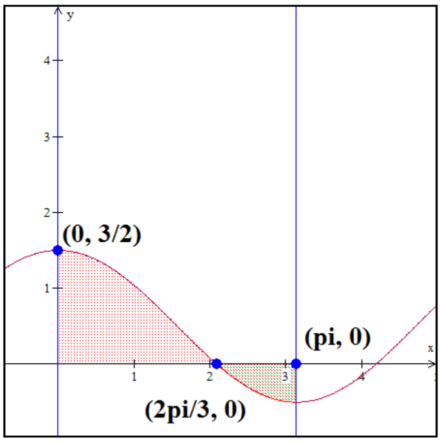
Beraz, [1]+[2] egiten badugu:

$$x\cdot\frac{\partial z}{\partial x}+y\cdot\frac{\partial z}{\partial y}=(x-y)\cos\left(\frac{y}{x-y}\right)+\left(\frac{x\,y}{(x-y)}-\frac{x\,y}{(x-y)}\right)\cdot\operatorname{sen}\left(\frac{y}{x-y}\right)=z$$

D) $y = \frac{1}{2} + \cos x$, absiza-ardatzak eta hurrengo zuzenek $x = 0$ eta $x = \pi$ mugatzen duten azalera kalkulatu.

(6 puntu)

Ebazpena



Bere azalera integralen bidez kalkulatuko dugu:

$$A = \int_0^{\frac{2\pi}{3}} \left(\frac{1}{2} + \cos x\right) dx - \int_{\frac{2\pi}{3}}^{\pi} \left(\frac{1}{2} + \cos x\right) dx = \left(\frac{1}{2}x + \operatorname{sen} x\right)\Bigg]_0^{\frac{2\pi}{3}} - \left(\frac{1}{2}x + \operatorname{sen} x\right)\Bigg]_{\frac{2\pi}{3}}^{\pi} =$$

$$= \left(\frac{\pi}{3} + \operatorname{sen}\frac{2\pi}{3}\right) - \left(\frac{\pi}{2} - \frac{\pi}{3} - \operatorname{sen}\frac{2\pi}{3}\right) = \frac{2\pi}{3} - \frac{\pi}{2} + 2\frac{\sqrt{3}}{2} = \frac{\pi}{6} + \sqrt{3} \quad u^2$$

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2. ORRIA (20 puntu)

A) Izan bitez $O = (0,0)$, $A = (0,2)$, $B = (2,2)$, $C = (4,2)$ eta $E = (6,0)$ puntuak. $[D]$ domeinua hurrengo eran mugatuta dago:

- $\overline{OA}$ zuzenaren segmentua zeinak *O* eta *A* puntuak lotzen dituen.
- $\overline{AB}$ zuzenaren segmentua zeinak *A* eta *B* puntuak lotzen dituen.
- $(3,2)$ zentrodun eta 1 erradiodun zirkunferentziaren beheko erdi-zirkulua.
- $\overline{CE}$ zuzenaren segmentua zeinak *C* eta *E* puntuak lotzen dituen.
- $\overline{EO}$ zuzenaren segmentua zeinak *E* eta *O* puntuak lotzen dituen.

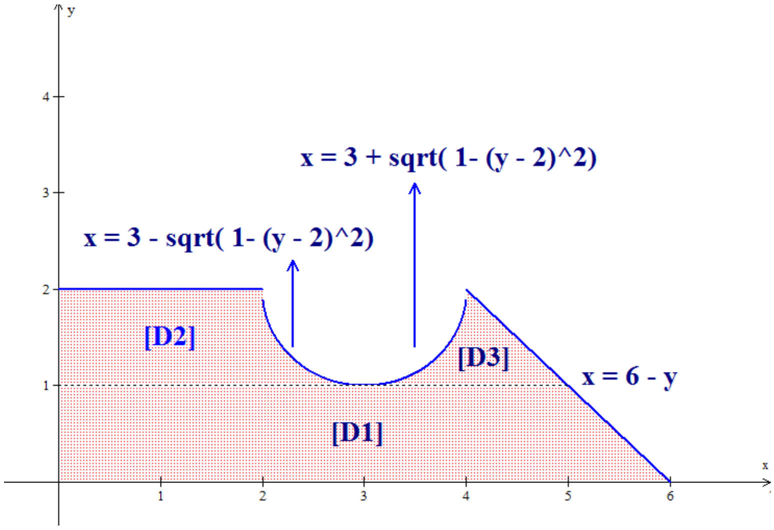
1.- $I = \iint_{[D]} f(x,y)\,dx\,dy$ integralean integrazio-limiteak bi era desberdinetan planteatu.

2.- Integral bikoitzak erabiliz, $[D]$ domeinu lauaren azalera kalkulatu, eta emaitza egiaztatu oinarrizko geometria erabiliz.

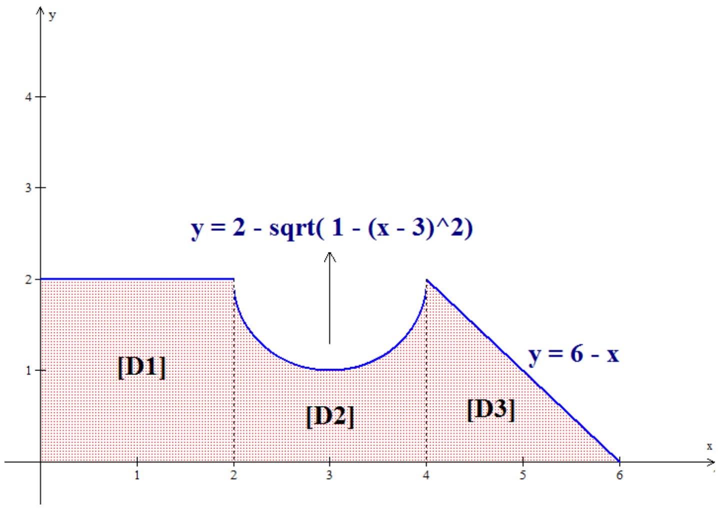
(6 puntu)

Ebazpena

Integralaren limiteak



$$\iint_D f(x,y)\,dx\,dy = \int_0^1 dy \int_0^{6-y} f(x,y)\,dx + \int_1^2 dy \int_0^{3-\sqrt{1-(y-2)^2}} f(x,y)\,dx + \int_1^2 dy \int_{3+\sqrt{1-(y-2)^2}}^{6-y} f(x,y)\,dx$$



$$\iint_D f(x,y)\,dx\,dy = \int_0^2 dx \int_0^2 f(x,y)\,dy + \int_2^4 dx \int_0^{2-\sqrt{1-(x-3)^2}} f(x,y)\,dy + \int_4^6 dx \int_0^{-x+6} f(x,y)\,dy$$

Azaleraren kalkulua

$$A = \iint_D dx\,dy = \int_0^2 dx \int_0^2 dy + \int_2^4 dx \int_0^{2-\sqrt{1-(x-3)^2}} dy + \int_4^6 dx \int_0^{-x+6} dy =$$

$$= \int_0^2 2\,dx + \int_2^4 \left(2-\sqrt{1-(x-3)^2}\right)dx + \int_4^6 (-x+6)\,dx =$$

$$\underset{*}{=} 2x\Big]_0^2 + \left(2x - \frac{1}{2}\left((x-3)\sqrt{1-(x-3)^2} + \operatorname{arcsen}(x-3)\right)\right)\Bigg]_2^4 + \left(-\frac{x^2}{2} + 6x\right)\Bigg]_4^6 =$$

$$= 4 + \left(\left(8 - \frac{\pi}{4}\right) - \left(4 + \frac{\pi}{4}\right)\right) + \left((-18+36) - (-8+24)\right) = 10 - \frac{\pi}{2} \quad u^2$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2 \operatorname{arcsen}\frac{x}{a}\right) + C \qquad (*)$$

Geometrikoki

$$A_D = A_{\text{cuadrado}} + \left(A_{\text{cuadrado}} - \frac{1}{2}A_{\text{círculo}}\right) + A_{\text{triángulo}} = 2\cdot 2 + \left(2\cdot 2 - \frac{\pi\cdot 1^2}{2}\right) + \frac{2\cdot 2}{2} = 10 - \frac{\pi}{2} \quad u^2$$

B) Hurrengo EDA sailkatu eta ebatzi: $\left(\dfrac{2x}{x^2+y^2+1} - 2y\right)dx + \left(\dfrac{2y}{x^2+y^2+1} - 2x\right)dy = 0$

(4 puntu)

<u>*Ebazpena*</u>

$$\begin{cases} X(x,y) = \dfrac{2x}{x^2+y^2+1} - 2y \\ Y(x,y) = \dfrac{2y}{x^2+y^2+1} - 2x \end{cases}$$

$$\frac{\partial X}{\partial y} = \frac{-4xy}{(x^2+y^2+1)^2} - 2 = \frac{\partial Y}{\partial x}$$

Beraz, EDA **zehatza** da.

Ebazpen orokorra hurrengoa da:

$$\int_a^x \left(\frac{2x}{x^2+y^2+1} - 2y\right)dx + \int_b^y \left(\frac{2y}{a^2+y^2+1} - 2a\right)dy = C$$

Kalkuluak sinplifikatzeko hurrengo erabiliko dugu: $a = 0$; $b = 0$:

$$\int_0^x \left(\frac{2x}{x^2+y^2+1} - 2y\right)dx + \int_0^y \frac{2y}{y^2+1}dy = C$$

$$\left[\ln\left|x^2+y^2+1\right| - 2xy\right]_0^x + \left[\ln\left|y^2+1\right|\right]_0^y = C$$

$$\ln\left|x^2+y^2+1\right|-2xy-\ln\left|y^2+1\right|+\ln\left|y^2+1\right|=C \quad \rightarrow \quad \boxed{\ln\left|x^2+y^2+1\right|-2xy=C}$$

C) Hurrengo EDA ebatzi**:** $y''-y=xe^x$

(5 puntu)

<u>Ebazpena</u>

1.- Koefiziente indeterminatuen metodoa

<u>Elkartutako ekuazio homogeneoaren soluzio orokorra</u>:

$$r^2-1=0 \quad \rightarrow \quad r=\pm 1 \quad \Rightarrow \quad y_h=C_1e^{-x}+C_2e^x$$

<u>Ekuazio osoaren soluzio partikularra:</u>

$$f(x)=xe^x \quad \rightarrow \quad Y=x(Ax+B)e^x$$

$x(Ax+b)e^x$ erabiliko dugu $(Ax+B)e^x$ erabili beharrean. Horrela, ekuazio homogeneoaren soluzioetako batekin sortuko litzatekeen bikoiztasuna ekiditen da. (Y)-ren koefizienteak identifikatzeko, (Y) eta bere deribatuak ekuazio osoan ordezkatzen dira.:

$$Y=(Ax^2+Bx)e^x \quad \rightarrow \quad Y'=(2Ax+B)e^x+(Ax^2+Bx)e^x=[Ax^2+(2A+B)x+B]e^x \quad \rightarrow$$

$$Y''=(2Ax+2A+B)e^x+[Ax^2+(2A+B)x+B]e^x=[Ax^2+(4A+B)x+(2A+2B)]e^x$$

$$Y''-Y=[Ax^2+(4A+B)x+(2A+2B)]e^x-(Ax^2+Bx)e^x=(4Ax+2A+2B)e^x\equiv x\,e^x$$

$$\rightarrow \quad \begin{cases} 4A=1 \\ 2A+2B=0 \end{cases} \quad \rightarrow \quad \begin{cases} A=1/4 \\ B=-1/4 \end{cases} \quad \rightarrow \quad Y(x)=\left(\frac{x^2}{4}-\frac{x}{4}\right)e^x$$

<u>Ekuazio osoaren soluzio orokorra</u>:

$$y=y_h+Y \quad \Rightarrow \quad \boxed{y(x)=C_1e^{-x}+C_2e^x+\frac{1}{4}(x^2-x)e^x}$$

2.- Parametroen aldakuntzaren metodoa

Parametroen aldakuntzaren metodoa erabiliz, ekuazio osoaren soluzio orokorra planteatzen da:

$$y=L_1(x)e^{-x}+L_2(x)e^x$$

(L_1') eta (L_2') hurrengo sisteman egonda:

$$\begin{cases} e^{-x}\, L_1'(x) + e^{x}\, L_2'(x) = 0 \\ -e^{-x}\, L_1'(x) + e^{x}\, L_2'(x) = x\, e^{x} \end{cases} \qquad \Rightarrow$$

$$L_1'(x) = \frac{\begin{vmatrix} 0 & e^{x} \\ xe^{x} & e^{x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix}} = \frac{-x\, e^{2x}}{1+1} = -\frac{x}{2} e^{2x}$$

$$L_2'(x) = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & xe^{x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix}} = \frac{x}{2}$$

$\int$ eragilea aplikatuz:

$$\boxed{L_1(x)} = \int L_1'(x)\, dx = -\frac{1}{2}\int xe^{2x} dx = \left\{ \begin{matrix} x = u & \Rightarrow & du = dx \\ e^{2x}\, dx = dv & \Rightarrow & v = e^{2x}/2 \end{matrix} \right\} =$$

$$= -\frac{1}{2}\left[\frac{x}{2} e^{2x} - \frac{1}{2}\int e^{2x} dx \right] = \boxed{-\frac{x}{4} e^{2x} + \frac{e^{2x}}{8} + A}$$

$$\boxed{L_2(x)} = \int L_2'(x)\, dx = \int \frac{x}{2} dx = \boxed{\frac{x^2}{4} + B}$$

Soluzio orokorra lortzen dugu:

$$\boxed{y} = L_1(x) e^{-x} + L_2(x) e^{-x} = \left[\left(-\frac{x}{4} + \frac{1}{8} \right) e^{2x} + A \right] \cdot e^{-x} + \left(\frac{x^2}{4} + B \right) \cdot e^{x} =$$

$$= Ae^{-x} + Be^{x} + e^{x} \left(-\frac{x}{4} + \frac{1}{8} + \frac{x^2}{4} \right) = \boxed{Ae^{-x} + Ke^{x} + \frac{1}{4}(x^2 - x) e^{x}}$$

D) Hurrengo ekuazio diferentzialen sistema ebatzi**:**

$$\begin{cases} x'(t) = 2x(t) - 3y(t) \\ y'(t) = y(t) - 2x(t) \end{cases} \qquad x(0) = 8,\ y(0) = 3$$

(5 puntu)

Ebazpena

Laplace eragilea aplikatuz eta hasierako baldintzak kontuan hartuz:

$$\begin{cases} pX(p) - x(0) = 2X(p) - 3Y(p) \\ pY(p) - y(0) = Y(p) - 2X(p) \end{cases} \rightarrow \begin{cases} (p-2)X(p) + 3Y(p) = 8 \\ 2X(p) + (p-1)Y(p) = 3 \end{cases}$$

Kramer-en erregela erabiliz sistema ebazteko:

$$X(p) = \frac{\begin{vmatrix} 8 & 3 \\ 3 & p-1 \end{vmatrix}}{\begin{vmatrix} p-2 & 3 \\ 2 & p-1 \end{vmatrix}} = \frac{8p-8-9}{(p-1)(p-2)-6} = \frac{8p-17}{p^2-3p-4} = \frac{8p-17}{(p+1)(p-4)}$$

Frakzio sinpleetan deskonposatuz:

$$\frac{8p-17}{(p+1)(p-4)} = \frac{A}{p+1} + \frac{B}{p-4}$$

$$8p - 17 \equiv A(p-4) + B(p+1) \quad \rightarrow \quad \begin{cases} p = -1 \ : \ -25 = -5A \\ p = 4 \ : \ 15 = 5B \end{cases} \rightarrow \begin{cases} A = 5 \\ B = 3 \end{cases}$$

$$\boxed{x(t)} = \mathfrak{L}^{-1}\left[X(p)\right] = 5 \cdot \mathfrak{L}^{-1}\left[\frac{1}{p+1}\right] + 3 \cdot \mathfrak{L}^{-1}\left[\frac{1}{p-4}\right] = \boxed{5e^{-t} + 3e^{4t}}$$

$$Y(p) = \frac{\begin{vmatrix} p-2 & 8 \\ 2 & 3 \end{vmatrix}}{(p+1)(p-4)} = \frac{3p-6-16}{(p+1)(p-4)} = \frac{3p-22}{(p+1)(p-4)}$$

Frakzio sinpleetan deskonposatuz:

$$\frac{3p-22}{(p+1)(p-4)} = \frac{C}{p+1} + \frac{D}{p-4}$$

$$3p-22 \equiv C(p-4)+D(p+1) \quad \rightarrow \quad \begin{cases} p=-1 & : \quad -25=-5C \\ p=4 & : \quad -10=5D \end{cases} \quad \rightarrow \quad \begin{cases} C=5 \\ D=-2 \end{cases}$$

$$\boxed{y(t)} = \mathfrak{L}^{-1}\left[Y(p)\right] = 5\cdot\mathfrak{L}^{-1}\left[\frac{1}{p+1}\right] - 2\cdot\mathfrak{L}^{-1}\left[\frac{1}{p-4}\right] = \boxed{5e^{-t}-2e^{4t}}$$