KALKULUA (INDUSTRIALAK)
AZTERKETA PARTZIALA 2017KO URTARRILAREN 13A

1. ORRIA **(200 puntu)**

A1) Izan bedi $w=\frac{z-2}{z+4}$, non $z,w\in\mathbb{C}$. w-k deskribatutako leku geometrikoa aurkitu hurrengo kasuetan:

1) w zenbaki erreala da
2) argumentua$(w)=\frac{\pi}{6}$

(40 p)

Ebazpena

Izan bedi $z=x+yi \;\rightarrow\; \begin{cases} z-2=(x-2)+yi \\ z+4=(x+4)+yi \end{cases}$

$$w=\frac{z-2}{z+4}=\frac{(x-2)+yi}{(x+4)+yi}=\frac{[(x-2)+yi][(x+4)-yi]}{[(x+4)+yi][(x+4)-yi]}=\frac{x^2+y^2+2x-8+6yi}{(x+4)^2+y^2}=$$

$$=\frac{x^2+y^2+2x-8}{(x+4)^2+y^2}+\frac{6y}{(x+4)^2+y^2}i=A+Bi$$

1) w zenbaki erreala da: $B=0 \;\rightarrow\; 6y=0 \;\rightarrow\; \boxed{y=0}$ (absiza-ardatza)

2) argumentua$(w)=\frac{\pi}{6}$:

$$\frac{6y}{x^2+y^2+2x-8}=\operatorname{tg}\frac{\pi}{6}=\frac{\sqrt{3}}{3} \;\rightarrow\; x^2+y^2+2x-8=\frac{18y}{\sqrt{3}}=6\sqrt{3}y$$

$$(x+1)^2+(y-3\sqrt{3})^2-1-27-8=0 \;\rightarrow\; \boxed{(x+1)^2+(y-3\sqrt{3})^2=36}$$

$(-1,3\sqrt{3})$ zentrodun eta 6 erradioko zirkunferentzia.

A2) Zehaztu $x \in \mathbb{R}$ hurrengoa bete dadin: $\ln(\mathrm{sh}^4 x - 4\,\mathrm{ch}^2 x) = 0$

(30 p)

Ebazpena

$$\ln(\mathrm{sh}^4 x - 4\,\mathrm{ch}^2 x) = 0 \quad \Rightarrow \quad \mathrm{sh}^4 x - 4\,\mathrm{ch}^2 x = 1$$

Trigonometria hiperbolikoaren oinarrizko formula kontuan hartuz:

$$\mathrm{ch}^2 x - \mathrm{sh}^2 x = 1$$

$$\mathrm{sh}^4 x - 4\,\mathrm{ch}^2 x - 1 = \mathrm{sh}^4 x - 4\,\mathrm{sh}^2 x - 5 = 0 \quad \Rightarrow$$

$$\mathrm{sh}^2 x = \frac{4 \pm \sqrt{16+20}}{2} = \frac{4 \pm 6}{2} = \begin{cases} 5 \\ -1 \quad \text{(absurdoa)} \end{cases}$$

$$\begin{cases} \mathrm{sh}\, x = \sqrt{5} & \Rightarrow \quad \boxed{x = \arg\mathrm{sh}(\sqrt{5}) = \ln(\sqrt{5} + \sqrt{6})} \\ \mathrm{sh}\, x = -\sqrt{5} & \Rightarrow \quad \boxed{x = \arg\mathrm{sh}(-\sqrt{5}) = \ln(-\sqrt{5} + \sqrt{6})} \end{cases}$$

B) Hurrengo eran definitutako funtzioa izanda: $y^x = x\cos(y)$. Lortu $\dfrac{dy}{dx}$

(50 p)

Ebazpena

Logaritmoak aplikatuz:

$$y^x = x\cos(y) \Rightarrow x\ln(y) = \ln\left(x\cos(y)\right) \Rightarrow x\ln(y) = \ln(x) + \ln\left(\cos(y)\right)$$

Inplizituki deribatuz:

$$\ln(y) + x\frac{y'}{y} = \frac{1}{x} - \frac{y'\,\mathrm{sen}(y)}{\cos(y)}$$

$$y'\left(\frac{x}{y} + \frac{\mathrm{sen}(y)}{\cos(y)}\right) = \frac{1}{x} - \ln(y)$$

$$y'\left(\frac{x\cos(y) + y\sin(y)}{y\cos(y)}\right) = \frac{1 - x\ln(y)}{x}$$

$$\boxed{y' = \frac{y\cos(y)\left(1 - x\ln(y)\right)}{x\left(x\cos(y) + y\sin(y)\right)}}$$

MATEMATIKA APLIKATUA

C1) Lortu eta grafikoki adierazi hurrengo funtzioaren domeinua:

$$f(x,y)=\ln\left(x^2-y^2+2x\right)$$

(30 p)

Ebazpena

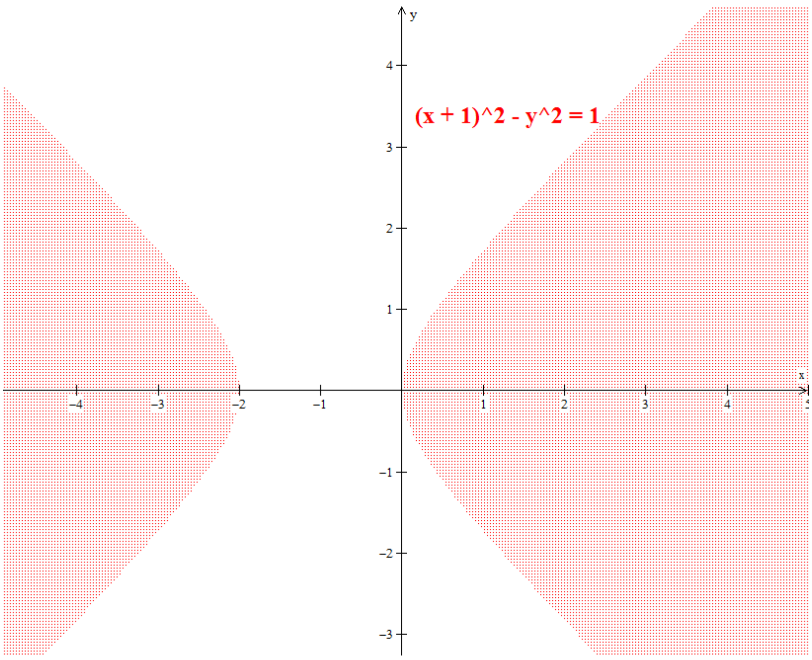
Logaritmo nepertarra existitu dadin:

$$x^2-y^2+2x>0 \quad \Rightarrow \quad (x+1)^2-y^2>1$$

Beraz, funtzioaren domeinua hurrengoa da:

$$D=\left\{(x,y)\in\mathbb{R}^2 \,/\, (x+1)^2-y^2>1\right\}$$

$(x+1)^2-y^2=1$ ekuazioak $(-1,0)$ puntuan zentroa duen hiperbola adierazten du. Beraz, domeinua grafikoki adieraziz:



C2) Hurrengo funtzioa emanda $z=(x-y)\cdot\cos\left(\dfrac{y}{x-y}\right)$, era sinplifikatuan lortu $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}$ adierazpenaren balioa:

(50 p)

Ebazpena

$$\frac{\partial z}{\partial x}=\cos\left(\frac{y}{x-y}\right)-(x-y)\operatorname{sen}\left(\frac{y}{x-y}\right)\cdot\left(\frac{-y}{(x-y)^2}\right)=\cos\left(\frac{y}{x-y}\right)+\frac{y}{(x-y)}\operatorname{sen}\left(\frac{y}{x-y}\right)$$

$$\frac{\partial z}{\partial y}=-\cos\left(\frac{y}{x-y}\right)-(x-y)\operatorname{sen}\left(\frac{y}{x-y}\right)\cdot\left(\frac{x-y-y(-1)}{(x-y)^2}\right)=-\cos\left(\frac{y}{x-y}\right)-\frac{x}{(x-y)}\operatorname{sen}\left(\frac{y}{x-y}\right)$$

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\cos\left(\frac{y}{x-y}\right)\cdot(1-1)+\frac{1}{(x-y)}\operatorname{sen}\left(\frac{y}{x-y}\right)(y-x)=-\operatorname{sen}\left(\frac{y}{x-y}\right)$$

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2. ORRIA (200 puntu)

A) Kalkulatu: $\int \ln(\text{sen}\, x)\cdot \text{sen}\, x\, dx$ $\qquad \int \frac{1}{x^5\cdot\sqrt{1+\frac{1}{x^2}}}\,dx$

(60 p)

Ebazpena

$$I=\int \ln(\text{sen}\, x)\cdot \text{sen}\, x\, dx = \left\| \begin{matrix} u=\ln(\text{sen}\, x) & du=\frac{\cos x}{\text{sen}\, x}dx \\ dv=\text{sen}\, x\, dx & v=-\cos x \end{matrix} \right\| =$$

$$=-\ln(\text{sen}\, x)\cdot\cos x+\int\frac{\cos^2 x}{\text{sen}\, x}dx=-\ln(\text{sen}\, x)\cdot\cos x+J$$

$$J=\int\frac{\cos^2 x}{\text{sen}\, x}dx=\int\frac{1-\text{sen}^2 x}{\text{sen}\, x}dx=\int\left(\frac{1}{\text{sen}\, x}-\text{sen}\, x\right)dx=\int\frac{1}{\text{sen}\, x}dx-\int\text{sen}\, x\, dx=H+\cos x$$

$$H=\int\frac{1}{\text{sen}\, x}dx=\left\| \begin{matrix} t=\text{tg}\frac{x}{2} & dx=\frac{2}{1+t^2}dt \\ \text{sen}\, x=\frac{2t}{1+t^2} & \end{matrix} \right\| =\int\frac{1+t^2}{2t}\cdot\frac{2}{1+t^2}dt=\int\frac{1}{t}dt=$$

$$=\ln|t|+K=\ln\left|\text{tg}\frac{x}{2}\right|+K_1$$

$$\boxed{I}=-\ln(\text{sen}\, x)\cdot\cos x+J=-\ln(\text{sen}\, x)\cdot\cos x+\cos x+H=$$

$$=\boxed{-\ln(\text{sen}\, x)\cdot\cos x+\cos x+\ln\left|\text{tg}\frac{x}{2}\right|+K}$$

$$I=\int\frac{1}{x^5\cdot\sqrt{1+x^{-2}}}dx=\int x^{-5}\left(1+x^{-2}\right)^{-1/2}dx=\begin{bmatrix} m=-5 & n=-2 \\ p=-\frac{1}{2}\notin\mathbb{Z} & \frac{m+1}{n}=2\in\mathbb{Z} \end{bmatrix}=\begin{pmatrix}\text{binomia}\\ 2^{\text{o}}\ \text{caso}\end{pmatrix}=$$

$$= \left\| \begin{matrix} x^{-2} = t \to x = t^{-1/2} \\ dx = -\frac{1}{2} t^{-3/2} dt \end{matrix} \right\| = \int t^{5/2} (1+t)^{-1/2} \left(-\frac{1}{2} \right) t^{-3/2} dt = -\frac{1}{2} \int t (1+t)^{-1/2} \, dt =$$

$$= \left\| \begin{matrix} 1+t = z^2 \\ dt = 2z \, dz \end{matrix} \right\| = -\frac{1}{2} \int \left(z^2 - 1 \right) z^{-1} \, 2z \, dz = -\int \left(z^2 - 1 \right) dz =$$

$$= z - \frac{z^3}{3} + K = (1+t)^{1/2} - \frac{(1+t)^{3/2}}{3} + K = \boxed{\left(1+x^{-2}\right)^{1/2} - \frac{\left(1+x^{-2}\right)^{3/2}}{3} + K}$$

B) Zehaztu a konstante erreal positiboaren balioa, hurrengo domeinuak

$$D = \left\{ (x,y) \in \mathbb{R}^2 \, / \, x^2 + y - 4 \leq 0 \ \wedge \ y \geq a \cdot x^2 \ \wedge \ y \geq 0 \right\}$$

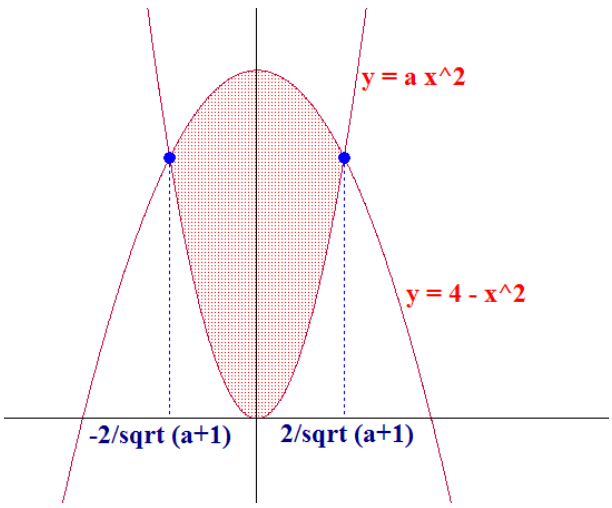
definitutako azalera $A = \frac{16}{3} \quad u^2$ izan dadin.

(80 p)

Ebazpena

Bi parabolak ditugu ardatz bertikala dutenak. Ebakidura puntuak hurrengoak dira:

$$\left. \begin{matrix} y = a \, x^2 \\ y = 4 - x^2 \end{matrix} \right\} \Rightarrow \quad x = \pm \frac{2}{\sqrt{a+1}}$$



Simetria kontuan izanda, azalera horrela kalkulatuko dugu:

$$A = 2 \left[\int_0^{\frac{2}{\sqrt{a+1}}} \left(4 - x^2\right) dx - \int_0^{\frac{2}{\sqrt{a+1}}} \left(a x^2\right) dx \right] = 2 \left[\left(4x - \frac{a+1}{3} x^3 \right) \Bigg|_0^{\frac{2}{\sqrt{a+1}}} \right] = \frac{16}{3} \quad \to$$

$$\frac{8}{\sqrt{a+1}}-\frac{8}{3\sqrt{a+1}}=\frac{8}{3} \quad\rightarrow\quad \frac{2}{\sqrt{a+1}}=1 \quad\rightarrow\quad 2=\sqrt{a+1} \quad\rightarrow\quad \boxed{a=3}$$

C) Hurrengo integral inpropioak kalkulatu:

$$\int_4^{\infty}\frac{1}{x(\ln x)^2}\,dx \qquad\qquad \int_0^{6}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx$$

(60 p)

Ebazpena

$$\int_4^{\infty}\frac{1}{x(\ln x)^2}\,dx=\lim_{b\to\infty}\int_4^{b}\frac{1}{x(\ln x)^2}\,dx \underset{*}{=} \lim_{b\to\infty}\left[-\frac{1}{\ln b}+\frac{1}{\ln 4}\right]=\frac{1}{\ln 4}$$

$$\int_4^{b}\frac{1}{x(\ln x)^2}\,dx=\left[\ln x=t \quad \frac{1}{x}dx=dt\right]=\int_{\ln 4}^{\ln b}\frac{1}{t^2}\,dt=-\frac{1}{t}\Big|_{\ln 4}^{\ln b}=-\frac{1}{\ln b}+\frac{1}{\ln 4} \qquad (*)$$

$$\int_0^{6}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx=\int_0^{2}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx+\int_2^{6}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx=$$

$$=\lim_{\varepsilon\to 0}\left[\int_0^{2-\varepsilon}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx+\int_{2+\varepsilon}^{6}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx\right]\underset{*}{=}$$

$$=\lim_{\varepsilon\to 0}\left[3\left(\sqrt[3]{(2-\varepsilon)^2-4}-\sqrt[3]{-4}\right)+3\left(\sqrt[3]{32}-\sqrt[3]{(2+\varepsilon)^2-4}\right)\right]=3\left(\sqrt[3]{4}+\sqrt[3]{32}\right)=3\left(\sqrt[3]{4}+2\sqrt[3]{4}\right)=9\sqrt[3]{4}$$

$$\int_0^{2-\varepsilon}\frac{2x}{\left(x^2-4\right)^{2/3}}\,dx=3\left(x^2-4\right)^{1/3}\Big|_0^{2-\varepsilon}=3\left[\sqrt[3]{(2-\varepsilon)^2-4}-\sqrt[3]{-4}\right] \qquad (*)$$

$$\int_{2+\varepsilon}^{6} \frac{2x}{\left(x^2-4\right)^{2/3}}\,dx = 3\left(x^2-4\right)^{1/3}\Big|_{2+\varepsilon}^{6} = 3\left[\sqrt[3]{32}-\sqrt[3]{\left(2+\varepsilon\right)^2-4}\right] \quad (*)$$