Integral lerromakurrak

Funtzio eskalarra

funtzio edo aplikazio $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$

Adibidea: $\vec{z} = (x, y, z)$

$w = f(x,y,z) = x^2 + y^2 + z^2 \quad \mathbb{R}^3 \rightarrow \mathbb{R}$

$z = f(x,y) = x + y \quad \mathbb{R}^2 \rightarrow \mathbb{R}$

bi aldagai errealeko funtzio erreala

$f(x) = x^2 \qquad y = f(x) \qquad \mathbb{R} \rightarrow \mathbb{R}$

Funtzio bektoriala

$\vec{f} : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$n = 1, 2 \ldots 3$ (nahi duguna)

Adibidea

$\vec{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \qquad \vec{f}(\vec{w}) = \vec{h} \quad \Rightarrow \quad f(x,y,z) = (x+y, \ 2yz)$

$\vec{w} \in \mathbb{R}^3 \quad ; \quad \vec{h} \in \mathbb{R}^2$

$\vec{w} = (x, y, z) \quad ; \quad \vec{h}\,(x+y, \ 2yz)$

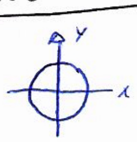
bikote ordenatua izan behar da

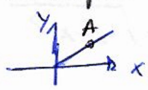
$\vec{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$\vec{f}(x, y) = (x+y, \ x-y)$

Kurben eta gainazalen parametrizazioa

Kurbak

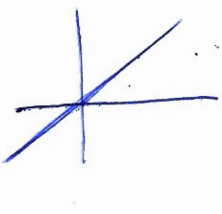




$A(x_1, y_1)$ bikote ordenatua

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

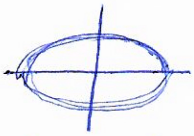
$x = t$
$y = t$ $\quad (x, y) = (t, t)$



bikotea t-ren menpe parametrizatu du.

Adibideak

$\cdot\; y = f(x) \quad \begin{cases} x = x \\ y = f(x) \end{cases}$ $\qquad\qquad$ $\cdot\; x = f(y) \quad \begin{cases} x = f(y) \\ y = y \end{cases}$



$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \quad \text{(ELIPSE)}$$

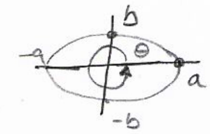
PARAMETRIZAZIO GEOMETRIKOA

(Koordenatu polarrak orokortua)

nos $\rho = 1$ den

$$\left.\begin{array}{l} x(\theta) = x = x_0 + a\cos\theta \\ y(\theta) = y = y_0 + b\sin\theta \end{array}\right\} \theta \in [0, 2\pi]$$

$\theta = \frac{\pi}{2} \left\{ \begin{array}{l} x = 0 \\ y = b \end{array} \right\} 0, b$

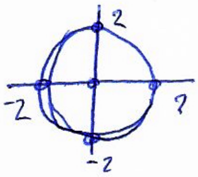


$\theta = \emptyset \left\{ \begin{array}{l} x = a \\ y = 0 \end{array} \right\} (a, 0)$

Adibidea

$x^2 + y^2 = 4$ Zirkunferentzia

$R = 2 \quad C(0,0)$



Parametrizazio geometrikoa

$x = x(\theta) = 2\cos\theta$

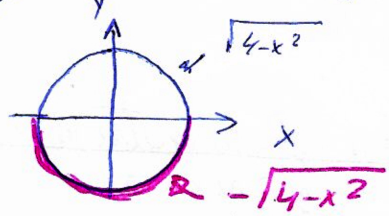
$y = y(\theta) = 2\sin\theta \qquad \theta \in [0, 2\pi]$

θ	0	$\pi/2$	π	$3\pi/2$	2π
$x = 2\cos\theta$	2	0	-2	0	2
$y = 2\sin\theta$	0	2	0	-2	0

Parametrizazio algebraiko

$x^2 + y^2 = 4 \Longrightarrow y = \pm\sqrt{4-x^2} \quad \begin{cases} x = x \\ y = y(x) = \pm\sqrt{4-x^2} \end{cases}$

$\left.\begin{array}{l} y = y \\ x = x(y) \end{array}\right\} \boxed{x = \pm\sqrt{4-y^2}}$



VI Gainazalak

$$\begin{cases} x = x(\lambda, \mu) \\ y = y(\lambda, \mu) \\ z = z(\lambda, \mu) \end{cases} \qquad z = f(x,y) \begin{cases} x = x \\ y = y \\ z = f(x,y) \end{cases}$$ PARAMETRIZAZIO ALGEBRAIKOA

a = b = c = 1 badira esferaren formula da

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1 \qquad \begin{cases} x = x_0 + a\cos\theta \sin\varphi \\ y = y_0 + b\sin\theta \sin\varphi \\ z = z_0 + c\cos\varphi \end{cases}$$

$\boxed{\text{koordenatu esferiko orokortuak}}$

$$\frac{\cancel{a^2}\cos^2\theta \sin^2\varphi}{\cancel{a^2}} + \frac{\cancel{b^2}\sin^2\theta\sin^2\varphi}{\cancel{b^2}} + \frac{\cancel{c^2}\cos^2\varphi}{\cancel{c^2}} = 1$$

$$\sin^2\varphi\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + \cos^2\varphi = 1$$

$$\sin^2\varphi + \cos^2\varphi = 1$$

$$1 = 1 \checkmark$$

XOY ardatzean
$\theta \in [0, 2\pi]$
$\varphi \in [0, \pi]$

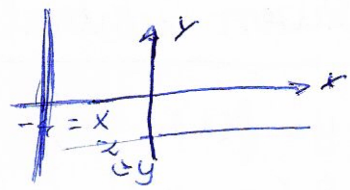
Adibideak

① $y = x+3$

Parametrizatu (algebraikoa)
$x = x$
$y = y(x) = x+3 \quad \forall x \in \mathbb{R} \begin{cases} dx = dx \\ dy = \underbrace{y'(x)}_{1} \cdot dx = dx \end{cases}$

② $y = -2$

Parametrizatu (algebraikoa)
$x = x$
$y = -2 \quad \forall x \in \mathbb{R} \begin{cases} dx = dx \\ dy = \emptyset \cdot dx = \emptyset \end{cases}$



③ $x = -4$

Parametrizazio (algebraikoa)
$\begin{cases} x = -4 \\ y = y \end{cases} \quad \forall y \in \mathbb{R} \begin{cases} dx = \underbrace{x'(y)}_{0} \cdot dy = 0 \\ dy = dy \end{cases}$

④ $\frac{x^2}{9} + y^2 = 1$

Parametrizazioa
$\begin{cases} x = 3\cos\theta \\ y = \sin\theta \end{cases} \quad \forall \theta\,[0, 2\pi] \begin{cases} dx = x'(\theta)\,d\theta = -3\sin\theta\,d\theta \\ dy = y'(\theta)\,d\theta = \cos\theta\,d\theta \end{cases}$

ekuazioan ordezkatu? 1 = 1 geratzen dela konprobatzen dugu

⊛ Kurba motak planoan

Kurba erregularra

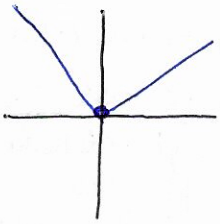
$\sim$ ERREGULARRA

$\wedge$ EZ-ERREGULAR

noiz da erregularra?

Baldin x, y, x' eta y' jarraituak badira. ERPINAK EZ BADITU

$$y = |x|$$



$\Rightarrow$ EZ ERREGULARRA

KURBA ITXIA :

$\bigcirc$ $\begin{cases} \text{NORANZKOA } \ominus \\ \text{NORANZKOA } \oplus \end{cases}$ $\Longrightarrow$ $\triangle$ ← ez da erregularra, zatika erregularra da, itxia ere da.

KURBA SINPLEA :

Puntu anizkoitzak ez dituenean

puntu anizkoitzak

← puntu beratik behin baino gehiagotan ez garenean pasatzen

$\Downarrow$

⚠ ez-sinplea

⊛ Zuzen ukitzailea

$$\left.\begin{array}{l} y = f(x) \\ P_0(x_0, y_0) \end{array}\right\} \quad y - y_0 = \underbrace{f'(x_0)}_{\text{malda}}(x - x_0)$$

$$m = f'(x_0) = y'(x_0)$$

PARAMETRIZATUTA DAGOENEAN

$$x = x(t) \qquad y = y(t)$$

$$t_0 \to \begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$$

$$\frac{y - y_0}{y'(t_0)} = \frac{x - x_0}{x'(t_0)} \longrightarrow y - y_0 = \frac{y'(t_0)}{x'(t_0)}(x - x_0)$$

non $m = \dfrac{y'(t_0)}{x'(t_0)}$

$$\begin{array}{l} y' = \dfrac{dy}{dt} \\ x' = \dfrac{dx}{dt} \end{array} \quad \left\{ \frac{y'}{x'} = \frac{\frac{dy}{\cancel{dt}}}{\frac{dx}{\cancel{dt}}} = \frac{dy}{dx} \right.$$

⊗ Integral lerromakurren kontzeptua

Integral lerromakurra → Funtzio eskalarra / Funtzio bektoriala → C kurba baten gainean integratuko dugu

↓

definitutako egon daiteke

↓ Planoan ↓ Espazioan

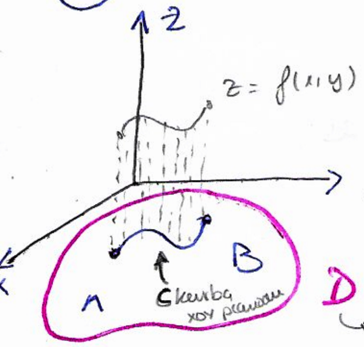
⊗ Funtzio eskalarraren integral lerromakurra planoan definitutako C kurbaren gainean.

$$f: D \subset \mathbb{R}^2 \to \mathbb{R}$$

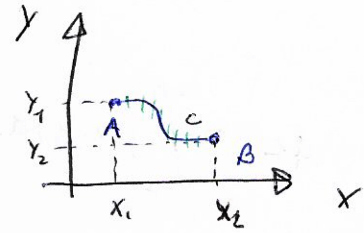
$$(x,y) \to f(x,y) = z$$

↳ bikote ordenatua ↳ z altuera izengo da

C kurba domeinuko A eta B puntuen artean definituta dago.



D ↳ XOY planoan



S_i: kurbaren arku luzera parametroa

Integral lerromakurraren definizioa:

$$\int_C f(x,y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{\infty} f(x_i, y_i)\, \Delta S_i$$

↳ kurba n zatitan zatituko dugu

↳ infinitu ΔS_i ditut

$\max \Delta S_i \to \emptyset$ (zati bakoitzaren luzera 0-ra doa)

Analogiaz hurrengoa esan dezaket:

$$\int_C f(x,y)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,y_i) \cdot \Delta x_i$$

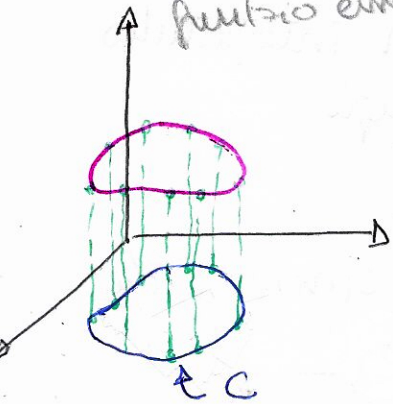
$$\int_C f(x,y)\, dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,y_i)\, \Delta y_i$$

$f: \mathbb{R}^2 \to \mathbb{R} \quad z = f(x,y)$

- KURBA ITXIA

KURBA ERREGULARRA EDO ZATIKA ERREGULARRA

funtzio eskalarra
2 aldagai errealeko
funtzio erreala



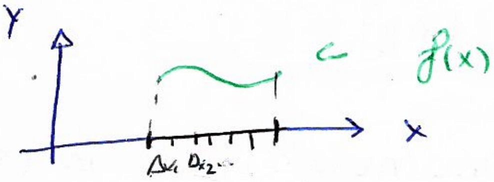
borobila C kurba itxia dela adierazten du

$$\oint_C f(x,y)\,ds$$ Noranzko positibo ⊕

$$\oint_C f(x,y)\,ds$$ Noranzko negativo ⊖

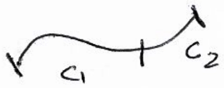
$$\int_C f(x,y)\,ds \to \begin{cases} f(x,y) \geq 0 \longrightarrow \text{'Kortinaren azalera'} \\ f(x,y) = 1 \longrightarrow C \text{ kurbaren luzera} \end{cases}$$

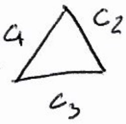
$$\lim_{n\to\infty} \sum_{i=1}^{n} \cancel{f(x_i,y_i)}^{1} \Delta s_i = L$$



⊗ PROPIETATEA

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \int_{C_3} f(x,y)\,ds$$

↳ zatika erregularra den kurba aplikagarria ere →  denean.



⊗ Integral lerromakunaren kalkulua

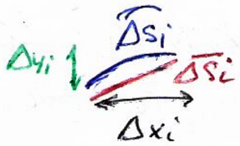
f, funtzio eskalarra, jarraitua C kurban

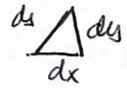
- C ekuazio parametrikoetan definituta, non s kurbaren luzera parametroa

$$C = \begin{cases} x = x(s) \\ y = y(s) \end{cases} \quad 0 \leq s \leq L$$

$$\boxed{I = \int_C f(x,y)\,ds = \int_0^L f[x(s), y(s)]}$$

- $C = \begin{cases} x = x(t) & dx = x'(t)\,dt \\ y = y(t) & dy = y'(t)\,dt \end{cases} \quad t_0 \leq t \leq t_1$





$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(x'(t)\,dt)^2 + (y'(t)\,dt)^2} = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$I = \int_C f(x,y)\,ds =$$

$$\boxed{I = \int_{t_0}^{t_1} f[x(t), y(t)] \cdot \sqrt{(x'(t))^2 + (y'(t))^2} \cdot dt}$$

⊗ Parametrización algebraica:

- $C \begin{cases} x = x & \rightarrow dx = dx \\ y = y(x) & \rightarrow dy = y'(x) \cdot dx \end{cases} \quad x_0 \leq x \leq x_1$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 + (y'(x)\,dx)^2} = \sqrt{1 + (y'(x))^2}\,dx$$

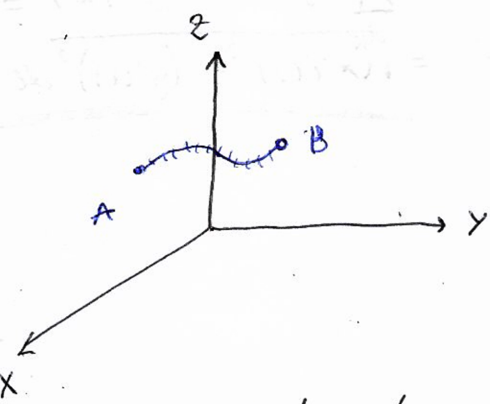
$$\boxed{I = \int_C f(x,y)\,ds = \int_{x_0}^{x_1} f[x, y(x)]\,\sqrt{1 + (y'(x))^2}\,dx}$$

- $C \begin{cases} x = x(y) & \rightarrow dx = x'(y)\,dy \\ y = y & \rightarrow dy = dy \end{cases} \quad y_0 \leq y \leq y_1$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(x'(y) \cdot dy)^2 + (dy)^2} = \sqrt{(x'(y))^2 + 1}\,dy$$

$$\boxed{I = \int_C f(x,y) = \int_{y_0}^{y_1} f[x(y), y] \cdot \sqrt{(x'(y))^2 + 1} \cdot dy}$$

⊛ Funtzio eskalarraren integrala lerromakurra espazioan definitutako C kurbaren gainean



$$\int_C f(x,y,z)\cdot ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,y_i,z_i)\,\Delta s_i$$

$$\max(\Delta s_i) \to 0$$

Integralaren kalkulua

f: funtzio jarraitua C kurban.

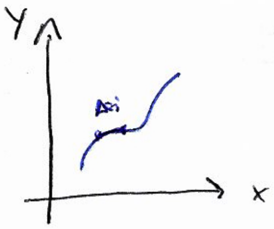
- C parametrizatu, s kurbaren luzera parametroaren arabera

$$C = \begin{cases} x = x(s) \\ y = y(s) \\ z = z(s) \end{cases} \qquad \underline{\underline{0 \le s \le L}}$$

$$\boxed{I = \int_C f(x,y,z)\,ds = \int_0^L f(x(s), y(s), z(s))\cdot ds}$$

- C parametrizatu

$$C = \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \qquad t_0 \le t \le t_1 \qquad I = \int_C f(x,y,z)\,ds = ⊛$$



Norma, bektorearen luzera

$$ds = \sqrt{(dx)^2+(dy)^2+(dz)^2} = \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\cdot dt$$

$$dx = x'(t)\cdot dt$$
$$dy = y'(t)\cdot dt$$
$$dz = z'(t)\cdot dt$$

$$⊛ \quad I = \int_{t_0}^{t_1} f(x(t), y(t), z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

* Parametrizarea algebraică:

- $C = \begin{cases} x = x \\ y = y(x) \\ z = z(x) \end{cases} \quad x_0 \leq x \leq x_1$

$$\boxed{I = \int_C f(x,y,z)\,ds = \int_{x_0}^{x_1} f(x, y(x), z(x))\sqrt{1 + (y'(x))^2 + (z'(x))^2}\,dx}$$

- $C = \begin{cases} x = x(y) \\ y = y \\ z = z(y) \end{cases} \quad y_0 \leq y \leq y_1$

$$\boxed{I = \int_C f(x,y,z)\,ds = \int_{y_0}^{y_1} f(x(y), y, z(y))\sqrt{(x'(y))^2 + 1 + (z'(y))^2}\,dy}$$

- $C = \begin{cases} x = x(z) \\ y = y(z) \\ z = z \end{cases} \quad z_0 \leq z \leq z_1$

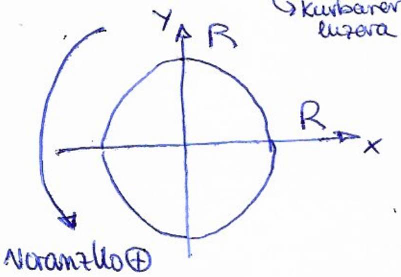
$$\boxed{I = \int_C f(x,y,z)\,ds = \int_{z_0}^{z_1} f(x(z), y(z), z)\cdot\sqrt{(x'(z))^2 + (y'(z))^2 + 1}\,dz}$$

Ariketak

kurbaren luzera kalkulatu. Horretarako integralaren funtzioa = 1 egingo dugu. (*)

① $\underbrace{x^2+y^2=R^2}_{\text{kurbaren luzera}}$ → kurba itxia ⟲

$$\oint_C f(x,y)\,ds$$



1) kurba parametrizatuko dugu (geometrikoa)

$$x = R\cos\theta \qquad dx = -R\sin\theta\,d\theta$$
$$y = R\sin\theta \qquad dy = R\cos\theta\,d\theta$$
$$\theta = [0, 2\pi]$$

$$I = \oint_C f(x,y)\,ds = \int_0^{2\pi} \cancel{f}(x(\theta), y(\theta))\sqrt{(x'(\theta))^2 + (y'(\theta))^2}\,d\theta =$$

(*) 1

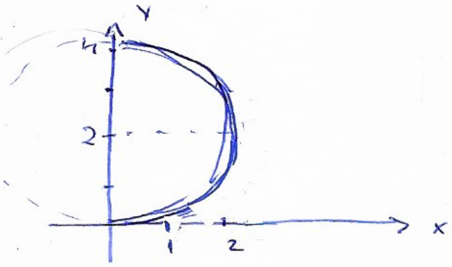
$$= \int_0^{2\pi} \sqrt{R^2\sin^2\theta + R^2\cos^2\theta}\,d\theta = \int_0^{2\pi} R\,d\theta = R[\theta]_0^{2\pi} = \underline{\underline{2\pi R}}$$

② $I = \int_C (x^2-y^2)\,ds$ non $C = \begin{cases} x^2+y^2 = 4y \\ \underline{x \geq 0} \end{cases}$ $\Rightarrow x^2+y^2-4y=0 \Rightarrow x^2+(y-2)^2=4$

$(y-2)^2 = y^2-4y+4$

⇓

Zirkunferentzia $\begin{cases} C(0,2) \\ R=2 \end{cases}$ → (x_0, y_0)



(*) PARAMETRIZAZIO GEOMETRIKOA

$$C = \begin{cases} x = 2\cos\theta \\ y = 2\sin\theta + y_0 = 2\sin\theta + 2 \end{cases}$$

$$\begin{cases} dx = -2\sin\theta\,d\theta \\ dy = 2\cos\theta\,d\theta \end{cases}$$

$\theta \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

parte de abajo del arculo — 1 parte de arriba

$$I = \int_C f(x,y)\,ds = \int_{-\pi/2}^{\pi/2} \left(4\cos^2\theta - (2+2\sin\theta)^2\right)\sqrt{(4\sin^2\theta + 4\cos^2\theta)}\,d\theta =$$

$$= 2\int_{-\frac{\pi}{2}}^{\pi/2} 4\cos^2\theta - (4 + 4\sin^2\theta + 8\sin\theta)\,d\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\underbrace{4\cos^2\theta - 4\sin^2\theta}_{} - 4 - 8\sin\theta)\,d\theta =$$

$(4\cos^2\theta - 4(1-\cos^2\theta) - 4 - 8\sin\theta)$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (8\cos^2\theta - 8 - 8\sin\theta)\,d\theta = 16\int_{-\frac{\pi}{2}}^{\pi/2} \left[\left(\frac{1+\cos 2\theta}{2}\right) - 1 - \sin\theta\right] d\theta = 16\left[\frac{\theta}{2} + \cancel{\frac{\sin 2\theta}{4}} - \theta + \cancel{\cos\theta}\right]_{-\pi/2}^{\pi/2} =$$

$$= 16\left[\frac{\pi}{4} - \frac{\pi}{3} + \frac{4}{\pi} - \frac{\pi}{2}\right] = 16\left[-\frac{\pi}{2}\right] = -8\pi$$

⊛ Funtzio bektorialaren integral lerromakurra planoan dagoen C kurbaren gainean.

t-ren tartea (eremua) $\rightarrow$ $h \leq t \leq k$

DESPLAZAMENDU BEKTOREA
t-ren menpe dago

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} = (x(t), y(t))$

$\hookrightarrow d\vec{r} = \vec{r}\,'(t)\, dt = (x'(t), y'(t))$

FUNTZIO BEKTORIALA

$\vec{F}(x,y) = X(x,y)\hat{i} + Y(x,y)\hat{j} = (X(x,y), Y(x,y))$

(x,y) → bikote ordenatua

$\vec{F}$ funtzio bektorialaren integral lerromakurra planoan dagoen C kurbaren gainean

$$\int_C \underbrace{\vec{F}\cdot d\vec{r}}_{\text{biderkadura eskalarra}} = \int_h^k \vec{F}\cdot \vec{r}\,'(t)\cdot dt = \int_h^k (X, Y) \underset{\text{biderkadura eskalarra}}{\cdot} (x'(t), y'(t))\, dt =$$

$$= \int_h^k \left[X \cdot x'(t) + Y \cdot y'(t)\right] dt = \int_h^k \left(X \cdot \underbrace{x'(t)\cdot dt} + Y \cdot \underbrace{y'(t)\cdot dt}\right) =$$

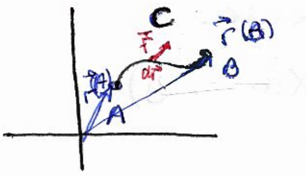
$$= \int_h^k (X\,dx + Y\,dy)$$

$$\boxed{\int_C \vec{F}\, d\vec{r} = \int_h^k \vec{F}\, \vec{r}\,'(t)\cdot dt = \int_C (X\,dx + Y\,dy)}$$

INTERPRETAZIO FISIKOA

$$W = \int_C \vec{F}\, d\vec{r}$$

W: lana
$\vec{F}$: indarra (funtzio bektoriala)



$$\int_{C_{A,B}} \vec{F}\, d\vec{r} = -\int_{C_{B,A}} \vec{F}\, d\vec{r}$$

⊛ Kalkulua egiteko:

$$\boxed{\int_C \vec{F}\, d\vec{r} = \int_C (X\,dx + Y\,dy + Z\,dz)}$$

→ badute ezer esaten hau egingo dugu

↺ ⊕ NORANTZA $\oint$

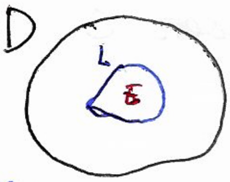
↻ ⊖ NORANTZA $\oint$

Domeinu

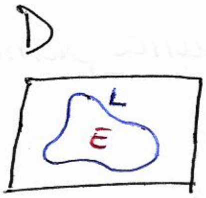
Azpimultzo sinpleki konexua

KONTZEPTUAK

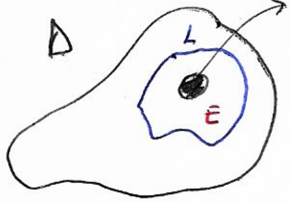
$D \subset \mathbb{R}^2$, sinpleki konexua dela esango dugu, baldin eta D-ren barruan edozein L kurba itxiak sortutako (E) eskualdea D-ren barruan badago.



D SINPLEKI KONEXUA



D SINPLEKI KONEXUA



zuloa, hau da ez dago D-ren barruan

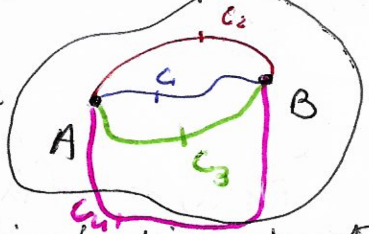
D bikoitzki konexua (doblemente conexo)

Bidearekiko independentzia

(x,y)-ren menpe daudela ere

$\vec{F}(X, Y)$

$\int_C \vec{F}\, d\vec{r} = \int_C (X\,dx + Y\,dy)$ bidearekiko independentea izango da D azpimultzoan baldin eta



$\int_{AB} (X\,dx + Y\,dy)$

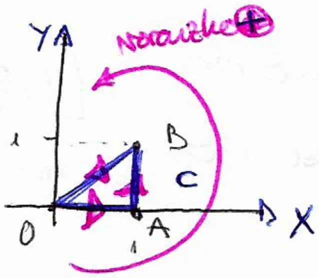
A eta B puntuen artean balio berdina bueltatzen badu edozein bide hartuta.

C_4 ez dago D-ren barne, eta ez dugu ebaluatu behar.

$$\int_{C_1} (X\,dx + Y\,dy) = \int_{C_2} (X\,dx + Y\,dy) = \int_{C_3} (X\,dx + Y\,dy)$$

ARIKETAK

④ kalkulatu $I = \oint_C y^2 dx + x^2 dy$ non $C = \{O(0,0),\ A(1,0)$ eta $B(1,1)\}$ erpinetako hiruki'a den. Noranzko ⊕ hartuko dugu ez diguletako ezer esaten.



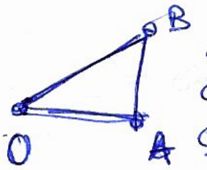
$\int_C \vec{F} d\vec{r}$

$\vec{F} = (X, Y) = (X(x,y), Y(x,y)) \Rightarrow f: \mathbb{R}^2 \to \mathbb{R}^2$ aplikazioa da

$X = y^2$

$Y = x^2$

C kurba zatika erregularra da.



$C_1 = \overline{OA}$, $C_2 = \overline{AB}$, $C_3 = \overline{BO}$

$I = \oint_C y^2 dx + x^2 dy = \underbrace{\int_{C_1} y^2 dx + x^2 dy}_{I_1} + \underbrace{\int_{C_2} y^2 dx + x^2 dy}_{I_2} + \underbrace{\int_{C_3} y^2 dx + x^2 dy}_{I_3}$

⊛ → Ordman

$I = I_1 + I_2 + I_3 = 0 + 1 - \frac{2}{3} = \frac{1}{3}$

I_1 kalkulatuko dugu, gure kurba $\overline{OA}$ zuzena da (C_1) → $y = 0$

C_1 kurba parametrizatuko dugu (algebraikoa)

$\begin{cases} x = x \\ y = 0 \end{cases} \quad x \in [0,1] \quad \begin{matrix} dx = dx \\ dy = 0 \end{matrix}$

$$\boxed{I_1 = \int_{C_1} y^2 dx + x^2 dy = \int_0^1 \cancel{0^2 dx} + \cancel{x^2 \cdot 0} = 0}$$

I_2 C_2 kurba ⇒ $\overline{AB}$ zuzena ⇒ $(x = 1)$

C_2 parametrizatu → $\begin{cases} x = 1 \\ y = y \end{cases} \quad y \in [0,1] \quad \begin{matrix} dx = 0 \\ dy = dy \end{matrix}$

$$\boxed{I_2 = \int_{C_2} y^2 dx + x^2 dy = \int_0^1 \cancel{y^2 \cdot 0} + 1 dy = 1}$$

I_3 C_3 kurba ⇒ $\overline{BO}$ zuzena ⇒ $(y = x)$

C_3 parametrizatu (berdin gaiju) → $\begin{cases} x = x \\ y = x \end{cases}$ aldagai independentea (x) da → $x \in [1, 0]$ (de derecha a izquierda (aldagai beheko)) $\quad \begin{matrix} dx = dx \\ dy = dx \end{matrix}$

$$\boxed{I_3 = \int_{C_3} y^2 dx + x^2 dy = \int_1^0 x^2 dx + x^2 dx = \left[\frac{2x^3}{3}\right]_1^0 = \frac{-2}{3}}$$

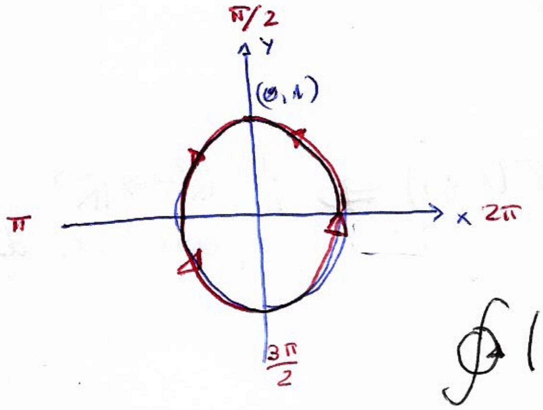
⊛

ARIKETAK

⑥ Ebaluatu $I = \int_C y^2 dx - x^2 dy$ enloju-orratzen kontrako noranzkoan → noranzko positibo ⊕

$x^2 + y^2 = 1$ ($C(0,0)$, $r = 1$) zirkunferentzian zehar $(0,1)$ puntutik $(0,1)$ puntura.

→ C kurba



⊗ Parametrizazio geometrikoa

$$C = \begin{cases} x = \cos\theta & dx = -\sin\theta \, d\theta \\ y = \sin\theta & dy = \cos\theta \, d\theta \end{cases} \qquad \theta \left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$$

$$\frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$$

$$\oint (y^2dx - x^2dy) = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \sin^2\theta \cdot (-\sin\theta)\, d\theta - (\cos^2\theta) \cdot \cos\theta \, d\theta =$$

$$= \int_{\frac{\pi}{2}}^{5\pi/2} \left((1-\cos^2\theta)(-\sin\theta) - (1-\sin^2\theta)\cos\theta\right) d\theta = \int_{\pi/2}^{5\pi/2} (-\sin\theta + \underbrace{\sin\theta\cos^2\theta} - \cos\theta + \sin^2\theta\cos\theta)\, d\theta =$$

$$= \left[\cos\theta - \frac{\cos^3\theta}{3} - \sin\theta + \frac{\sin^3\theta}{3}\right] = 0$$

→ 2. teoremak ⊖ du esan nahi integral lerroarekin bidearekiko independentea dela

$\ast$ gradientea $\Rightarrow$ grad$(U) = \nabla u$

Funtzio potentziala

$\vec{F}$-ren U funtzio potentziala dela esaten da baldin eta $\boxed{\nabla U = \vec{F}}$ $\begin{cases} U'_x = X \\ U'_y = Y \end{cases}$

U-ren deribatu partziala x-rekiko

U-ren deribatu partziala y-rekiko

$\circledast$ $\text{grad}(U) = \nabla U = \nabla \vec{U} = U'_x \hat{\imath} + U'_y \hat{\jmath} = (U'_x, U'_y)$

$\circledast$ $\vec{F} = X(x,y)\,\hat{\imath} + Y(x,y)\,\hat{\jmath} = (X, Y)$

$$\int_C \vec{F}\,d\vec{r} = \int_C X(x,y)\,dx + Y(x,y)\,dy$$

$$X'_y = Y'_x \Rightarrow \exists U(x,y)$$

(funtzio potentziala existitu)

1. TEOREMA

$\vec{F} = X(x,y)\,\hat{\imath} + Y(x,y)\,\hat{\jmath} = (X, Y)$ jarraitua $D \subset \mathbb{R}^2$ azpimultzoan/Domeinuan

existitzen da adierazten du

$$\boxed{\int_C \vec{F}\,d\vec{r} \text{ bidearekiko independentea} \iff \exists U}$$

Kasu honetan, $\boxed{\int_{AB} \vec{F}\,d\vec{r} = U(B) - U(A)}$

FROGAPENA:

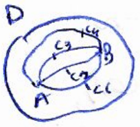
$$\int_{AB} \vec{F}\,d\vec{r} = \int_{AB} \nabla U\,d\vec{r} = \int_{AB} \underbrace{U'_x\,dx + U'_y\,dy}_{dU} = \int_{AB} dU = [U]_A^B = U(B) - U(A)$$

erabili

U-ren diferentzial totala

$$\int_{AB} X\,dx + Y\,dy$$

hauek erabili

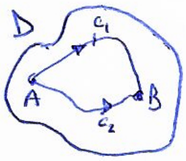


$$\boxed{U(x,y) = \int_a^x X(x,y)\,dx + \int_b^y Y(a,y)\,dy}$$

2. TEOREMA

$\vec{F} = X(x,y)\,\hat{\imath} + Y(x,y)\,\hat{\jmath} = (X(x,y), Y(x,y))$ jarraitu $D \subset \mathbb{R}^2$-n

$\int_{AB} \vec{F}\,d\vec{r}$ bideareliko independentea bada $\Rightarrow \oint_c \vec{F}\,d\vec{r} = \emptyset$



$\int_{AB} \vec{F}\,d\vec{r}$ Bideareliko independente

$\boxed{\int_{c_1} \vec{F}\,d\vec{r} = \int_{c_2} \vec{F}\,d\vec{r}}$

$$\oint_c \vec{F}\,d\vec{r} = \int_{AA} \vec{F}\,d\vec{r} = \int_{AB} \vec{F}\,d\vec{r} + \int_{BA} \vec{F}\,d\vec{r} =$$

Norazko negatiboa

$$= \int_{c_1} \vec{F}\,d\vec{r} - \int_{c_2} \vec{F}\,d\vec{r} = 0$$ nahi dugulako

3. TEOREMA

non D sinpleki konexua den (zulorik ez)

$\vec{F} = X(x,y)\,\hat{\imath} + Y(x,y)\,\hat{\jmath} = (X(x,y), Y(x,y))$ eta bere lehenengo deribatu partzialak jarraituak badira $D \subset \mathbb{R}^2$-n, eta $\int_c \vec{F}\,d\vec{r}$ bideareliko independentea bada $\Longleftrightarrow$ $\boxed{X'_y = Y'_x}$

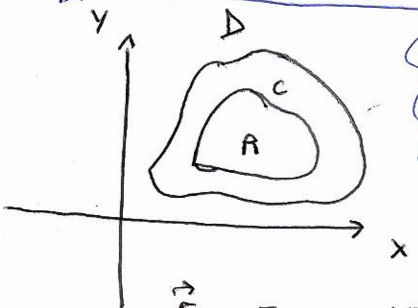
$$\boxed{\int_{AB} \vec{F}\,d\vec{r} \text{ bideareliko independentea} \Longleftrightarrow X'_y = Y'_x}$$

GREEN-EN TEOREMA

D domeinua definituta dago XY planoan eta C kurba D barruan dago. C kurba erregularra (edo zatika erregularra), itxia eta sinplea da, eta C kurbak mugaturiko eskualdea D barruan dago. Izan bitez X, Y eta euren 1. deribatu partzialak jarraituak D-n. Orduan:

$$\boxed{\oint_c \vec{F}\,d\vec{r} = \oint_c X\,dx + Y\,dy = \iint_R (Y'_x - X'_y)\,dx\,dy}$$

R: C kurbak mugaturiko eskualdea



$\vec{F} = X(x,y)\,\hat{\imath} + Y(x,y)\,\hat{\jmath}$

OHARRA

3. teorema

$\oint_c \vec{F}\,d\vec{r}$ bideareliko independentea bada $\Rightarrow Y'_x = X'_y$

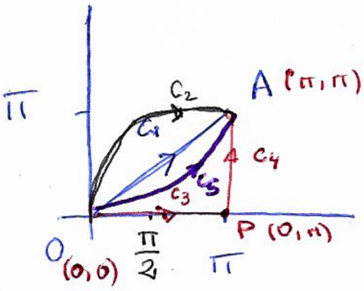
orduan $\oint \vec{F}\,d\vec{r} = \iint_R (Y'_x - X'_y)\,dx\,dy = \emptyset$

ARIKETAK

⑨ Kalkulatu $I = \int_C (x+y)\,dx + (x-y)\,dy$ $O(0,0)$ eta $A(\pi,\pi)$ puntuen artean, C hurrengo kurba izanik:

O eta A artean bidearekiko independentea da

a) Esandako puntuak elkartzen dituen zuzena



$\overline{OA} \Rightarrow \boxed{y = x} \longrightarrow C_1 = \begin{cases} x = x & dx = dx \\ y = x & dy = 1\,dx \end{cases} \quad x \in [0,\pi]$

$\int_{C_1} (x+y)\,dx + (x-y)\,dy = \int_0^\pi (x+x)\cdot dx + \underbrace{(x-x)}_{\emptyset}\cdot dx = \int_0^\pi 2x\,dx =$

$= [x^2]_0^\pi = \pi^2$

b) $y = x + \sin x$ C_2

parametrizatu

$C_2 = \begin{cases} x = x & dx = dx \\ y = x + \sin x & dy = (\cos x + 1)\,dx \end{cases} \quad x \in [0,\pi]$

$\int_{C_2} (x+y)\,dx + (x-y)\,dy = \int_0^\pi (x + x + \sin x)\,dx + (\cancel{x} - \cancel{x} - \sin x)(1 + \cos x)\,dx =$

$(\cos x)' = -\sin x$

$= \int_0^\pi (2x + \sin x - \sin x - \sin x \cos x)\,dx = \left[x^2 + \frac{\cos^2 x}{2}\right]_0^\pi = \pi^2 + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} = \pi^2$

c) $\overline{OP}$ eta $\overline{PA}$ non $P(0,\pi)$ den (zatika erregularra da) C_3

parametrizatu

$(y=0) \Rightarrow$ zuzena $C_3 = \begin{cases} x = x & dx = dx \\ y = 0 & dy = \emptyset \end{cases} \quad x \in [0,\pi]$

$(x=\pi) \Rightarrow$ zuzena $C_4 = \begin{cases} x = \pi & dx = \emptyset \\ y = y & dy = dy \end{cases} \quad y \in [0,\pi]$

$\int_C (x+y)\,dx + (x-y)\,dy = \int_{C_3} (x+y)\,dx + (x-y)\,dy + \int_{C_4} (x+y)\,dx + (x-y)\,dy =$

$= \int_0^\pi (x+0)\,dx + \cancel{(x-0)\,\emptyset} + \int_0^\pi \cancel{(\pi+y)\cdot\emptyset} + (\pi - y)\,dy = \left[\frac{x^2}{2}\right]_0^\pi + \left[\frac{(-\pi-y)^2}{2}\right]_0^\pi =$

$= \frac{\pi^2}{2} + \frac{\pi^2}{2} = \pi^2$

d) $y = \frac{x^2}{\pi}$ PARABOLA C_5

$$C_5 \begin{cases} x = x & dx = dx \\ y = \frac{x^2}{\pi} & dy = \frac{2x}{\pi}dx \end{cases} \qquad x \in [0, \pi]$$

$$\int_{C_5} (x+y)\,dx + (x-y)\,dy = \int_0^{\pi} \left(x + \frac{x^2}{\pi}\right) dx + \left(x - \frac{x^2}{\pi}\right)\frac{2x}{\pi}dx =$$

$$= \int_0^{\pi} \left(x + \frac{x^2}{\pi} + \frac{2x^2}{\pi} - \frac{2x^3}{\pi^2}\right) dx = \int_0^{\pi}\left(x + \frac{3x^2}{\pi} - \frac{2x^3}{\pi^2}\right) dx =$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{\pi} - \frac{x^4}{2\pi^2}\right]_0^{\pi} = \frac{\pi^2}{2} + \frac{\pi^3}{\pi} - \frac{\pi^4}{2\pi^2} = \cancel{\frac{\pi^2}{2}} + \pi^2 - \cancel{\frac{\pi^2}{2}} = \pi^2$$

⊛ Bideareikiko independentea al da? susmoa daukagu kurba desberdinak erabiliz berdina lortu dugu eta.

$$\vec{F} = \underbrace{(x+y)}_{X(x,y)}\vec{i} + \underbrace{(x-y)}_{Y(x,y)}\vec{j} = (x+y,\ x-y)$$

1. TEOREMA

$$\Rightarrow \int_{OA} \vec{F}\,d\vec{r} = U(A) - U(O)$$

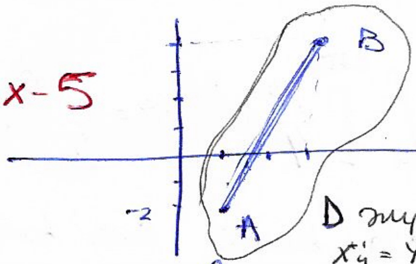
$$\begin{aligned} X'_y &= 1 \\ Y'_x &= 1 \end{aligned} \Big\{ X'_y = Y'_x \Rightarrow \exists U \Rightarrow \int_{OA} \vec{F}\,d\vec{r} \text{ Bideareikiko independentea}$$

$$\int_{C_1} \vec{F}\,d\vec{r} = \int_{C_2} \vec{F}\,d\vec{r} = \int_{C_3} \vec{F}\,d\vec{r} = \ldots$$

$$U(x,y) = \int_0^x (x+y)\,dx + \int_0^y -y\,dy = \left[\frac{x^2}{2} + xy\right]_0^x + \left[\frac{-y^2}{2}\right]_0^y = \frac{x^2}{2} + xy - \frac{y^2}{2}$$

$$\int_{OA} \vec{F}\,d\vec{r} = U(A) - U(O) = \cancel{\frac{\pi^2}{2}} + \pi^2 - \cancel{\frac{\pi^2}{2}} = \pi^2$$

(10) $\displaystyle\int_{A(1,-2)}^{B(3,4)} \frac{y\,dx - x\,dy}{x^2}$ C KURBA $y = 3x-5$



D sinpleki konexua $X'_y = Y'_x$ ele jarraituak D-n

$$X(x,y) = \frac{y}{x^2} \qquad Y(x,y) = \frac{-1}{x}$$

$$\begin{cases} X'_y = \dfrac{1}{x^2} \\ Y'_x = \dfrac{1}{x^2} \end{cases} \quad \boxed{X'_y = Y_x} \Rightarrow \exists U \Rightarrow \int_{AB} \vec{F}\,d\vec{r} = U(B) - U(A)$$

$$\Updownarrow$$

$\int_{AB} \vec{F}\,d\vec{r}$ 3 TEOREMA bid. ind.

a) Funtzio potentziala erabiliz:

$$U(x,y) = \int_1^x \frac{y}{x^2}\,dx + \int_0^y \left(\frac{-1}{1}\right) dy = \left[-\frac{y}{x}\right]_1^x - \left[\frac{y}{x}\right]_0^y = \frac{-y}{x} + y - y = \frac{-y}{x}$$

$$\int_{AB} \frac{y\,dx - x\,dy}{x} = U(B) - U(A) = U(3,4) - U(1,-2) =$$

$$= \frac{-4}{3} - 2 = \frac{-4}{3} - \frac{6}{3} = \frac{-10}{3}$$

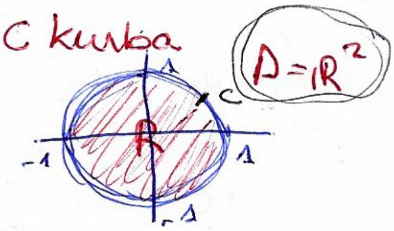
b) Zuzena parametrizatuz:

$$C = \begin{cases} x = x \\ y = 3x-5 \end{cases} \quad \begin{matrix} dx = dx \\ dy = 3dx \end{matrix} \quad x \in [1,3]$$

$$\int_1^3 \left[\frac{3x-5}{x^2}\,dx - \frac{3dx}{x}\right] = \int_1^3 \left(\frac{3}{x} - \frac{5}{x^2} - \frac{3}{x}\right) dx = \left[\frac{5}{x}\right]_1^3 = \frac{5}{3} - 5 = \frac{5-15}{3} = \frac{-10}{3}$$

(11) Kalkulatu $I = \oint_C \left[(2x^3 - y^3)\,dx + (x^3+y^3)\,dy\right]$, C kurba $x^2+y^2=1$ zirkunferentzia izanik. $\to C(0,0)$, $R=1$

$y = \pm\sqrt{1-x^2}$



$$\begin{cases} X'_y = -3y^2 \\ Y'_x = 3x^2 \end{cases}$$

GREEN TEOREMA

$$\oint_C (X\,dx + Y\,dy) = \iint_R (Y'_x - X'_y)\,dx\,dy = \iint_R (3x^2 + 3y^2)\,dx\,dy =$$

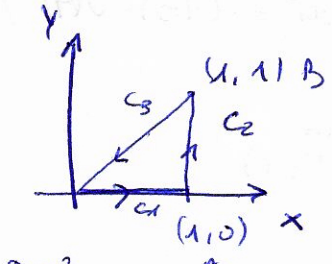
$$= \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3x^2+3y^2)\,dy = \left\| \begin{matrix} \text{Koordenatu polarretara} \\ x = \rho\cos\theta \\ y = \rho\sin\theta \\ J(\rho,\theta) = \rho \end{matrix} \right\| = \int_0^{2\pi} d\theta \int_0^1 \rho \cdot 3\rho^2\,d\rho =$$

$$= \cancel{3}\int_0^{2\pi}\left[\frac{\rho^4}{\cancel{4}}\right]_0^1 d\theta = 3\int_0^{2\pi}\left(\frac{1}{4}\right)d\theta = \frac{3}{4}2\pi = \frac{3\pi}{2}$$

4. ariketa beste era batean

$I=\oint_c y^2dx+x^2dy$ $\quad X(x,y)=y^2 \quad X'_y=\frac{dX}{dy}=2y$

$Y(x,y)=x^2 \quad Y'_x=\frac{dY}{dx}=2x$



$D\in\mathbb{R}^2$

GREEN TEOREMA APLIKATUZ: $I=\int_{c_1}+\int_{c_2}+\int_{c_3}=I_1+I_2+I_3=1/3$

- C kurba zatika erregularra, itxia, sinplea ✓
- X, Y eta euren 1. deribatu partzialak jarraituak ✓
- C kurbak mugatutako R eskualdea D barnean

$$I=\oint_c y^2dx+x^2dy=\iint_R(Y'_x-X'_y)\,dx\,dy$$

B mismo

$$I=\int_0^1dx\int_0^x(2x-2y)dy=\int_0^1\left[2xy-y^2\right]_0^x dx=\int_0^1(2x^2-x^2)dx=\left[\frac{x^3}{3}\right]_0^1=1/3$$

7. ariketa: kalkulatu $I=\oint_c xy\,dx+y\,dy$ non $C: y^2+4x^2-8x=0$

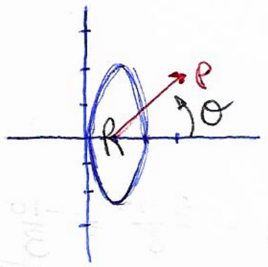
$y^2+4(x^2-2x)=0 \Rightarrow$

$y^2+4((x-1)^2-1)=0 \Rightarrow$

$\frac{y^2}{2}+(x-1)^2=1 \Rightarrow$ ELIPSEA $c(1,0)$, $a=1$, $b=2$

$\rho^2\cos\theta+\frac{4\rho^2\sin^2\theta}{4}=1$

$\rho=1$



GREEN TEOREMA

$X(x,y)=xy \qquad \frac{dX}{dy}=X'_y=x$

$Y(x,y)=y \qquad Y'_x=0$

$$\oint_c(xy\,dx+y\,dy)=\iint_R(Y'_x-X'_y)\,dx\,dy=\iint_R(-x)\,dx\,dy=$$

Polarretara pasatu

$x=1+\rho\cos\theta$

$y=2\rho\sin\theta$

$J(\rho,\theta)=2\rho$

$=$

θ-ren noranzko ⊕

$$=-\int_0^{2\pi}d\theta\int_0^1 2\rho(1+\rho\cos\theta)\,d\rho=-2\int_0^{2\pi}\left[\frac{\rho^2}{2}+\frac{\rho^3}{3}\cos\theta\right]_0^1 d\theta=(-2)\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{3}\cos\theta\right)d\theta=$$

$$(-2)\left[\frac{\theta}{2}+\frac{\overset{0}{\cancel{\sin\theta}}}{3}\right]_0^{2\pi}=(-2)\pi=-2\pi$$

7. ariketaren jarraipena

PARAMETRIZAZIOA EGINEZ (geometrikoa)

$$C: \begin{cases} x = 1+\cos\theta & dx = -\sin\theta \\ y = 2\sin\theta & dy = +2\cos\theta \end{cases} \qquad \theta \in [0, 2\pi]$$

$$\oint_C xy\,dx + y\,dy = \int_0^{2\pi} (1+\cos\theta)\,2\sin\theta + 2\sin\theta \cdot 2\cos\theta \,d\theta =$$

$$\int_0^{2\pi} (-2\sin^2\theta - 2\sin^2\theta\cos\theta + 4\sin\theta\cos\theta)\,d\theta =$$

$$\int_0^{2\pi} \left(-\cancel{2}\right)\left(\frac{1-\cos 2\theta}{\cancel{2}}\right) - 2\sin^2\theta\cos\theta + 4\sin\theta\cos\theta)\,d\theta = \left[-\theta + \frac{\sin 2\theta}{2} - \frac{2\sin^3\theta}{3} + \frac{4\sin^2\theta}{2}\right]_0^{2\pi} =$$

$= \underline{-2\pi} \Leftarrow$ lehen izan dugun soluzio bera.

8. ariketa

$\oint_c (x^2dx + y\,dy)$, $C: y^2 + 4x^2 - 8x = 0$ ← 7. ariketan erabili dugun berdina, horregatik ez dut berriro grafikatu

zuzenean

1. era
$$\begin{cases} X(x,y) = x^2 & \frac{dX}{dy} = X'_y = 0 \\ Y(x,y) = y & Y'_x = 0 \end{cases} \quad \boxed{X'_y = Y'_x} \Rightarrow \int_c \vec{F}\,d\vec{r} \text{ Bid. ind.} \Rightarrow$$

3. Teorema aplika daiteke.

$\oint_c \vec{F}\,d\vec{r} = 0$ esango du.

Ez badugu aurrekoa ikusten, parametrizazio geometrikoa:

2. era
$$C: \begin{cases} x = 1+\cos\theta & dx = -\sin\theta\,d\theta \\ y = 2\sin\theta & dy = 2\cos\theta\,d\theta \end{cases} \qquad \theta \in [0, 2\pi]$$

$$\oint_c x^2dx + y\,dy = \int_0^{2\pi} (1+\cos\theta)^2 (-\sin\theta)\,d\theta + 2\sin\theta\, 2\cos\theta\,d\theta =$$

$$= \int_0^{2\pi} ((1+\cos^2\theta + 2\cos\theta)\sin\theta + 4\sin\theta\cos\theta)\,d\theta = \int_0^{2\pi} (-\sin\theta - \sin\theta\cos^2\theta - 2\sin\theta\cos\theta + 4\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} (-\sin\theta - \sin\theta\cos^2\theta + 2\sin\theta\cos\theta)\,d\theta = \left[\cos\theta + \frac{\cos^3\theta}{3} + \sin^2\theta\right]_0^{2\pi} = \underline{\underline{0}} \text{ ateratzen du.}$$

12. ariketa

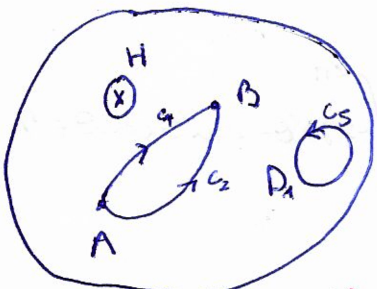
$I = \int_c [P(x,y)dx + Q(x,y)dy]$ $\underbrace{}_{\vec{F}}$

$D = \mathbb{R}^2 - \{H\}$

P, Q eta euren 1. deribatu partzialak jarraituak $\frac{dP}{dy} = \frac{dQ}{dx}$ dakigu.

$D = \mathbb{R}^2 - \{H\}$

$\boxed{\text{Gainera } \int_{C_1, AB} = 1}$



a) $\oint_{C_5} [P(x,y)dx + Q(x,y)dy] = \emptyset$ $\boxed{\frac{dP}{dy} = \frac{dQ}{dx}}$

$\Downarrow$ 3. teorema

$\int_c \vec{F}d\vec{r}$ Bid. ind.

$\int_{C_5} \vec{F}d\vec{r} = \emptyset$ ematen du.

hemen erabili

b) $\int_{C_2, AB} \vec{F}d\vec{r} = 1$

2. TEOREMA

$\int_c \vec{F}d\vec{r}$ Bid ind

$\int_{C_1, AB} = \int_{C_2, AB} = \int_{C_3, AB} = 1$

$\int_{C_1 AA} \vec{F}d\vec{r} = 0$ $= \boxed{\int_{C_1, AB} - \int_{C_2, AB}} \rightarrow \int_{C_1, AB} = \int_{C_2, AB} = 1$

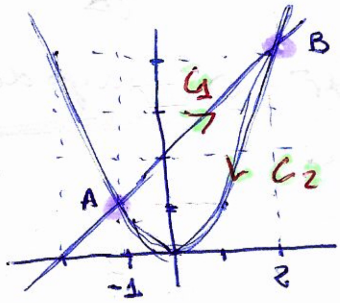
14. ariketa

funtzio bektoriala

$I = \oint_c (1+xy)dx + (x-4)dy$

$\circlearrowleft$ $\ominus$ NORANTZKOAN

C kurba itxia $\Rightarrow$ $\begin{cases} y = 2+x \\ y = x^2 \end{cases}$



EBAKIDURA PUNTUAK

A (-1, 1)

B (2, 4)

$\begin{cases} y = 2+x \\ y = x^2 \end{cases}$ $x^2 = 2+x$

$x^2 - x - 2 = 0$

$\frac{1 \pm \sqrt{1 - 4 \cdot 1(-2)}}{2} = \frac{1 \pm 3}{2}$ $\begin{matrix} \nearrow x = 2 \\ \searrow x = -1 \end{matrix}$

Deribatu partzialak

$P(x,y) = 1 + xy$ $\quad \frac{dP}{dy} = x$

$Q(x,y) = x - y$ $\quad \frac{dQ}{dx} = 1$

1. deribatuak jarraituak $\Downarrow$

GREEN ERABILI DAITEKE, baina parametrizazioz ebatziko dugu.

14. ariketaren jarraipena

PARAMETRIZAZIOA 2⊖ NORANTZKOAN *fruitu bektoriala*

$$I = \int_{C_1, AB} (1+xy)\,dx + (x-y)\,dy + \int_{C_2, BA} (1+xy)\,dx + (x-y)\,dy = I_1 + I_2$$

$\boxed{I_1}$ C_1 parametrizatu (algebraikoki)

$$C_1: \begin{cases} x = x \\ y = 2+x \end{cases} \quad dx = dx \quad dy = dx \quad x \in [-1, 2]$$

x nondik nora doan

$$I_1 = \int_{-1}^{2} (1+x(2+x))\,dx + (\cancel{x}-2-\cancel{x})\,dx = \int_{-1}^{2} (1+2x+x^2-2)\,dx = \int_{-1}^{2} (x^2+2x-1)\,dx =$$

$$= \int_{-1}^{2} (x-1)^2\,dx = \left[\frac{(x-1)^3}{3}\right]_{-1}^{2} = \frac{1}{3} + \frac{8}{3} = \frac{9}{3} = 3$$

$\boxed{I_2}$ C_2 parametrizatu (algebraikoki)

$$C_2: \begin{cases} x = x \\ y = x^2 \end{cases} \quad dx = dx \quad dy = 2x\,dx \quad x \in [2, -1]$$

x nondik nora doan

$$I_2 = \int_{2}^{-1} (1+x\cdot x^2)\,dx + (x-x^2)\,2x\,dx = \int_{2}^{-1} (1+x^3+2x^2-2x^3)\,dx =$$

$$= \int_{2}^{-1} (1+2x^2-x^3)\,dx = \left[x + \frac{2x^3}{3} - \frac{x^4}{4}\right]_{2}^{-1} = -1 - \frac{2}{3} - \frac{1}{4} - \left(2 - \frac{16}{3} - 4\right) =$$

$$= 1 - 6 - \frac{1}{4} = -5 - \frac{1}{4} = -21/4$$

Orduan

$$I = I_1 + I_2 = 3 - \frac{21}{4} = \frac{-9}{4}$$

16. ariketa (azterketan kurba planoan) Hau espazioan da.

$\vec{F} = X(x,y,z)\vec{i} + Y(x,y,z)\vec{j} + Z(x,y,z)\vec{k} = (X, Y, Z)$

$$I = \int_C (x-y+z)\,dx + (3x^2+z)\,dy + (x+z-1)\,dz$$

funtzio bektoriala da, $\int_C \vec{F}\,d\vec{r}$ adierazi daiteke edo $\int_C X\,dx + Y\,dy$

eta espazioan $\int_C X\,dx,\ Y\,dy + Z\,dz$.

$$\begin{cases} X(x,y,z) = x-y+z \\ Y(x,y,z) = 3x^2+z \\ Z(x,y,z) = x+z-1 \end{cases}$$

$\vec{F}: D \subset \mathbb{R}^3 \to \mathbb{R}^3$

Funtzio bektoriala

adibide bat
$K(x,y,z) = x^2+y^2+z^2$
$K: D \subset \mathbb{R}^2 \to \mathbb{R}$
funtzio eskalarra

$\to A(0,0,0)$ $\to B(1,1,1)$
$t=0$ $t=1$
Atik Bra

C: parametrizatuta $\begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases}$

$dx = dt$
$dy = 2t\,dt$ $\quad t \in [0,1]$
$dz = 3t^2\,dt$

$$I = \int_0^1 (t-t^2+t^3)\,dt + (3t^2+t^3)\,2t\,dt + (t+t^3-1)\,3t^2\,dt =$$

$$= \int_0^1 (t-t^2+t^3+6t^3+2t^4+3t^3+3t^5-3t^2)\,dt = \int_0^1 \left[t-4t^2+10t^3+2t^4+3t^5\right]dt =$$

$$= \left[\frac{t^2}{2} - \frac{4t^3}{3} + \frac{5t^4}{2} + \frac{2t^5}{5} + \frac{t^6}{2}\right]_0^1 = \left[\frac{1}{2} - \frac{4}{3} + \frac{5}{2} + \frac{2}{5} + \frac{1}{2}\right] =$$

$$= \frac{7}{2} + \frac{2}{5} - \frac{4}{3} = \frac{105-40+12}{30} = \boxed{\frac{77}{30}}$$