TDT4171 Assignment 2

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1 Part A

- What is the set of unobserved variable(s) for a given time-slice t (denoted $\mathbf{X}_t$ in the book)?
  Answer:
  $$X_t : \{Rain_t\}$$

- What is the set of observable variable(s) for a given time-slice t (denoted $\mathbf{E}_t$ in the book)?
  Answer:
  $$E_t : \{Umbrella_t\}$$

- Present the *dynamic model* $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$ and the *observation model* $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$ as matrices.
  Answer:

  For the dynamic model, we have the conditional probabilities

  $P(R_t|R_{t-1}) = 0.7$, $P(R_t|R'_{t-1}) = 0.3$, $P(R'_t|R_{t-1}) = 0.3$, $P(R'_t|R'_{t-1}) = 0.7$

  For the observation model, we have the conditional probabilities

  $P(U_t|R_t) = 0.9$, $P(U_t|R'_t) = 0.2$, $P(U'_t|R_t) = 0.1$, $P(U'_t|R'_t) = 0.8$

  The resulting matrices can then be written as,

  $$\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1}) = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

  $$\mathbf{P}(\mathbf{E}_t|\mathbf{X}_t) = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

- Which assumptions are encoded in this model? Are the assumptions reasonable for this particular domain?
  Answer:

  We make several assumptions when utilizing this model.
  The **Markov assumption** is that the current state is only dependent on a bounded subset of previous states, in this case the weather on the previous day. This means we assume that a first-order Markov process describes the model sufficiently. This is not reasonable, as the weather on a given day is most likely dependent on other variables, ex. season, location etc.
  We assume the process is **stationary**, meaning that the dynamic/transition model is fixed for all t. As the weather on a given day most likely depends on ex. the season of the year, this assumption is not reasonable for all t. It *may* however be reasonable to assume a stationary process on a small subset of t.
  The final assumption is the **Sensor Markov assumption**, meaning the evidence is only dependent on the current state. In this case it means that the probability of observing an umbrella on a given day t only depends on the weather on day t. This may be a reasonable assumption, as the current weather often is the only variable deciding whether to use an umbrella or not. One could argue that if it was raining consecutively for ex. 100 days straight, one might bring an umbrella just out of habit the next day. In this case, the assumption may not be reasonable.

2 Part B

Implement filtering using the Forward operation (Equation 15.5 and Equation 15.12). Note that this can be done with simple matrix operations in the HMM.

- Verify your implementation by calculating $\mathbf{P}(\mathbf{X}_2|\mathbf{e}_{1:2})$, where $e_{1:2}$ is the evidence that the Umbrella was used both on day 1 and day 2. The desired result is (confer the slides from the lecture available on blackboard) is that the probability of rain at day 2 (after the observations) is 0.883.
- Use your program to calculate the probability of rain at day 5 given the sequence of observations $e_{1:5}$ = {Umbrella1 = true, Umbrella2 = true, Umbrella3 = false, Umbrella4 = true, Umbrella5 = true}. Document your answer by showing all normalized forward messages (in the book the un-normalized forward messages are denoted $f_{1:k}$ for k = 1, 2, . . . , 5).

  Answer:

  With the initial probability of rain on day 0 set to 0.5 and using the FORWARD-algorithm appended, we get the sequence:

$$f_{1:1} = [0.818, 0.182]$$

$$f_{1:2} = [0.883, 0.117]$$

$$f_{1:3} = [0.191, 0.809]$$

$$f_{1:4} = [0.731, 0.269]$$

$$f_{1:5} = [0.867, 0.133]$$

  The probability of rain at day 5 is thus <u>0.867</u>.

3 Part C

Implement smoothing using the Forward-Backward algorithm (Figure 15.4). Note that also this can be done with simple matrix operations.

- Verify your implementation by calculating $\mathbf{P}(\mathbf{X}_1|\mathbf{e}_{1:2})$ where $e_{1:2}$ is the evidence that the umbrella was used the first two days (as in Part (b) of the assignment). The desired result is $\mathbf{P}(\mathbf{X}_1|\mathbf{e}_{1:2}) = \langle 0.883, 0.117 \rangle$.
- Use your Forward-Backward algorithm to calculate the probability of rain at day 1 given the sequence of observations e1:5 = Umbrella1 = true, Umbrella2 = true, Umbrella3 = false, Umbrella4 = true, Umbrella5 = true, i.e., $\mathbf{P}(\mathbf{X}_1|\mathbf{e}_{1:5})$. Document your answer by showing all backward messages ($b_{k+1:t}$ for k = 1, 2, . . . , 5).

  Answer:
  With the initial probability of rain on day 0 set to 0.5 and using the FORWARD-BACKWARD-algorithm appended, we get the sequence:

  $$b_{6:6} = [1.000, 1.000]$$

  $$b_{5:6} = [0.690, 0.410]$$

  $$b_{4:6} = [0.459, 0.244]$$

  $$b_{3:6} = [0.091, 0.150]$$

  $$b_{2:6} = [0.066, 0.045]$$

  With these backward messages we get a probability of rain at day 5 to <u>0.867</u>