02424 Week 8

Exercise 1

This exercise is related to the random effects type of models in Chapter 5. Let $\boldsymbol{U}$, $\boldsymbol{Y}$ andy $\boldsymbol{Z}$ be random variables, and recall that

$$\begin{aligned} Var[\boldsymbol{Y}] &= E[Var[\boldsymbol{Y}|\boldsymbol{U}]] + Var[E[\boldsymbol{Y}|\boldsymbol{U}]] \quad (1) \\ Cov[\boldsymbol{Y}, \boldsymbol{Z}] &= E[Cov[\boldsymbol{Y}, \boldsymbol{Z}|\boldsymbol{U}]] + Cov[E[\boldsymbol{Y}|\boldsymbol{U}], E[\boldsymbol{Z}|\boldsymbol{U}]] \quad (2) \end{aligned}$$

Consider now the linear regression model where

$$Y = U\theta + \varepsilon$$

where U and ε are mutually independent random variables with mean μ_U and $\mu_\varepsilon = 0$, and variance σ_U^2 and σ_ε^2, respectively. We assume that θ is known.

Question 1

Explain the difference between the above described assumptions and the normally used assumption for linear regression type of problems?

Question 2

Find expressions for the conditional mean $E[Y|U]$ and the conditional variance $Var[Y|U]$.

Question 3

Assume we have a given $U = u$. Find then the prediction $E[Y|U = u]$, and what is the variance $Var[Y|U = u]$.

Question 4

Finally, find the marginal mean and variance of Y. Explain the various contributions to the marginal variance of Y.

Exercise 2

As mentioned in the book on page 171 is sometimes of interest to estimate the random effects. This can be done by writing down the joint density for the observable random variables $\boldsymbol{Y}$ and the unobservable $\boldsymbol{\mu}$. This is often called the *hierarchical likelihood*:

$$L(\mu, \sigma^2, \gamma, \boldsymbol{\mu}) = f(\boldsymbol{y}|\boldsymbol{\mu}) f(\boldsymbol{\mu})$$

$$L = \frac{1}{\sigma^N} \exp\left\{ -\frac{SSE + \sum_i n_i(\overline{y}_i - \mu_i)^2}{2\sigma^2} \right\} \frac{1}{(\gamma\sigma^2)^{k/2}} \exp\left\{ -\frac{\sum_i(\mu_i - \mu)^2}{2\gamma\sigma^2} \right\}$$

Problem: Derive the equation above for the hierarchical likelihood.