02424 Week 2

The main purpose of this collection of small exercises is to illustrate various important aspects in relation to estimation techniques in Chapter 2 of the lecture notes and in particular to the likelihood methods. Most of the problems are simplified in order to illustrate the concepts using rather simple analytical techniques, however, in practice numerical methods are often required.

Some of the illustrated concepts are the Minimum Variance Unbiased Estimator (MVUE), Likelihood functions for iid samples, maximum likelihood estimate (MLE), Fisher Information Matrix, Score Function, Cramer-Rao lower bound (CRLB).

Exercise 1

The following are heart rate measurements (beats/minute) of one person measured throughout the day.

```
71  74  82  76  91  82  82  75  79  82  72  90
```

Assume that the data are an iid sample from $N(\theta, \sigma^2)$, where σ^2 is assumed to be known at the observed sample variance. Sketch the likelihood function for θ if

a) the whole data are reported.

b) only the sample mean $\bar{y}$ is reported.

Find the MLE, $\widehat{\theta}$, and the Hessian (for case a)) using the `optim` function in R.

Exercise 2

The measurements $y_1, y_2, ..., y_n$ are an iid sample from the Poisson distribution with density

$$f(y) = \frac{\lambda^y \exp(-\lambda)}{y!}.$$

a) Write down the combined likelihood function, the log-likelihood function, $l'_\lambda(\lambda; \mathbf{y})$ and $j(\lambda; \mathbf{y})$.

b) Derive the MLE, $\widehat{\lambda}$, and calculate the observed information.

Exercise 3

The following data are number of customers arriving at a cafe per 10 minutes:

```
4 6 3 7 2 4
```

Assume that the data are an iid sample from the Poisson distribution.
Plot the log-likelihood function and the quadratic approximation. Set the maximum of the log-likelihood to zero and check a range of λ such that the log-likelihood is approximately between between -4 and 0. Do the same plot again but this time not on log-scale.

Exercise 4

Question 1 and 2 from Exercise 2.1 in the textbook, but with the important change in Question 2 that T_w should be defined as:

$$T_w = wY_1 + (1-w)10Y_2$$

Exercise 5

Consider a sample consisting of n i.i.d. observations from a Gamma distribution, ie. $Y_i \sim G(\alpha, \beta)$.

Question 1

Explain how the χ^2 distribution is obtained as special cases of the Gamma distribution.

Question 2

Write down the log-likelihood function.

Question 3

Assume now that α is known. Write down the score function.

Question 4

Find the maximum likelihood estimate (MLE) for β.

Exercise 6

Suppose $y_1, ..., y_n$ are an iid sample from the exponential distribution with density

$$g(y) = \beta^{-1} exp(-y/\beta) = \lambda \exp(-\lambda y)$$

where λ is the rate – or the inverse scale.

Question 1

Derive the MLE.

Exercise 7

Assume $y_1, ..., y_n$ are an iid sample from $N(\mu, \sigma^2)$, where μ is assumed known.

Question 1

Write down the log-likelihood function.

Question 2

Find the score function for estimating σ.

Question 3

Find the observed information.

Question 4

Find the expected information.

Question 5

Find the Cramer-Rao lower bound (CRLB) for estimating σ^2.

Question 6

Do you know any estimator/statistic which has the variance equal to the CRLB?