02424 Week 4

Exercise 3.2

We consider the model

$$Y_t = \beta x_t + \epsilon \tag{1}$$

Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^T$ and $\boldsymbol{x} = [x_1, ..., x_n]$

Question 1:

Assume $V[\epsilon_t] = \sigma^2/x_t^2$, but that $Cor[\epsilon_i, \epsilon_j] = 0$, for $i \neq j$. The unweighted least square estimator is

$$\begin{aligned}
\hat{\beta}^* =&(\boldsymbol{x}^T\boldsymbol{x})^{-1}\boldsymbol{x}^T\boldsymbol{Y} && (2)\\
=&\frac{1}{\sum_{t=1}^n x_t^2}\sum_{t=1}^n x_t Y_t && (3)
\end{aligned}$$

the expected value of $\hat{\beta}^*$ is

$$\begin{aligned}
E[\hat{\beta}^*] =&\frac{1}{\sum_{t=1}^n x_t^2}\sum_{t=1}^n x_t E[Y_t] && (4)\\
=&\frac{1}{\sum_{t=1}^n x_t^2}\sum_{t=1}^n x_t E[x_t\beta + \epsilon_t] && (5)\\
=&\frac{1}{\sum_{t=1}^n x_t^2}\sum_{t=1}^n x_t x_t \beta = \beta && (6)
\end{aligned}$$

Hence the estimator is unbiased. The variance of this estimator is

$$\begin{aligned}
V[\hat{\beta}^*] =&\frac{1}{\left(\sum_{t=1}^n x_t^2\right)^2}\sum_{t=1}^n x_t^2 V[Y_t] && (7)\\
=&\frac{1}{\left(\sum_{t=1}^n x_t^2\right)^2}\sum_{t=1}^n x_t^2 V[x_t\beta + \epsilon_t] && (8)\\
=&\frac{1}{\left(\sum_{t=1}^n x_t^2\right)^2}\sum_{t=1}^n x_t^2 \sigma^2/x_t^2 && (9)\\
=&\frac{n\sigma^2}{\left(\sum_{t=1}^n x_t^2\right)^2} && (10)
\end{aligned}$$

Question 2:

For the weighted least square estimator we need the weight matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1/x_1^2 & 0 & ... & & 0 \\ 0 & 1/x_2^2 & 0 & ... & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & ... & 0 & 1/x_n^2 \end{bmatrix}; \quad \boldsymbol{\Sigma}^{-1} = \begin{bmatrix} x_1^2 & 0 & ... & & 0 \\ 0 & x_2^2 & 0 & ... & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & ... & 0 & x_n^2 \end{bmatrix} \tag{11}$$

The weighted least square estimator become

$$\hat{\beta} =(\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x})^{-1}\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{Y} \tag{12}$$

$$=\frac{1}{\sum_{t=1}^n x_t^4}\sum_{t=1}^n x_t^3 Y_t \tag{13}$$

and the expected value become

$$E[\hat{\beta}] =\frac{1}{\sum_{t=1}^n x_t^4}\sum_{t=1}^n x_t^3 E[Y_t] \tag{14}$$

$$=\frac{1}{\sum_{t=1}^n x_t^4}\sum_{t=1}^n x_t^3 E[\beta x_t + \epsilon] \tag{15}$$

$$=\frac{1}{\sum_{t=1}^n x_t^4}\sum_{t=1}^n x_t^4 \beta = \beta \tag{16}$$

the variance of the estimator become

$$V[\hat{\beta}] =\frac{1}{\left(\sum_{t=1}^n x_t^4\right)^2}\sum_{t=1}^n x_t^6 V[Y_t] \tag{17}$$

$$=\frac{1}{\left(\sum_{t=1}^n x_t^4\right)^2}\sum_{t=1}^n x_t^6 V[\beta x_t + \epsilon] \tag{18}$$

$$=\frac{1}{\left(\sum_{t=1}^n x_t^4\right)^2}\sum_{t=1}^n x_t^6/x_t^2\sigma^2 \tag{19}$$

$$=\frac{\sigma^2}{\sum_{t=1}^n x_t^4} \tag{20}$$

Question 3:

The ratio between the two estimators is

$$\frac{V[\hat{\beta}]}{V[\hat{\beta}^*]} =\frac{\sigma^2}{\sum_{t=1}^n x_t^4}\left(\frac{n\sigma^2}{\left(\sum_{t=1}^n x_t^2\right)^2}\right)^{-1} \tag{21}$$

$$=\frac{n}{\sum_{t=1}^n x_t^4}\left(\frac{\sum_{t=1}^n x_t^2}{n}\right)^2 \tag{22}$$

by Jensen's inequality $\frac{\sum_{t=1}^{n} x_t^4}{n} \geq \left(\frac{\sum_{t=1}^{n} x_t^2}{n}\right)^2$, and hence the ration is less than 1 implying that $V[\hat{\beta}] \leq V[\hat{\beta}^*]$.

Question 4:

We calculated the unweighted estimatro under Q1, and since $E[\epsilon]$ is still zero, the estimator is still unbiased. For the variance we need

$$\boldsymbol{\Sigma} = \sigma^2 \begin{bmatrix} 1 & \rho & 0 & ... & & 0 \\ \rho & 1 & \rho & 0 & ... & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & ... & 0 & \rho & 1 \end{bmatrix} = \sigma^2 \boldsymbol{\Sigma} \tag{23}$$

$$\begin{aligned} V[\hat{\beta}] =& \frac{1}{\left(\sum_{t=1}^{n} x_t^2\right)^2} \boldsymbol{x}^T V[Y_t] \boldsymbol{x} \\ =& \frac{1}{\left(\sum_{t=1}^{n} x_t^2\right)^2} \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} \\ =& \frac{\sigma^2}{\left(\sum_{t=1}^{n} x_t^2\right)^2} \boldsymbol{x}^T \boldsymbol{R} \boldsymbol{x} \end{aligned}$$

After some matrix algebra one can arrive at

$$V[\hat{\beta}] = \sigma^2 \left(\frac{1}{\sum_{t=1}^{n} x_t^2} + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i x_j \right)$$