Advanced Time Series Analysis: Computer Exercise 2

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Part 1

There has been simulated $n = 3000$ samples of noise where $\epsilon_t \sim \mathcal{N}(0, 1)$. ϵ_t is used as noise input for simulations in part one.

The equation below shows the used parameters in the selected system: a SETAR(2,1,1) model. The parameters for the two regimes are defined by eqn. 1 and eqn. 2.

$$a_0 = [0.125, -0.125] \tag{1}$$

$$a_1 = [0.6, -0.4] \tag{2}$$

Simulation of the SETAR(2,1,1)

The SETAR(2,1,1) model is given by eqn. 3.

$$X_t = a_0^{(J_t)} + \sum_{i=1}^{k_{(J_t)}} a_i^{(J_t)} X_{t-i} + \epsilon^{(J_t)} \tag{3}$$

where J_t are the regime processes. The complete model are defined in eqn. 4.

$$X_t = \begin{Bmatrix} a_{0,1} + a_{1,1} X_{t-1} + \epsilon_t & for & X_{t-1} \leq 0 \\ a_{0,2} + a_{1,2} X_{t-1} + \epsilon_t & for & X_{t-1} > 0 \end{Bmatrix} \tag{4}$$

A simulation of the model (eqn. 4) is showed in fig. 1.

Estimate the parameters using conditional least squares

The following code snippet contain three functions: `Setar()`, `RSSSetar` and `PESetar` calculates the conditional means, the squared residuals and the total squared prediction error respectively.

```r
# calculation of the conditional mean for the SETAR(2,1,1) model
Setar <- function(par, model) {
    # init conditional mean vector
    e_mean <- rep(NA, length(model))
    # loop through observations
    for (t in 2:length(model)) {
        if (model[t - 1] <= 0) {
            e_mean[t] <- par[1] + par[2] * model[t - 1]
```

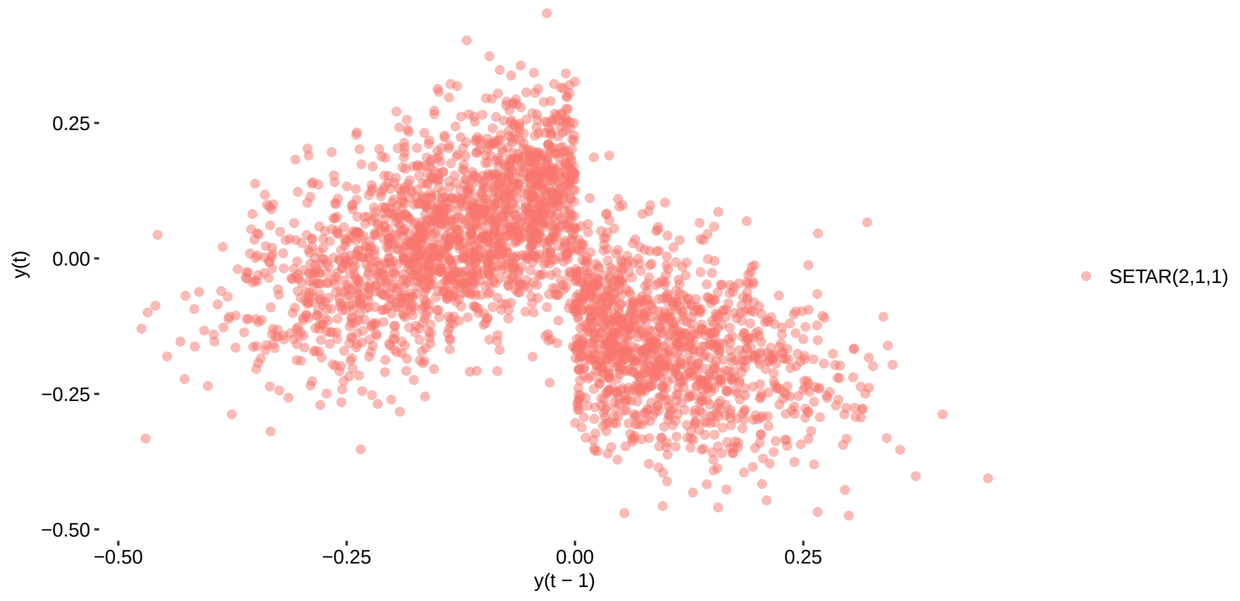


Figure 1: One simulation of a SETAR(2,1,1) model.

```
        } else {
            e_mean[t] <- par[3] + par[4] * model[t - 1]
        }
    }
    # return the conditional mean vector
    return(e_mean)
}

RSSSetar <- function(par, model) {
    # conditional mean
    e_mean <- Setar(par, model)
    # calculate and return the squared residuals
    return((model - e_mean)^2)
}

# summed prediction error
PESetar <- function(par, model) {
    # conditional mean
    e_mean <- Setar(par, model)
    # calculate and return the objective function value
    return(sum((model - e_mean)^2, na.rm = TRUE))
}
```

The native R function `optim()` has been applied to estimate the parameters of the simulated process. The objective cost function is the total squared prediction error, which needs to be minimized.

eqn. 5 and eqn. 6 have been used as initial parameter input to optimization function. The "initial" conditional mean is computed with the initial parameters (eqn. 5 and eqn. 6) and illustrated in fig. 2 with the label: `M(x) initial`.

Table 1: Table shows the real parameters, the estimated parameters and their percentage diviation.

Parameter	Real value	Optimized value	Change in (%)
a0_1	0.125	0.1267946	1.4153889
a1_1	0.600	0.6127632	2.0828869
a0_2	-0.125	-0.1239285	-0.8645787
a1_2	-0.400	-0.4062514	1.5388065

$$a_0 = [0.1, -0.02] \tag{5}$$

$$a_1 = [0.4, -0.25] \tag{6}$$

The estimated parameters are listed in table 1.

The percentage deviation from the real parameters are within ±2.0828869%. The optimized squared prediction error (30.0575719) is smaller than the real squared error (30.0607994) which is a subject of overfitting. Dividing data into train and test with proper cross validation techniques can solve the issue of overfitting.

Figure 2 illustrate the conditional means for the SETAR(2,1,1) model with initial parameters (eqn. 5 and eqn. 6), the real and estimated parameters (table 1).

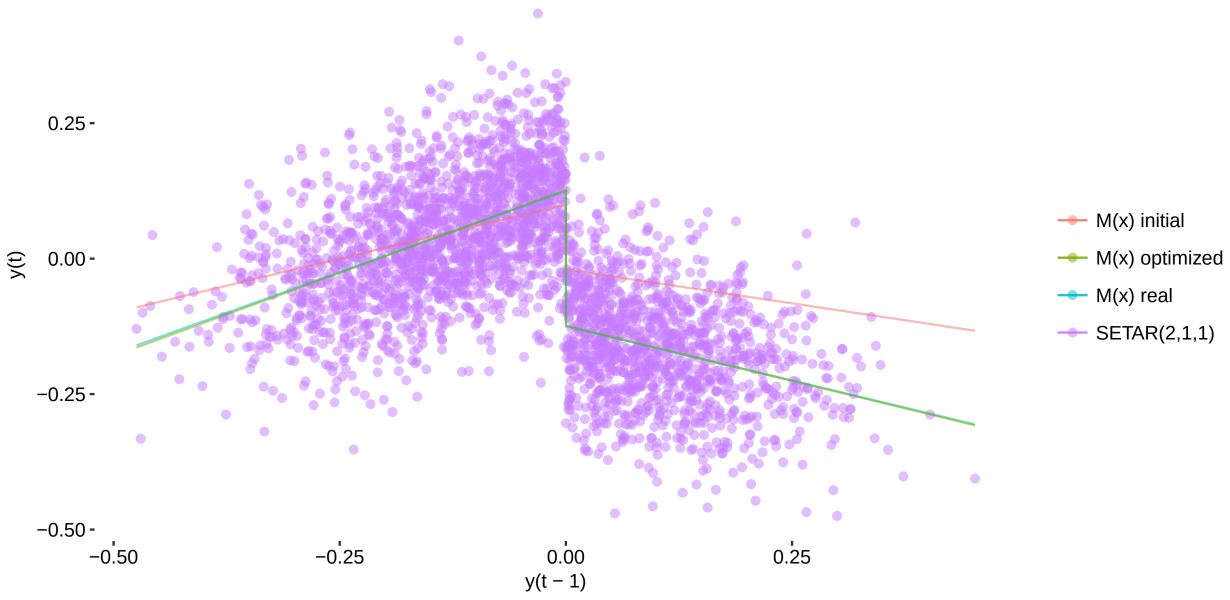


Figure 2: Simulations of a SETAR(2,1,1) model with the initial, real and optimized parameters. The vertical lines in the transition of the regimes does not exist.

The initial parameters are not a good representation of the simulated SETAR model, as illustrated in fig. 2 with the label `M(x) initial`. But after estimating the parameters s.t. a minimization of the squared prediction error, the estimated parameters is a reasonable representation of the SETAR model.

Part 2

It has been chosen to create a grid based upon the two slope parameters, eqn. 2. The grid is based on a matrix with the size $(51, 51)$, where the center element, the intersection of the diagonal and off-diagnoal, is the optimal parameter value, and is highligthed with a blue dot. There is also illustrated a red dot, which shows the optimal parameters for the given subset of the simulation. The first axis is $a_{1,1}$ parameter and the second axis in the $a_{1,2}$ parameter.

The the diviation of the parameter values are $\pm 30\%$ (`max_change_p`) of the optimal value. The following function has been used to plot the contours for the different subsets: `plot_contours()`. General to the contour plots, close contour lines indenticate a higher slope of the curvature of the obejctive function.

```r
# only change the slope par[2] and par [4]
plot_contours <- function(model, par = optimal_PE$par, change_p = max_change_p,
    nplot = resolution) {
    # create squence of nplot values with the optimized parameter(s) in center
    par_2_seq <- seq(par[2] - par[2] * change_p, par[2] + par[2] * change_p,
        len = nplot)
    par_4_seq <- seq(par[4] - par[4] * change_p, par[4] + par[4] * change_p,
        len = nplot)

    # create grid
    loess_melt <- expand.grid(par_2_seq, par_4_seq)

    # caculate values
    loess_melt$value <- sapply(1:nrow(loess_melt), function(i) {
        return(PESetar(par = c(par[1], loess_melt$Var1[i], par[3], loess_melt$Var2[i]),
            model = model))
    })

    # find optimal subset value
    min_val <- loess_melt[which.min(loess_melt$value), ]

    # return contour plot
    return(ggplot(loess_melt, aes(x = Var1, y = Var2, z = value)) + stat_contour(geom = "contour",
        alpha = 1/2, binwidth = 0.005) + geom_point(aes(x = par[2], y = par[4],
        color = paste0("optim,\nval=", round(PESetar(par = par, model = model),
            digits = 4))), alpha = 1/2) + geom_point(aes(x = min_val$Var1, y = min_val$Var2,
        color = paste0("best subset,\nval=", round(min_val$value, digits = 4))),
        alpha = 1/2) + labs(x = "a_1,1", y = "a_1,2", color = "parameter and value") +
        theme_TS())
}
```

The contour plots illustatres the objective function (total squared prediction error) with changes in each changeable parameter. The objective function is convex which means there is only one local/global minima.

N = 1:3000

Figure 3 shows the contour plot for the complete simulated model. The optimal parameters gives the global minima of the objective function, and the two dots are identical as illustared.

The $a_{1,1}$ paramter will result in a higher objective function with equally changes in both parameter.

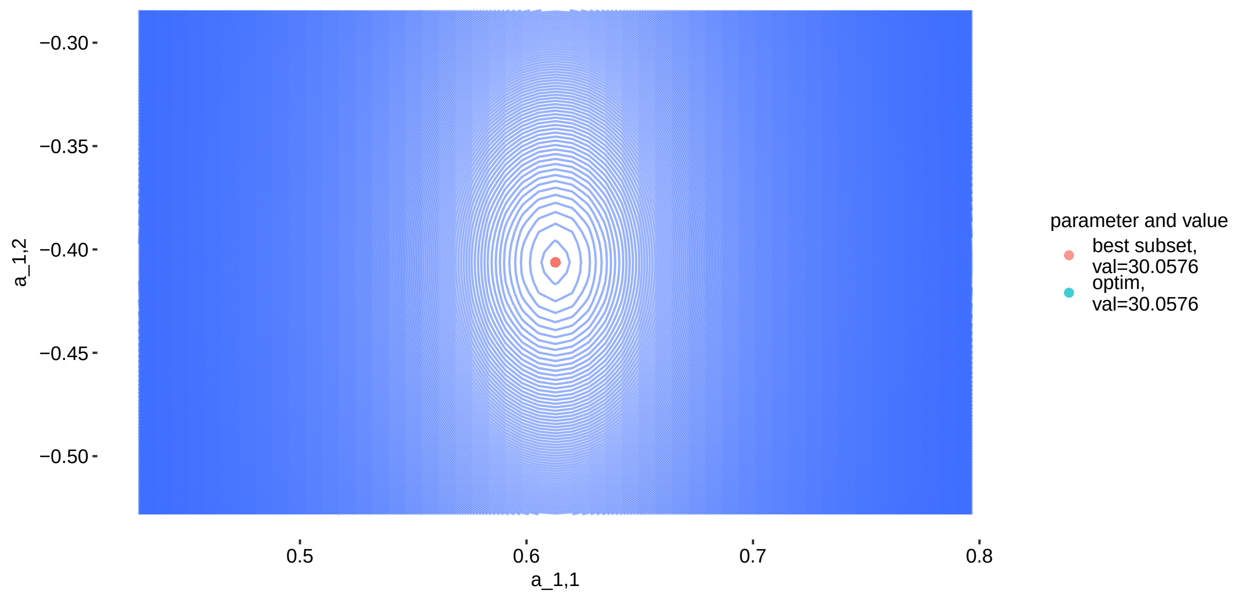


Figure 3: Contour plot the slope parameters for N = 1:3000.

N = 1:300

Figure 4 shows the contour plot with only $N = 1 : 300$ samples of the simulated model considered. The number of samples are decreased with a factor of 100 similar to the objective function.

The optimal parameters does not represent the minimum value of the of the objective function. There is a clear speration of the two dots and the their values are different, see labels in fig. 4.

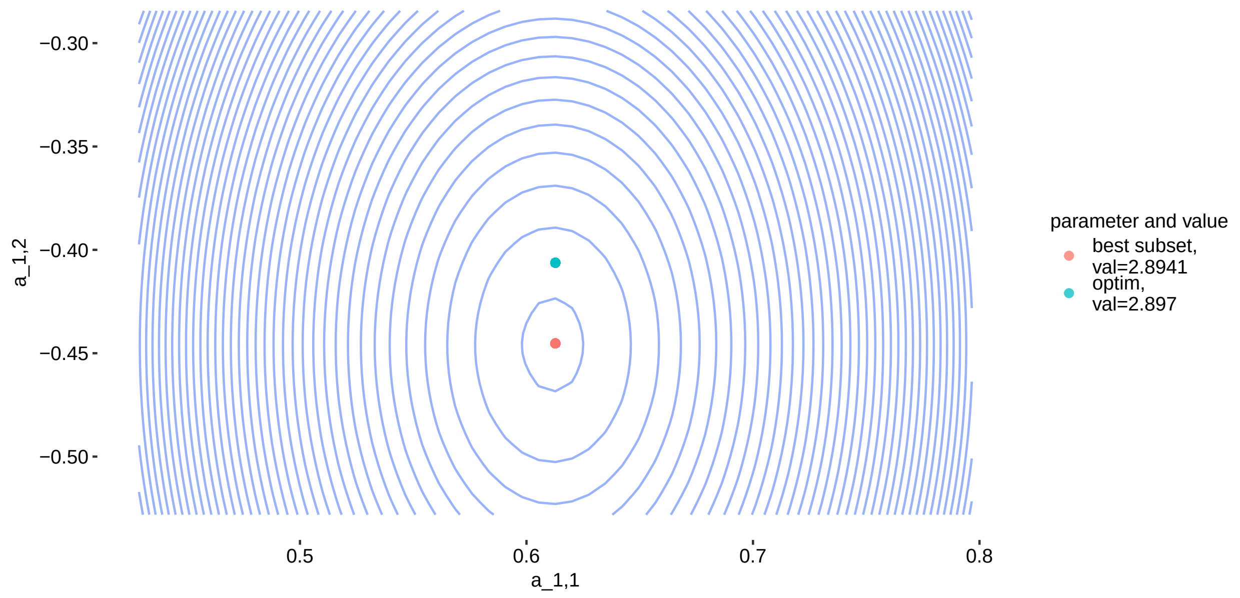


Figure 4: Contour plot the slope parameters for N = 1:300.

N = 1:30

Figure 5 shows a contour plot of the first 30 samples. It is possible to see that the best set of parameter, in the given subset, is not within ±2.0828869% of the optimized parameters. In contrast to fig. 5 there red dot is placed on the boundary of the grid which indicates that the global minimum is not obtained.

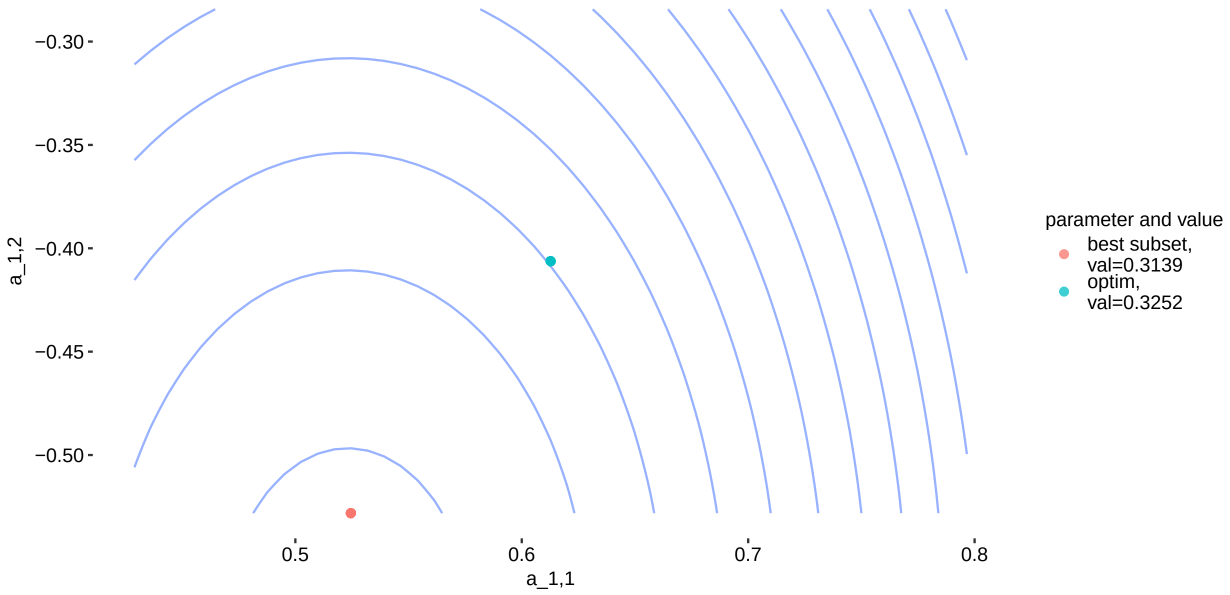


Figure 5: Contour plot the slope parameters for N = 1:30.

N = 1001:1300

Figure 6 uses same number of simulated samples but in an different "localtion" of the simulated data. The optimal subset parameters is not within ±30% of the optimized parameters which is opposite to fig. 4.

N = 1001:1030

The best subset parameters in fig. 7 are not within ±30% of the optimized parameters.

Findings

- The optimization of the parameters must occur with respect to the wanted (sub)set of the simulated data.
- The biggest changes for the optimal subset parameters is for the $a_{1,2}$ parameter. In figures 4 and 5 the best subset $a_{1,2}$ parameter is smaller then the optimized value of $a_{1,2}$.
  Opposite in figures 6 and 7, the best subset $a_{1,2}$ parameter must be higher than the optimized $a_{1,2}$ parameter.
- The total squared prediction error scales well with the number of considered simulated data. This can indenticate that the residuals are normally distritubed over the entrie simulated process.

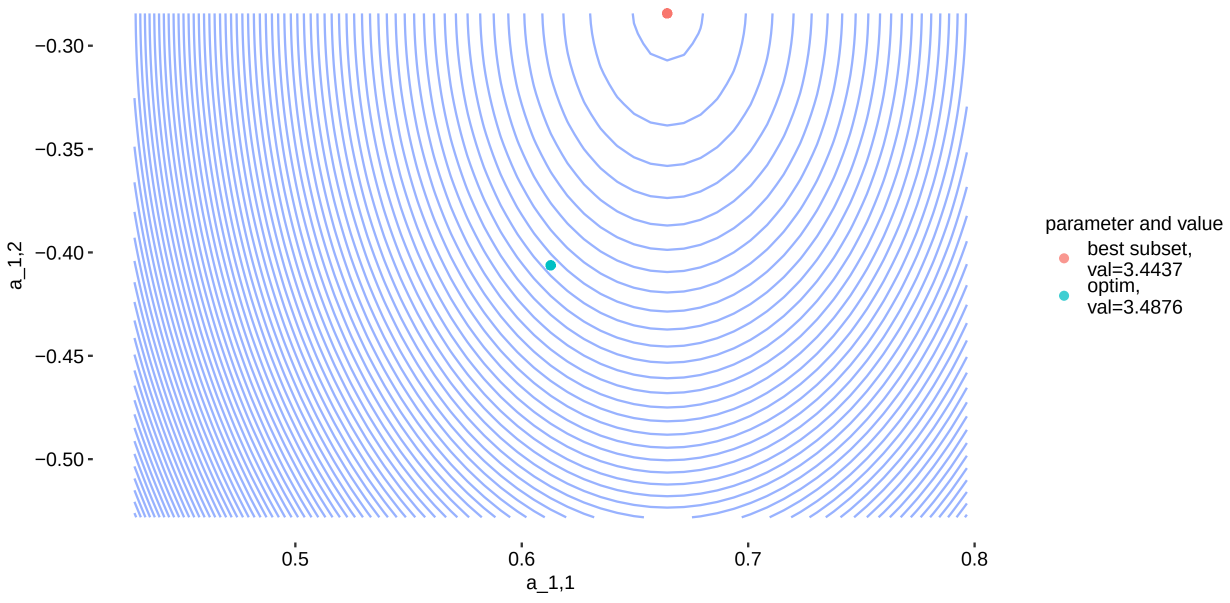


Figure 6: Contour plot the slope parameters for N = 1001:1300.

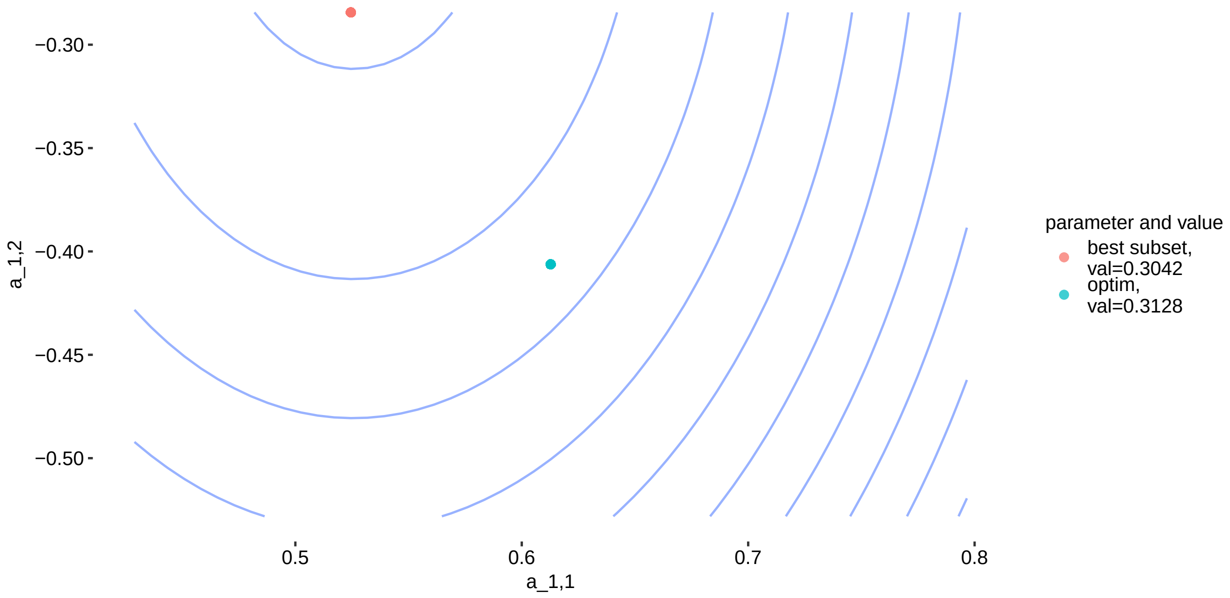


Figure 7: Contour plot the slope parameters for N = 1001:1030.

Part 3

The chosen model for the doubly stochastic model, is an AR(2)-AR(4) model, see eqn. 7.

$$\begin{aligned} Y_t &= \sum_{k=1}^{2} \left(\Phi_{t-(1-k)} Y_{t-k}\right) + \epsilon_t \\ \Phi_t - \mu &= \sum_{n=1}^{4} \left(\phi_n \left(\Phi_{t-n} - \mu\right)\right) + \zeta_t \\ \Phi_t &= \sum_{n=1}^{4} \left(\phi_n \left(\Phi_{t-n} - \mu\right)\right) + \zeta_t + \underbrace{\mu \left(1 - \sum_{n=1}^{4} (\phi_n)\right)}_{\delta_t} \end{aligned} \tag{7}$$

Equation 8 shows eqn. 7 in a reparameterizated state space format.

$$\begin{aligned} \begin{pmatrix} \Phi_t \\ \Phi_{t-1} \\ \Phi_{t-2} \\ \Phi_{t-3} \\ \delta_t \end{pmatrix} &= \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi_{t-1} \\ \Phi_{t-2} \\ \Phi_{t-3} \\ \Phi_{t-4} \\ \delta_{t-1} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \delta_t \\ Y_t &= \begin{pmatrix} Y_{t-1} & Y_{t-2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi_t \\ \Phi_{t-1} \\ \Phi_{t-2} \\ \Phi_{t-3} \\ \delta_t \end{pmatrix} + e_t \end{aligned} \tag{8}$$

Simulate

The initial parameters for the simulation are given in eqn. 9.

$$\begin{aligned} n &= 500 \\ \mu &= 0.1 \\ \zeta &= 0.01 \\ \epsilon &= 0.4 \\ \delta_t &\sim \mathcal{N}(\mu, \zeta) \\ e_t &\sim \mathcal{N}(\mu, \epsilon) \end{aligned} \tag{9}$$

Comment

delta som state i stedet for constant -> estimate

Tænk hvor havd der sker i den underlæggende proces ?

hvordan er stationary conditions?

Fordele ved at se delta som et state..

Tjek for stabilitet

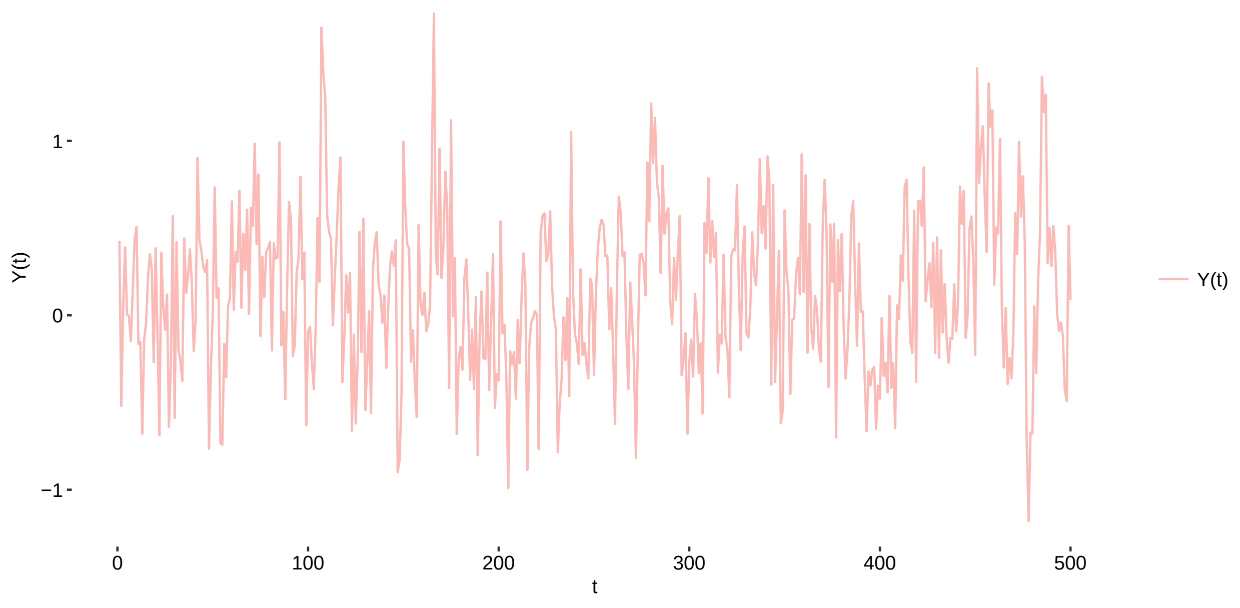


Figure 8: Simulated process of Y(t).

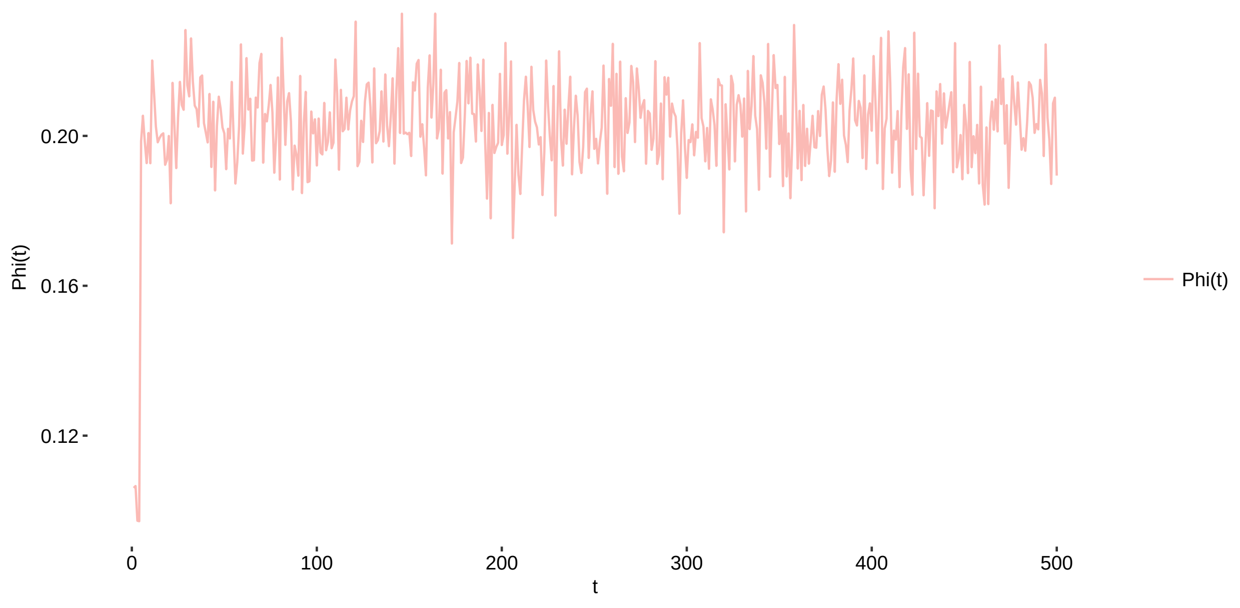


Figure 9: Simulated process of Phi(t).

Part 4

Following simple state space model is given, eqn. 10.

$$x_{t+1} = ax_t + v_t y_t \quad = x_t + e_t \tag{10}$$

where a is an unknown parameter and v_t and e_t are mutually uncorrelated white noise processes with their variences σ_v^2 and σ_e^2.

Part 4a

The model from eqn. 10 is on state space form in eqn. 11

$$\begin{aligned} x_{t+1} &= a x_t + v_t \\ y_t &= x_t + e_t \end{aligned} \tag{11}$$

Simulate

Simulate X time series where $a = 0.4$, $\sigma_v^2 = \sigma_e^2 = 1$ with zero mean

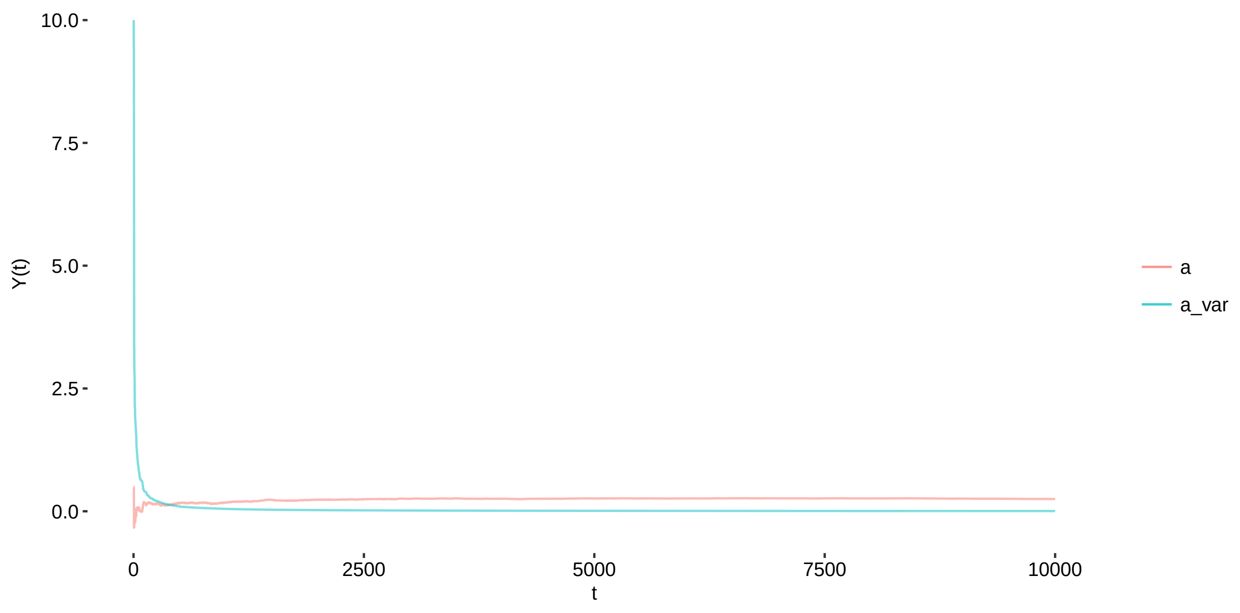
Rewrite

$$\begin{aligned} \begin{pmatrix} x_{t+1} \\ a_{t+1} \end{pmatrix} &= \begin{pmatrix} a_t & 0 \\ 0 & a_t \end{pmatrix} \begin{pmatrix} x_t \\ 1 \end{pmatrix} + \begin{pmatrix} v_t \\ 0 \end{pmatrix} \\ y_t &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ 1 \end{pmatrix} + e_t \end{aligned} \tag{12}$$

Part 4b

```r
##-----------------------------------------------------------------
## EKF algorithm for use in Part 4 of computer exercise 2 in Advanced Time
## Series Analysis
##-----------------------------------------------------------------

ext_kalman <- function(y, aInit = 0.5, aVarInit = 1, sigma.v = 1) {
    ## aInit : The starting guess of the AR coefficient estimate aVarInit : The
    ## initial variance for estimation of the AR coefficient sigma.v : Standard
    ## deviation of the system noise of x in the filter

    # Initialize---- Init the state vector estimate
    zt <- c(0, aInit)
    # Init the variance matrices
    Rv <- matrix(c(sigma.v^2, 0, 0, 0), ncol = 2)
    # sigma.e : Standard deviation of the measurement noise in the filter
    Re <- 1

    # Init the P matrix, that is the estimate of the state variance
    Pt <- matrix(c(Re, 0, 0, aVarInit), nrow = 2, ncol = 2)
    # The state is [X a] so the differentiated observation function is
    Ht <- t(c(1, 0))
    # Init a vector for keeping the parameter a variance estimates
    aVar <- rep(NA, length(y))
    # and keeping the states
    Z <- matrix(NA, nrow = length(y), ncol = 2)
    Z[1, ] <- zt

    ## The Kalman filtering----
    for (t in 1:(length(y) - 1)) {
        # Derivatives (Jacobians)
        Ft <- matrix(c(zt[2], 0, zt[1], 1), ncol = 2)  # F_t-1
        # Ht does not change

        ## Prediction step
        zt = c(zt[2] * zt[1], zt[2])  #z_t|t-1 f(z_t-1|t-1)
        Pt = Ft %*% Pt %*% t(Ft) + Rv  #P_t|t-1

        ## Update step
        res = y[t] - zt[1]  # the residual at time t
        St = Ht %*% Pt %*% t(Ht) + Re  # innovation covariance
        Kt = Pt %*% t(Ht) %*% St^-1  # Kalman gain
        zt = zt + Kt * res  # z_t|t
        Pt = (diag(2) - Kt %*% Ht) %*% Pt  #P_t|t

        ## Keep the state estimate
        Z[t + 1, ] <- zt
        ## Keep the P[2,2], which is the variance of the estimate of a
        aVar[t + 1] <- Pt[2, 2]
    }
    return(list(zt = zt, Pt = Pt, Rv = Rv, aVar = aVar, Z = Z))
}
```

Check for converges in worst case



a = 0.5

state	sigma_v^2	sigma_a	a mean	a sd	a_var mean	a_var sd
1	10	1	0.2227370	0.0204999	0.0152562	0.0004328
2	1	1	0.3955557	0.0292888	0.0008427	0.0000681
3	10	10	0.2188578	0.0208357	0.0154694	0.0004450
4	1	10	0.3955225	0.0292323	0.0008432	0.0000693

a = -0.5

state	sigma_v^2	sigma_a	a mean	a sd	a_var mean	a_var sd
1	10	1	0.2227370	0.0204999	0.0152562	0.0004328
2	1	1	0.3955557	0.0292888	0.0008427	0.0000681
3	10	10	0.2188578	0.0208357	0.0154694	0.0004450
4	1	10	0.3955225	0.0292323	0.0008432	0.0000693

hvordan påvirker størrelsen ad sigma_v2 og hvordan påvirkes

variance of the system?

Improvements

do regulizing of the sigma vector.. add some to the diagonal