NECESSARILY NON-ANALYTIC INDUCTION PROOFS——SUMMARY OF SOME RESULTS

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§1 Introduction
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Some mathematical facts seem to not lend themselves to a "straightforward" induction proof—the proof attempt gets "stuck" at the induction step—but instead a "straightforward" induction proof of a "stronger" fact succeeds. The fact we originally wanted to prove then follows from this "stronger" fact. This proof method is usually called something like "strengthening of the induction hypothesis". However, there need not always be a precise sense in which the "stronger" fact is, in fact, stronger. Thus, following Hetzl and Wong (2017), I use the more general terminology "non-analytic induction proofs" for such proofs. Consequently, I will use "analytic induction proofs" to refer to ordinary "straightforward" induction proofs.

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Faced with a non-analytic induction proof, a natural question is whether the non-analyticity is necessary—that is, could we have proved what we wanted to prove by analytic induction, without the "detour" proving some other fact? Hetzl and Wong have made precise sense of such questions in the setting of first-order arithmetic.

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In §2 we look at two examples of non-analytic induction proofs, proving the two facts:

  (F1) For all natural numbers n: $1+3+5+\cdots+(2n-1) = k^2$ for some natural number
       k.

  (F2) For all natural numbers n: $1+2^{-2}+3^{-2}+\cdots+(n+1)^{-2} < 2$.

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In §3 we look at more examples of non-analytic induction proofs: correctness proofs for "tail-recursive" definitions of multiplication, of exponentiation, and of the factorial.

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In §4 I present a slight generalization and reformulation of some of the notions introduced by Hetzl and Wong.

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Using the notions introduced in §4 I show that there are precise senses in which
the example facts may only be proved by non-analytic induction.

— In §5 I show that (F1) may only be proved by non-analytic induction.

— In §6 I show that (F2) may only be proved by non-analytic induction.

— In §7 and §8 I show that the correctness properties from §3 may only be
proved by non-analytic induction.

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In §9 I discuss the results and present some ideas for future work.

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The lazy, or busy, reader may wish to skip §§6–8, as the core ideas of §§6–8 are
present in simpler form in §5. Then §3 may be skipped as well since it is a
prerequisite only for §7 and §8.

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§2 Two results proved by non-analytic induction
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For our first case, we have that the sum of any initial segment of the odd natural numbers is a perfect square, that is,

for all natural numbers n: $1+3+5+\cdots+(2n-1) = k^2$ for some natural number k,

or, avoiding ellipsis notation, the following.

**

DEFINITION 2.1. $f_2 : \mathbb{N} \to \mathbb{N}$ is defined recursively by

$f_2(0) := 0$,

$f_2(n+1) := f_2(n)+2n+1$.

**

FACT 2.2. For all natural numbers n: $f_2(n) = k^2$ for some natural number k.

PROOF ATTEMPT. "Straightforward" induction:

— Base case: We have

$f_2(0) = 0$ (by Definition 2.1)

$= 0^2$

so $f_2(0) = k^2$ for k = 0.

— Induction step:

$f_2(n+1) = f_2(n)+2n+1$ (by Definition 2.1)

$= k^2+2n+1$ (for some k, by induction hypothesis).

But k^2+2n+1 is not a perfect square for all natural numbers k and n so how do we proceed from here?

ACTUAL PROOF. It suffices to prove the following stronger fact, of which Fact 2.2 is a logical consequence.

$f_2(n) = n^2$ for all natural numbers n.

(This fact is not a logical consequence of Fact 2.2, so it is stronger than Fact 2.2 in at least that sense.)

We prove this fact by induction.

— Base case:

$$f_2(0) = 0 \quad \text{(by Definition 2.1)}$$

$$= 0^2.$$

— Induction step:

$$f_2(n+1) = f_2(n)+2n+1 \quad \text{(by Definition 2.1)}$$

$$= n^2+2n+1 \quad \text{(by induction hypothesis)}$$

$$= (n+1)^2. \qquad\qquad \square$$

**

For our second case, we have

$$1+2^{-2}+3^{-2}+\cdots+(n+1)^{-2} < 2 \text{ for all natural numbers } n,$$

or, avoiding ellipsis notation, the following.

**

DEFINITION 2.3. $g_2 : \mathbb{N} \to \mathbb{Q}$ is defined recursively by

$$g_2(0) := 1,$$

$$g_2(n+1) := g_2(n)+(n+2)^{-2}.$$

**

FACT 2.4. $g_2(n) < 2$ for all natural numbers n.

PROOF ATTEMPT. "Straightforward" induction:

— Base case:

$$g_2(0) = 1 \quad \text{(by Definition 2.3)}$$

$$< 2.$$

— Induction step:

$$g_2(n+1) = g_2(n)+(n+2)^{-2} \quad \text{(by Definition 2.3)}$$

$$< 2+(n+2)^{-2} \quad \text{(by induction hypothesis).}$$

But $2+(n+2)^{-2} \not< 2$ for any natural number n so how do we proceed from here?

ACTUAL PROOF. It clearly suffices to prove the stronger fact

$g_2(n) \leq 2-(n+1)^{-1}$ for all natural numbers n.

(In what sense is this fact stronger than Fact 2.4? It is at least stronger in the sense that for all $g : \mathbb{N} \to \mathbb{Q}$,

$g(n) \leq 2-(n+1)^{-1}$ for all n

implies

$g(n) < 2$ for all n,

while the converse implication does not hold for all $g : \mathbb{N} \to \mathbb{Q}$.)

We prove this fact by induction.

— Base case:

$g_2(0) = 1$ (by Definition 2.3)

        ≤ 1

        $= 2-(0+1)^{-1}$.

— Induction step:

$g_2(n+1) = g_2(n)+(n+2)^{-2}$ (by Definition 2.3)

        $\leq 2-(n+1)^{-1}+(n+2)^{-2}$ (by induction hypothesis)

        $= 2-(n+2)^{-1}-(n+1)^{-1}(n+2)^{-2}$ (by elementary arithmetic)

        $\leq 2-(n+2)^{-1}$

        $= 2-((n+1)+1)^{-1}$.   □

$**$

§3 Tail-recursion

For practical reasons one might want to implement recursive functions such that they are "tail-recursive". I am not overly familiar with "tail-recursion" but for present purposes what matters is that we have non-standard "tail-recursive" definitions of common functions. We will consider tail-recursive definitions of multiplication, of exponentiation, and of the factorial. For each definition one naturally desires a correctness proof. It turns out that such correctness proofs seem to require non-analytic induction.

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The technique we will use to convert the standard recursive definitions to tail-recursive definitions is to introduce an extra "accumulator" argument.

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DEFINITION 3.1.

(a) Multiplication $\times : \mathbb{N}^2 \to \mathbb{N}$ is defined recursively by

$$n \times 0 := 0,$$

$$n \times (m+1) := n \times m + n.$$

(b) Exponentiation $-^- : \mathbb{N}^2 \to \mathbb{N}$ is defined recursively by

$$n^0 := 1,$$

$$n^{m+1} := n^m \times n.$$

(c) The factorial $! : \mathbb{N} \to \mathbb{N}$ is defined recursively by

$$0! := 1,$$

$$(n+1)! := (n+1) \times n!.$$

(d) The function m-acc $: \mathbb{N}^3 \to \mathbb{N}$ is defined recursively by

$$\text{m-acc}(a,n,0) := a,$$

$$\text{m-acc}(a,n,m+1) := \text{m-acc}(a+n,n,m),$$

(e) The function e-acc $: \mathbb{N}^3 \to \mathbb{N}$ is defined recursively by

$$\text{e-acc}(a,n,0) := a,$$

$$\text{e-acc}(a,n,m+1) := \text{e-acc}(a \times n,n,m),$$

(e) The function f-acc : $\mathbb{N}^2 \to \mathbb{N}$ is defined recursively by

$$f\text{-acc}(a,0) := a,$$

$$f\text{-acc}(a,m+1) := f\text{-acc}(a \times (m+1), m).$$

**

EXAMPLE.

$$m\text{-acc}(0,5,3) = m\text{-acc}(0,5,2+1)$$

$$= m\text{-acc}(0+5,5,2) \qquad \text{(by Definition 3.1d)}$$

$$= m\text{-acc}(0+5,5,1+1)$$

$$= m\text{-acc}(0+5+5,5,1) \qquad \text{(by Definition 3.1d)}$$

$$= m\text{-acc}(0+5+5,5,0+1)$$

$$= m\text{-acc}(0+5+5+5,5,0) \qquad \text{(by Definition 3.1d)}$$

$$= 0+5+5+5 \qquad\qquad \text{(by Definition 3.1d)}$$

$$= 5 \times 3.$$

**

FACT 3.2. For all natural numbers n and m:

(a) $n \times m = m\text{-acc}(0,n,m)$,

(b) $n^m = e\text{-acc}(1,n,m)$,

(c) $n! = f\text{-acc}(1,n)$.

PROOF ATTEMPT. We try to prove (a) by induction on m. The base case is immediate. For the induction step we have

$$n \times (m+1) = n \times m + n \qquad \text{(by Definition 3.1a)}$$

$$= m\text{-acc}(0,n,m) + n \quad \text{(by induction hypothesis)}$$

and

$$m\text{-acc}(0,n,m+1) = m\text{-acc}(0+n,n,m) \quad \text{(by Definition 3.1d)}$$

$$= n\text{-acc}(n,n,m).$$

It seems we cannot get any further and that we are "stuck". Similarly one gets "stuck" when attempting "straightforward" induction proofs of (b) and (c).

ACTUAL PROOF. (a)-(c) are immediate consequences of the following stronger facts. For all natural numbers a, n and m:

(a') m-acc(a,n,m) = a+n×m,

(b') e-acc(a,n,m) = a×n^m,

(c') f-acc(a,n) = a×n!.

We prove (a'). The proof of (c') is similar and the proof of (b') is exactly the same up to substitution of function symbols and constants. For (a'), induction on m gives the following.

— Base case:

$$m\text{-}acc(a,n,0) = a \qquad \text{(by Definition 3.1d)}$$
$$= a+0$$
$$= a+n×0 \quad \text{(by Definition 3.1a)}.$$

— Induction step:

$$m\text{-}acc(a,n,m+1) = m\text{-}acc(a+n,n,m) \text{ (by Definition 3.1d)}$$
$$= (a+n)+n×m \qquad \text{(by induction hypothesis)}$$
$$= a+(n×m+n) \qquad \text{(by associativity and commutativity)}$$
$$= a+n×(m+1) \qquad \text{(by Definition 3.1a).} \qquad \square$$

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The above proof uses additive identity of 0 and associativity and commutativity of addition. These properties are in fact not necessary. We state and prove a more general fact as follows.

**

ASSUMPTION 3.3.

— Let A be any set.

— Let a_0 be an element in A.

— Let • : A^2 → A.

**

DEFINITION 3.4.

  (a) The function $\bullet\bullet$: $A\times\mathbb{N} \to A$ is defined recursively by

$$a\bullet\bullet 0 := a_0,$$

$$a\bullet\bullet(n+1) := (a\bullet\bullet n)\bullet a.$$

  (b) The function acc : $A^2\times\mathbb{N} \to A$ is defined recursively by

$$acc(a,b,0) := a,$$

$$acc(a,b,n+1) := acc(a\bullet b,b,n).$$

**

LEMMA 3.5. For all a and b in A and for all natural numbers n:

  $acc(a,b,n)\bullet b = acc(a\bullet b,b,n).$

PROOF. Induction on n.

  — Base case:

$$acc(a,b,0)\bullet b = a\bullet b \qquad \text{(by Definition 3.4b)}$$

$$= acc(a\bullet b,b,0) \quad \text{(by Definition 3.4b)}.$$

  — Induction step:

$$acc(a,b,n+1)\bullet b = acc(a\bullet b,b,n)\bullet b \quad \text{(by Definition 3.4b)}$$

$$= acc((a\bullet b)\bullet b,b,n) \quad \text{(by induction hypothesis)}$$

$$= acc(a\bullet b,b,n+1) \quad \text{(by Definition 3.4b)}. \qquad \square$$

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FACT 3.6. For all a in A and all natural numbers n:

$$a \bullet\bullet n = acc(a_0, a, n).$$

PROOF. With Lemma 3.5 at our disposal, "straightforward" induction on n will work.

— Base case:

$$a \bullet\bullet 0 = a_0 \qquad \text{(by Definition 3.4a)}$$

$$= acc(a_0, a, 0) \quad \text{(by Definition 3.4b)}.$$

— Induction step:

$$a \bullet\bullet (n+1) = (a \bullet\bullet n) \bullet a \qquad \text{(by Definition 3.4a)}$$

$$= acc(a_0, a, n) \bullet a \quad \text{(by induction hypothesis)}$$

$$= acc(a_0 \bullet a, a, n) \quad \text{(by Lemma 3.5)}$$

$$= acc(a_0, a, n+1) \quad \text{(by Definition 3.4b)}. \qquad \square$$

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REMARK. The above proof is an analytic induction proof. However, as is shown in §8, without first proving, by induction, something like Lemma 3.5, an analytic induction proof is not possible.

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§4 Definitions
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This section presents a reformulation and slight generalization of some of the
notions introduced by Hetzl and Wong (2017). I choose to introduce some
non-conventional machinery, allowing for rich languages in order to be able to
conveniently and transparently represent mathematical facts and proofs.

**

DEFINITION 4.1.

   — The *full (first-order) language of arithmetic*, notation $\mathscr{L}[full]$, is the
     first-order language with the following vocabulary. For each natural
     number n:

         — n is a constant symbol,

         — for each function $f : \mathbb{N}^{n+1} \to \mathbb{N}$: f is a function symbol of arity n+1,

         — for each relation $P \subseteq \mathbb{N}^{n+1}$: P is a relation symbol of arity n+1.

   — The *minimal (first-order) language of arithmetic*, notation $\mathscr{L}[min]$, is
     the $\mathscr{L}[full]$-reduct with signature $\langle 0,1,+\rangle$.

   — The *standard (first-order) language of arithmetic*, notation $\mathscr{L}_0$, is the
     $\mathscr{L}[full]$-reduct with signature $\langle 0,1,+,\times,<\rangle$.

   — A first-order language is a *(first-order) language of arithmetic* if and
     only if it is an $\mathscr{L}[min]$-expansion.

   — Let L be a language of arithmetic.

         — If L is an $\mathscr{L}[full]$-reduct then L is *pure*; otherwise L is *impure*.

         — pure(L) := L∩$\mathscr{L}[full]$ is the *pure part* of L.

         — impure(L) := L-$\mathscr{L}[full]$ is the *impure part* of L.

**

Thus any pure language of arithmetic has as symbols natural numbers and
operations and relations on natural numbers. This is just a convenient "trick"
which allows us to easily define the standard model for these languages.

**

DEFINITION 4.2.

— Let L be a pure language of arithmetic.

  — The *standard L-model* has domain ℕ and each symbol interpreted as
    itself.

  — Let M be an L-model. If M is isomorphic to the standard L-model then
    M is *standard*; otherwise M is *non-standard*.

  — The L-theory *true L-arithmetic* is the theory of the standard
    L-model.

  — Any subset of true L-arithmetic is a *theory of L-arithmetic*.

— Let L be an impure language of arithmetic, let M be an L-model and let T
  be an L-theory.

  — M is *standard* if its pure(L)-reduct is; otherwise M is
    non-standard.

  — T is a *theory of L-arithmetic* if and only if T is true in some
    standard L-model.

                                    **

REMARKS. Let L be an impure language.

— There are infinitely many non-isomorphic standard L-models—one for each
  way of expanding the standard pure(L)-model to L. (Similarly, there are
  infinitely many standard L-models that are not elementary equivalent. Thus
  there are also infinitely many non-compatible theories of L-arithmetic.)

— For an L-theory T there may be none, one, or many non-isomorphic standard
  models of T (and T is a theory of L-arithmetic if and only if there is at
  least one such standard model).

                                    **

NOTATIONS.

- $\mathbb{N}_0$ is the standard $\mathscr{L}_0$-model.

- $\mathbb{N}[\min]$ is the standard $\mathscr{L}[\min]$-model.

- $\mathbb{N}[full]$ is the standard $\mathscr{L}[full]$-model.

- T_0 is true $\mathscr{L}_0$-arithmetic.

- $T[\min]$ is true $\mathscr{L}[\min]$-arithmetic.

- $T[full]$ is true $\mathscr{L}[full]$-arithmetic.

                                    **

DEFINITION 4.3. Let L be a language of arithmetic and let φ(x) be an L-formula in the free variable x. The *induction instance* for φ(x) is the L-sentence

$$\text{IND}(φ) :\equiv φ(0) \wedge \forall x(φ(x) \to φ(x+1)) \to \forall x.φ(x).$$

                                    **

DEFINITION 4.4. Let L be a language of arithmetic and let T be an L-theory. Let φ(x) be an L-formula in the free variable x. Say that *T proves ∀x.φ(x) by necessarily non-analytic induction* if and only if there is an L-formula ψ(x) in the free variable x such that

(1) $T, \text{IND}(φ) \nvdash \forall x.φ(x)$,

(2) $T \vdash φ(0)$,

(3) $T \vdash ψ(0)$,

(4) $T \vdash \forall x: ψ(x) \to ψ(x+1)$,

(5) $T \vdash \forall x.ψ(x) \to \forall x.φ(x)$.

Under conditions (1)–(5) we also say that *ψ(x) witnesses that T proves ∀x.φ(x) by necessarily non-analytic induction*.

                                    **

REMARKS.

- The above definition does not require T to be a theory of arithmetic (that is, T could contain sentences that are false in all standard L-models). I will of course require T to be a theory of arithmetic when using this definition to make precise sense of the necessary non-analyticity of non-analytic induction proofs (of arithmetic) found in the wild. But I do

not rule out that there might be interesting uses of the above definition
with T not a theory of arithmetic.

- As we saw when proving Fact 3.6, an analytic induction proof may be
  possible once an appropriate lemma has been proved (by induction). Such
  cases could qualify as necessarily non-analytic induction proofs under
  Definition 4.4 (and indeed, the proof of Fact 3.6 does, as shown in §8).
  This is because for any two induction axioms there is always an induction
  axiom logically equivalent to their conjunction. Thus any proof along the
  lines of "induction proof of a lemma followed by an analytic induction
  proof" could be rewritten as a single non-analytic induction proof (but
  that proof would probably not look very "natural"). In light of this,
  perhaps I should rethink the terminology, since it might seem a bit
  inappropriate to call proofs such as the proof of Fact 3.6 for
  "non-analytic induction proofs".

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An alternative way of phrasing condition (1), provided by the following lemma,
might be more intuitive.

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LEMMA 4.5. (1) and (2) are equivalent to (2) and

 (1') $T \nvdash \forall x: \varphi(x) \rightarrow \varphi(x+1)$.

PROOF. The direction from left (that is, from (1) and (2)) to right (that is, to
(1') and (2)) is immediate. The direction from right to left is an easy (but
non-trivial) exercise in propositional logic.   □

**

REMARK. I chose (1) instead of (1') in the definition precisely because the
direction (1) ⇒ (1') is immediate (given (2)) while the direction (1') ⇒ (1)
(given (2)) is not.

**

When trying to prove that typical T and $\varphi(x)$ satisfy the above definition, the
hardest part will be—as far as I can tell at least—to show condition (1) (or
condition (1')). Therefore it makes sense to break out a lemma that captures the
idea of using non-standard models to establish (1).

**

LEMMA 4.6.

    — Let L be a language of arithmetic.

    — Let T be an L-theory.

    — Let $\varphi(x)$ be an L-formula in the free variable x.

    — Suppose $T \vdash \varphi(n)$ for all numerals n.

The following are then equivalent.

    — $T, \text{IND}(\varphi) \nvdash \forall x.\varphi(x)$.

    — There is a non-standard L-model $M \vDash T$ and a non-standard number c in M
       such that $M \vDash \varphi(c)$ and $M \nvDash \varphi(c+1)$.

PROOF. Since $T \vdash \varphi(0)$, Lemma 4.5 gives that

    $T, \text{IND}(\varphi) \nvdash \forall x.\varphi(x)$

is equivalent to

    $T \nvdash \forall x: \varphi(x) \rightarrow \varphi(x+1)$,

which by soundness and completeness is equivalent to

    there is $M \vDash T$ and c in M such that $M \vDash \varphi(c)$ and $M \nvDash \varphi(c+1)$

which, since $T \vdash \varphi(n)$ for all numerals n, is equivalent to

    there is nonstandard $M \vDash T$ and a non-standard number c in M such that
    $M \vDash \varphi(c)$ and $M \nvDash \varphi(c+1)$.   □

<div align="center">**</div>

REMARK. The direction from the existence of a non-standard "countermodel" to
$T, \text{IND}(\varphi) \nvdash \forall x.\varphi(x)$ holds even if we drop the requirement

  (*) $T \vdash \varphi(n)$ for all numerals n.

This direction is the one we will use so we need not worry about (*) when
applying the lemma (but nevertheless, (*) will hold every time we apply the
lemma).

<div align="center">**</div>

§5 Fact 2.2 must be proved by non-analytic induction
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DEFINITION 5.1. The language L_5 of arithmetic is $\mathscr{L}[full]$ expanded with the unary function symbol 'f'.

<center>**</center>

REMARK. L_5 is thus an impure language, with $impure(L_5) = \{f\}$ (and with $pure(L_5) = \mathscr{L}[full]$).

<center>**</center>

DEFINITION 5.2.

   — The L_5-sentences $DEF_0(f)$ and $DEF_+(f)$ are defined by

$$DEF_0(f) :\equiv f(0) = 0.$$

$$DEF_+(f) :\equiv \forall x.\ f(x+1) = f(x)+2x+1.$$

   — The theory T_5 of L_5-arithmetic is defined by

$$T_5 :\equiv T[full], DEF_0(f), DEF_+(f).$$

   — The L_5-formulas $\varphi_5(x)$ and $\psi_5(x)$ are defined by

$$\varphi_5(x) :\equiv \exists y.\ f(x) = y^2,$$

$$\psi_5(x) :\equiv f(x) = x^2.$$

<center>**</center>

The reader who finds the following fact puzzling or even contradictory might want to jump ahead to the remark after the proof.

<center>**</center>

FACT 5.3. $\psi_5(x)$ witnesses that T_5 proves $\forall x.\varphi_5(x)$ by necessarily non-analytic induction.

PROOF. We need to show (1)–(5) of Definition 4.4. (2)–(5) are easy. To show (1) we appeal to Lemma 4.6 and exhibit a non-standard L_5-model $M \vDash T_5$ with an element c such that $M \vDash \varphi_5(c)$ and $M \nvDash \varphi_5(c+1)$. The following construction is due to Matt Kaufmann (private communication).

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Let $M_0 \vDash T[full]$ be a non-standard $\mathscr{L}[full]$-model and let c be any non-standard number in M_0. We expand M_0 to an L_5-model M by defining the interpretation $f^M : M_0 \rightarrow M_0$ as follows.

$\qquad f^M(n) := n^2-9c^2$ if $n = 5c+z$ for some integer z,

$\qquad f^M(n) := n^2$ otherwise.

We need to verify that this definition is well-formed—that is, we need to verify that n^2-9c^2 is never negative when $n = 5c+z$ for some integer z. We have

$$f^M(5c+z) = (5c+z)^2-9c^2$$

$$= 25c^2+10zc+z^2-9c^2$$

$$= 16c^2+10zc+z^2$$

$$= c(16c+10z)+z^2$$

which is positive since

$$c > 0 \quad \text{(since c is non-standard)},$$

$$16c+10z > 0 \quad \text{(since c is non-standard and z is not)},$$

$$z^2 \geq 0 \quad \text{(since } z^2 \text{ is c square).}$$

With f^M so defined it is easy to verify that $M \vDash T_5$. We have $M \vDash \varphi_5(5c)$ since

$$\varphi_5(5c) = (5c)^2-16c^2$$

$$= 25c^2-16c^2$$

$$= 9c^2$$

$$= (3c)^2.$$

It remains to show $M \nvDash \varphi_5(5c+1)$. We have

$$\varphi_5(5c+1) = (5c+1)^2-16c^2$$

$$= 25c^2+10c+1-16c^2$$

$$= 9c^2+10c+1.$$

This is not a perfect square in M since

$$\forall x \forall y: x > 0 \rightarrow 9x^2+10x+1 \neq y^2$$

is a theorem of $T[full]$. Thus we have $M \nvDash \varphi_5(5c+1)$. $\qquad\qquad\qquad\qquad$ □

**

REMARK. Fact 5.3 might seem contradictory. After all, there is a unique standard L_5-model, namely $\mathbb{N}[\text{full}]$ with f interpreted by f_2 (from §2), and in this model $\forall x.\varphi_5(x)$ is of course true. The catch is that T_5 is weaker than the theory of this standard model, since T_5 does not contain any induction axioms involving the symbol f. In particular, T_5 does not prove that f is equal to f_2 (which, being a function $\mathbb{N} \to \mathbb{N}$, is a symbol of $\mathscr{L}[\text{full}]$ and thus also of L_5). For more discussion of this phenomenon see §9.

**

The following alternative proof of Fact 5.3 might be useful for the reader, since it demonstrates the proof idea used in §6, but in a simpler setting. The alternative proof is roughly the following. Suppose we have a non-standard model $M \vDash T_5\text{-DEF}_0(f)$ with a non-standard c such that $M \vDash \varphi_5(c)$ and $M \nvDash \varphi_5(c+1)$. Whether $M \vDash \text{DEF}_0(f)$ holds depends only on the values f^M takes on the standard numbers. Furthermore, whether $\text{DEF}_+(f)$ holds on the non-standard numbers does not depend on the values f^M takes on the standard numbers. Thus, by reinterpreting (if necessary) f only on the standard numbers of M, we can turn M into a model $M' \vDash T_5$. Since f^M and $f^{M'}$ will then agree on the non-standard numbers, we will have $M' \vDash \varphi_5(c)$ and $M' \nvDash \varphi_5(c+1)$. Thus, appealing to Lemma 4.6 (and the fact that conditions (2)-(5) of Definition 4.4 are easy), we are done. Thus to prove Fact 5.3 it suffices to construct such a non-standard model M, which can be done by applying the compactness theorem in the usual way. In more detail, we do as follows.

**

DEFINITION 5.4.

  — L_5' is L_5 expanded with the constant symbol 'c'.

  — The L_5'-theories T_5' and T_5'' are defined by

      $T_5' := T_5, \varphi_5(c), \neg\varphi_5(c+1), c>0, c>1, c>2, \ldots$

      $T_5'' := T_5'\text{-DEF}_0(f).$

**

LEMMA 5.5. If T_5'' has a model then so does T_5'.

PROOF. Suppose $M \vDash T_5''$. We construct an L_5'-model N from M by reinterpreting f:

$$f^N(0) := 0,$$

$$f^N(n+1) := f^N(n)+2n+1 \quad \text{if n is standard,}$$

$$f^N(n) := f^M(n) \qquad \text{if n is non-standard.}$$

Then we have $N \vDash T_5'$, as the reader can easily verify.   □

                                **

To construct the model required to apply Lemma 5.5 we will use the compactness theorem. For each natural number m we will provide an L_5'-expansion $N(m) \vDash T_5''$ of $\mathbb{N}[full]$ with $c^{N(m)} = m$. This shows that any finite fragment of T_5'' has a model (namely, $N(m)$ for a large enough m). Thus the whole of T_5'' has a model, by the compactness theorem. A key to the construction of the models $N(m)$ is that since $DEF_0(f)$ is omitted from T_5'', we need not set $f^{N(m)}(0) := 0$. It turns out that setting $f^{N(m)}(0) := 2m+1$ (while obeying $DEF_+(f)$) we get that $f^{N(m)}(m)$ is a perfect square while $f^{N(m)}(m+1)$ is not.

                                **

DEFINITION 5.6. For each natural number m, the function $\beta_m : \mathbb{N} \to \mathbb{N}$ is defined recursively by

$$\beta_m(0) := 2m+1,$$

$$\beta_m(n+1) := \beta_m(n)+2n+1.$$

                                **

LEMMA 5.7. For all natural numbers m:

$$\beta_m(m) = (m+1)^2.$$

PROOF. See §B.   □

                                **

LEMMA 5.8. For all natural numbers m:

$$\beta_m(m+1) \text{ is not a perfect square.}$$

PROOF. See §B.   □

                                **

DEFINITION 5.9. For any natural number m, N(m) is the L_5'-expansion of $\mathbb{N}[full]$ defined by

$$f^{N(m)} := \beta_m,$$

$$c^{N(m)} := m.$$

<div align="center">**</div>

LEMMA 5.10. For all natural numbers m:

  (1) $N(m) \vDash T[full]$,

  (2) $N(m) \vDash DEF_+(f)$,

  (3) $N(m) \vDash \varphi(c)$,

  (4) $N(m) \vDash \neg\varphi(c+1)$.

PROOF. We have (1) since N(m) is an $\mathbb{N}[full]$-expansion. We have (2) immediately from definitions (Definitions 5.2, 5.4, 5.6 and 5.9). We have (3) and (4) by Lemmas 5.7 and 5.8, respectively.   □

<div align="center">**</div>

ALTERNATIVE PROOF OF FACT 5.3. By Lemmas 4.6 and 5.5 (and since (2)-(5) of Definition 4.4 are easy) it suffices to exhibit a model of T_5''. Let T be a finite fragment of T_5''. Then we have $N(m) \vDash T$ for m sufficiently large, by Lemma 5.10 (and Definitions 5.2, 5.4 and 5.9). Thus any finite fragment of T'' has a model and thus, by compactness, so does T_5.   □

<div align="center">**</div>

§6 Fact 2.4 must be proved by non-analytic induction
───

Since Fact 2.4 is a statement involving rationals, a little work is needed to phrase it as a natural statement in first-order arithmetic (that is, without involving any coding).

<div align="center">**</div>

DEFINITION 6.1. $g_6 : \mathbb{N} \to \mathbb{N}$ and $h_6 : \mathbb{N} \to \mathbb{N}$ are defined recursively by

$$g_6(0) := 1,$$

$$g_6(n+1) := (n+2)^2 g_6(n) + h_6(n),$$

$$h_6(0) := 1,$$

$$h_6(n+1) := (n+2)^2 h_6(n).$$

<div align="center">**</div>

By induction, we have

$$g_2(n) = g_6(n)/h_6(n)$$

so the inequality $g_2(n) < 2$ can be rewritten as

$$g_6(n) < 2h_6(n).$$

Similarly, the inequality $g_2(n) \leq 2-(n+1)^{-1}$ becomes

$$(n+1)g_6(n) \leq (2n+1)h_6(n).$$

<div align="center">**</div>

All in all, we have the following rephrasings of Fact 2.4 and its proof attempt and proof.

<div align="center">**</div>

LEMMA 6.2. For all natural numbers n, if $(n+1)g_6(n) \leq (2n+1)h_6(n)$ then $g_6(n) < 2h_6(n)$.

PROOF. Suppose $(n+1)g_6(n) \leq (2n+1)h_6(n)$. Since $(2n+1)h_6(n) < 2(n+1)h_6(n)$ we then have $(n+1)g_6(n) < 2(n+1)h_6(n)$ so we must have $g_6(n) < 2h_6(n)$.   □

<div align="center">**</div>

FACT 6.3. $g_6(n) < 2h_6(n)$ for all n.

PROOF ATTEMPT. Induction on n.

— Base case:

$$g_6(0) = 1 \qquad \text{(by Definition 6.1)}$$

$$< 2$$

$$= 2 \times 1$$

$$= 2h_6(0) \quad \text{(by Definition 6.1)}.$$

— Induction step:

$$g_6(n+1) = (n+2)^2 g_6(n) + h_6(n) \quad \text{(by Definition 6.1)}$$

$$< (n+2)^2 \times 2h_6(n) + h_6(n) \quad \text{(by induction hypothesis)}$$

But for any n we have $(n+2)^2 \times 2h_6(n) + h_6(n) \not< 2h_6(n+1) = 2(n+2)^2 h_6(n)$. Thus we are "stuck".

ACTUAL PROOF. By Lemma 6.2, it suffices to prove

$$(n+1)g_6(n) \le (2n+1)h_6(n) \text{ for all n.}$$

We prove this by induction on n.

— Base case:

$$(0+1)g_6(0) = (0+1) \times 1 \qquad \text{(by Definition 6.1)}$$

$$= 1$$

$$\le 1$$

$$= (2 \times 0 + 1) \times 1$$

$$= (2 \times 0 + 1)h_6(0) \quad \text{(by Definition 6.1)}.$$

— Induction step: Our induction hypothesis is

(IH) $(n+1)g_6(n) \le (2n+1)h_6(n)$

and we must show

(GOAL) $(n+2)g_6(n+1) \le (2n+3)h_6(n+1)$.

We have

$$(n+2)g_6(n+1) = (n+2)((n+2)^2 g_6(n)+h_6(n)) \quad \text{(by Definition 6.1)}$$

$$= (n+2)^3 g_6(n)+(n+2)h_6(n)$$

and

$$(2n+3)h_6(n+1) = (2n+3)(n+2)^2 h_6(n) \quad\quad\quad \text{(by Definition 6.1)}$$

$$= (2n+3)(n+2)(n+2)h_6(n)$$

$$= (2n^2+7n+6)(n+2)h_6(n)$$

$$= (2n^2+7n+5)(n+2)h_6(n)+(n+2)h_6(n)$$

so (GOAL) is equivalent to

$$(n+2)^3 g_6(n) \le (2n^2+7n+5)(n+2)h_6(n),$$

which is equivalent to

$$(n+2)^2 g_6(n) \le (2n^2+7n+5)h_6(n),$$

which we have by

$$(n+2)^2 g_6(n) = ((n+1)+1)^2 g_6(n)$$

$$= ((n+1)^2+2(n+1)+1)g_6(n)$$

$$= ((n+1)^2+2(n+1))g_6(n)+g_6(n)$$

$$= (n+3)(n+1)g_6(n)+g_6(n)$$

$$\le (n+3)(2n+1)h_6(n)+g_6(n) \quad\quad \text{(by (IH))}$$

$$\le (n+3)(2n+1)h_6(n)+2h_6(n) \quad\quad \text{(by Lemma 6.2 and (IH))}$$

$$= (2n^2+7n+5)h_6(n). \quad\quad\quad\quad\quad\quad\quad \square$$

**

REMARK. I find it a bit curious that the proof of Fact 2.4 uses only one application of the induction hypothesis, while the above proof of Fact 6.3 uses two applications. Is there a proof of Fact 6.3 which, in some sensible precise sense, uses only one application of the induction hypothesis?

**

DEFINITION 6.4.

  — The language L_6 of arithmetic is $\mathscr{L}$[full] expanded with the unary function
    symbols 'g' and 'h'.

  — The L_6-sentences $DEF_0(g)$, $DEF_+(g)$, $DEF_0(h)$ and $DEF_+(h)$ are defined by

        $DEF_0(g) :\equiv g(0) = 1,$

        $DEF_+(g) :\equiv \forall x.\ g(x+1) = (x+2)^2 \times g(x)+h(x),$

        $DEF_0(h) :\equiv h(0) = 1,$

        $DEF_+(h) :\equiv \forall x.\ h(x+1) = (x+2)^2 \times h(x).$

  — The theory T_6 of L_6-arithmetic is defined by

        $T_6 :\equiv T[full],DEF_0(g),DEF_+(g),DEF_0(h),DEF_+(h).$

  — The L_6-formulas $\varphi_6(x)$ and $\psi_6(x)$ are defined by

        $\varphi_6(x) :\equiv g(x) < 2 \times h(x),$

        $\psi_6(x) :\equiv (x+1) \times g(x)+h(x) \leq 2 \times (x+1) \times h(x).$

                              **

FACT 6.5. $\psi_6(x)$ witnesses that T_6 proves $\forall x.\varphi_6(x)$ by necessarily non-analytic
induction.

                              **

To prove Fact 6.5 we will proceed similarly to the alternative proof of
Fact 5.3.

                              **

DEFINITION 6.6.

  — L_6' is L_6 expanded with the constant symbol 'c'.

  — The L_6'-theories T_6' and T_6'' are defined by

        $T_6' :\equiv T_6,\varphi_6(c),\neg\varphi_6(c+1),c>0,c>1,c>2,\ldots$

        $T_6'' :\equiv T_6'-DEF_0(g).$

                              **

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LEMMA 6.7. If T_6'' has a model then so does T_6'.

PROOF. Similar to the proof of Lemma 5.5.   □

$$**$$

To easier make sense of the following two definitions, first recall that
$g_2 : \mathbb{N} \to \mathbb{Q}$ was recursively defined (in §2) such that

for all n: $g_2(n) = 1+2^{-2}+3^{-2}+\cdots+(n+1)^{-2}$.

Then recall Euler's famous solution to "the Basel problem" (Lemma C.1):

$g_2(n) \to \pi^2/6$ as $n \to \infty$.

$$**$$

DEFINITION 6.8. For each natural number m, $\xi_m : \mathbb{N} \to \mathbb{R}$ is defined by

$\xi_m(n) := 2+(m+1)^{-1}+g_2(n)-\pi^2/6$.

$$**$$

DEFINITION 6.9. For each natural number m, $\gamma_m : \mathbb{N} \to \mathbb{R}$ is defined recursively by

$\gamma_m(0) := 3+(m+1)^{-1}-\pi^2/6$,

$\gamma_m(n+1) := (n+2)^2\gamma_m(n)+h_6(n)$.

$$**$$

LEMMA 6.10. For all natural numbers m:

$\gamma_m(n)/h_6(n) = \xi_m(n)$.

PROOF. See §C.   □

$$**$$

LEMMA 6.11. For each natural number m there is a unique natural number n such
that

$\xi_m(n) < 2$

and

$\xi_m(n+1) \not< 2$.

PROOF. See §C.   □

**

DEFINITION 6.12.

$$\mu \; : \quad \mathbb{N} \to \mathbb{N}$$

$\mu(m) :=$ the n such that $\xi_m(n) < 2$ and $\xi_m(n+1) \not< 2$.

**

LEMMA 6.13. μ is unbounded.

PROOF. See §C.

**

We now reuse the notation 'N(m)' (that is, Definition 5.9 does not apply anymore).

**

DEFINITION 6.14. For each natural number m, N(m) is the L_6'-expansion of $\mathbb{N}[full]$ defined by

$$g^{N(m)} := \gamma_m,$$

$$h^{N(m)} := h_6,$$

$$c^{N(m)} := \mu(m).$$

**

LEMMA 6.15. For all natural numbers m:

(a) $N(m) \vDash T[full]$,

(b) $N(m) \vDash DEF_+(g)$,

(c) $N(m) \vDash DEF_0(h)$,

(d) $N(m) \vDash DEF_+(h)$,

(e) $N(m) \vDash \varphi_6(c)$,

(f) $N(m) \vDash \neg\varphi_6(c+1)$.

PROOF. We have (a) since N(m) is an $\mathbb{N}[full]$-expansion. We have (b) by Definitions 6.4, 6.9 and 6.14. We have (c) and (d) by Definitions 6.1, 6.4 and 6.14. By Lemma 6.10 (and Definitions 6.4 and 6.14), (e) and (f) are equivalent to $\xi_m(\mu(m)) < 2$ and $\xi_m(\mu(m)+1) \not< 2$, respectively, which we have by

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Definition 6.12.   □

                                                    **

LEMMA 6.16. T_6'' has a model.

PROOF. Let T be a finite fragment of T_6''. By Lemmas 6.13 and 6.15 (and by
Definitions 6.4, 6.6 and 6.14), we have $N(m) \vDash T$ for m sufficiently large. Thus
any finite fragment of T_6'' has a model and thus, by compactness, T_6'' has a
model.   □

                                                    **

PROOF OF FACT 6.5. Conditions (2)-(5) of Definition 4.4 are easy. Condition (1)
follows from Lemmas 4.6, 6.7 and 6.16.   □

                                                    **

Is there a "more direct" construction of a non-standard model satisfying the
conditions of Lemma 4.6? I have not been able to come up with one. We might try
to adapt the method used to prove Fact 5.3—that is, we might try to cleverly
expand an arbitrary non-standard $\mathscr{L}[full]$-model $M_0 \vDash T[full]$ to an L_6-model
$M \vDash T_6$ such that there is a non-standard c such that

$$M \vDash \varphi_6(c)$$

and

$$M \nvDash \varphi_6(c+1).$$

Thus we need a c in M_0 and interpetations $g^M, h^M : M_0 \to M_0$ such that

  (a) $g^M(0) = h^M(0) = 1$,

  (b) for all n in M_0:

        — $g^M(n+1) = (n+2)^2 g^M(n) + h^M(n)$,

        — $h^M(n+1) = (n+2)^2 h^M(n)$,

  (c) $g^M(c) < 2h^M(c)$,

  (d) $g^M(c+1) \nless 2h^M(c+1)$.

By (c) we must have $h^M(c) \neq 0$. Then we must have

$$h^M(c) = h^M((c-1)+1)$$

$$= (c+1)^2 h^M(c-1) \qquad \text{(by (b))}$$

$$= (c+1)^2 h^M((c-2)+1)$$

$$= (c+1)^2 c^2 h^M(c-2) \qquad \text{(by (b))}$$

$$= (c+1)^2 c^2 (c-1)^2 h^M(c-3) \qquad \text{(by (b))}$$

$$= (c+1)^2 c^2 (c-1)^2 (c-2)^2 h^M(c-4) \qquad \text{(by (b))}$$

$$\vdots$$

Thus it is necessary that $h^M(c)$ is non-zero and divisible by each of

$$(c+1)^2, c^2, (c-1)^2, (c-2)^2, \ldots$$

We get similar requirements on g^M. This makes it tricky to come up with explicit definitions that simultaneously satisfy (a)-(d)—or at least, I have not been able to do so.

<div align="center">**</div>

§7 Fact 3.2c must be proved by non-analytic induction
───

DEFINITION 7.1.

    — The language L_7 of arithmetic is $\mathscr{L}$[full] expanded with the binary function
      symbol 'acc$_7$'.

    — The L_7-sentence DEF(acc$_7$) is defined by

          DEF(acc$_7$) :≡

           $\forall$a. acc$_7$(a,0) = a $\land$ $\forall$a$\forall$x. acc$_7$(a,x+1) = acc$_7$(a×(x+1),x).

    — The theory T_7 of L_7-arithmetic is defined by

          T_7 :≡ T[full],DEF(acc$_7$).

    — The L_7-formulas φ_7(x) and ψ_7(x) are defined by

          φ_7(x) :≡ x! = acc$_7$(1,x),

          ψ_7(x) :≡ $\forall$a. acc$_7$(a,x) = a×x!.

                                 **

FACT 7.2. ψ_7(x) witnesses that T_7 proves $\forall$x.φ_7(x) by necessarily non-analytic
induction.

                                 **

To prove Fact 7.2 we will proceed as in the proof of Fact 5.3.
Conditions (2)–(5) of Definition 4.4 are straightforward. To prove (1) we take
any non-standard model M_0 ⊨ T[full] and expand it to an L_7-model M, interpreting
acc$_7$ such that we get M ⊨ T_7 and a non-standard c such that M ⊨ φ_7(c) and
M ⊭ φ_7(c+1).

                                 **

ASSUMPTION 7.3.

    — Let M_0 ⊨ T[full] be a non-standard $\mathscr{L}$[full]-model.

    — Let c be a non-standard number in M_0.

                                 **

We will expand M_0 to an L_7-model M ⊨ T_7 such that M ⊨ φ_7(c) and M ⊭ φ_7(c+1). The
defining equations for acc$_7$ (that is, the sentence DEF(acc$_7$)) fix all its
interpretations acc$_7^M$(a,n) where n is standard. For non-standard n we have quite

some leeway. To achieve $M \vDash \varphi_7(c)$ and $M \nvDash \varphi_7(c+1)$ we will put

    $acc_7{}^M(1,c) := c!$

and

    $acc_7{}^M(1,c+1) := 0.$

These together with the second conjunct of DEF(acc_7) then gives

    $c! = acc_7{}^M(1,c)$

       $= acc_7{}^M(c,c-1)$

       $= acc_7{}^M(c(c-1),c-2)$

       $= acc_7{}^M(c(c-1)(c-2),c-3)$

          $\vdots$

respectively

    $0 = acc_7{}^M(1,c+1),$

       $= acc_7{}^M(c+1,c),$

       $= acc_7{}^M((c+1)c,c-1),$

       $= acc_7{}^M((c+1)c(c-1),c-2),$

       $= acc_7{}^M((c+1)c(c-1)(c-2),c-3),$

          $\vdots$

The defining equations impose no further restrictions so we can put
$acc_7{}^M(n,m) := 0$ for all yet to be defined cases.

                              **

We will now make the construction in the preceding paragraph precise.

                              **

DEFINITION 7.4.

— The set $A \subseteq M_0^2$ is defined by

$$A := \{(a,n) : n \text{ is standard}\},$$

— The set $B \subseteq M_0^2$ is defined inductively by

$$\frac{}{(1,c) \text{ is in } B} \qquad \frac{(a,n+1) \text{ is in } B}{(a(n+1),n) \text{ is in } B} \ .$$

— The set $C \subseteq M_0^2$ is defined by

$$C := M_0^2 - (A \cup B).$$

<div align="center">**</div>

REMARK. The elements of B are thus

$(1,c)$,

$(c,c-1)$,

$(c(c-1),c-2)$,

$(c(c-1)(c-2),c-3)$,

$\vdots$

<div align="center">**</div>

LEMMA 7.5. A, B and C partitions M_0^2.

PROOF. Clearly:

— $A \cup B \cup C = M_0^2$.

— C is disjoint from A and from B.

It thus remains to show that A and B are disjoint. If (a,n) is in A then n is standard but if (a,n) is in B then n is non-standard. Thus A and B have no common elements and thus they are disjoint.   □

<div align="center">**</div>

By Lemma 7.5 the following definition is well-formed.

<div align="center">**</div>

DEFINITION 7.6. $acc_7^M : M_0^2 \rightarrow M_0$ is defined by

    $acc_7^M(a,n) := a \times n!$ if n is standard,

    $acc_7^M(a,n) := c!$ if (a,n) is in B,

    $acc_7^M(a,n) := 0$ if (a,n) is in C.

<div align="center">**</div>

LEMMA 7.7. Let X be A, B or C. For all a and n in M_0, (a,n+1) is in X if and only if (a(n+1),n) is.

PROOF. It suffices to show this for X = A and X = B (since then it must hold for C, by definition of C).

  — Case X = A:

      (a,n+1) is in A iff n+1 is standard

                      iff n is standard

                      iff (a(n+1),n) is in A.

  — Case X = B: The 'only if' direction holds by definition. For the 'if' direction, suppose (a(n+1),n) is in B. Case splitting on the inductive definition of B gives the following.

      — Case (a(n+1),n) = (1,c): Then n = c and a(n+1) = 1, so a(c+1) = 1 so a = c+1 = 1, so c = 0. Since c is non-standard this is a contradiction so this case is not possible.

      — Case (a(n+1),n) = (a'(n'+1),n') for some a' and n' in M_0 such that (a',n'+1) is in B: Then a(n+1) = a'(n'+1) and n = n' so a(n+1) = a'(n+1) so a = a' (by the appropriate well-known theorem of T[full]). Thus (a,n+1) = (a',n'+1) and since (a',n'+1) is in B so is (a,n+1). □

<div align="center">**</div>

LEMMA 7.8. For all a and n in M_0, if (a,n) is in B then n ≤ c.

PROOF. Induction on "(a,n) is in B".

  — Case (a,n) = (1,c): Immediate.

  — Case (a,n) = (a'(n+1),n) for some a' in M_0 and with induction hypothesis n+1 ≤ c: Immediate. □

<div align="center">**</div>

LEMMA 7.9.

   (1) $M \vDash T_7$.

   (2) $M \vDash \varphi_7(c)$.

   (3) $M \nvDash \varphi_7(c+1)$.

PROOF.

   (1) We need to show:

      (a) $M \vDash T[\text{full}]$,

      (b) $M \vDash \forall a.\ acc_7(a,0) = a$,

      (c) $M \vDash \forall a \forall x.\ acc_7(a,x+1) = acc_7(a \times (x+1), x)$.

   (a) holds since M is an M_0-expansion. (b) holds by Definition 7.6. For
   (c), suppose a and n are in M_0. We need to show

      (*) $acc_7{}^M(a,n+1) = acc_7{}^M(a(n+1),n)$.

   By Lemma 7.5 it suffices to consider the following cases.

      — Case (a,n+1) is in A:

          $acc_7{}^M(a,n+1) = a(n+1)!$ (by Definition 7.6)

                    $= a((n+1)n!)$ (by Definition 3.1c)

                    $= (a(n+1))n!$ (by associativity)

                    $= acc_7{}^M(a(n+1),n)$ (by Definition 7.6)

     — Case (a,n+1) is in B: Then, by Lemma 7.7, (a(n+1),n) is in B as well
       so, by Definition 7.6, both sides of (*) are c! and thus (*) holds.

     — Case (a,n+1) is in C: Similar to the previous case.

   (2) We need to show

      $c! = acc_7{}^M(1,c)$.

   Since (1,c) is in B, this holds by Definition 7.6.

   (3) We need to show

      (*) $(c+1)! \neq acc_7{}^M(1,c+1)$.

c+1 is non-standard so (1,c+1) is not in A. We have c+1 > c, so by
Lemma 7.8, (1,c+1) is not in B. Thus (1,c+1) is in C and thus, by
Definition 7.6, we have acc,^{M}(1,c+1) = 0. Thus (*) holds since M ⊨ T[full]
and

$$\forall x. \ x! \neq 0$$

is a theorem of T[full].

<div align="center">**</div>

PROOF OF FACT 7.2. We need to show (1)-(5) of Definition 4.4. (1) follows from
Lemmas 4.6 and 7.9. (2)-(5) are easy to verify.   □

<div align="center">**</div>

§8 Facts 3.2a, 3.2b and 3.6 must be proved by non-analytic induction
───

In this section I present a general result which shows that facts like
Facts 3.2a and 3.2b must be proved by non-analytic induction. Since Fact 3.6 is
a generalization of Facts 3.2a and 3.2b, this means that Fact 3.6 too must be
proved by non-analytic induction. The proof method is basically the same as for
Fact 7.2, only a bit more complicated.

<center>**</center>

We now reuse '•' and '••'. They were used in §3 but those usages does thus not
apply anymore.

<center>**</center>

ASSUMPTION 8.1.

   — Let $\bullet : \mathbb{N}^2 \to \mathbb{N}$.

   — Let n_0 be a natural number.

   — Let $M_0 \vDash T[full]$ be non-standard $\mathscr{L}[full]$-model.

   — Let c be a non-standard number in M_0.

<center>**</center>

REMARK. In §3 we had the more general $\bullet : A \times A \to A$, with A an arbitrary set.
Since the notions introduced in §4 only covers arithmetic we now consider only
the case $A = \mathbb{N}$, but I do not think there should be any problems generalizing to
a more general setting (once the notions in §4 have been appropriately
generalized).

<center>**</center>

DEFINITION 8.2. $\bullet\bullet : \mathbb{N}^2 \to \mathbb{N}$ is defined recursively by

$$n \bullet\bullet 0 := n_0,$$

$$n \bullet\bullet (m+1) := (n \bullet\bullet m) \bullet n.$$

<center>**</center>

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DEFINITION 8.3.

— The language L_8 of arithmetic is $\mathscr{L}[full]$ expanded with the ternary
   function symbol 'acc$_8$'.

— The L_8-sentence DEF(acc$_8$) is defined by

   DEF(acc$_8$) :≡

   $\forall a \forall x.\ acc_8(a,x,0) = a \land \forall a \forall x \forall y.\ acc_8(a,x,y+1) = acc_8(a \bullet x, x, y).$

— The theory T_8 of L_8-arithmetic is defined by

   $T_8 := T[full], DEF(acc_8).$

— The L_8-formula $\varphi_8(y)$ is defined by

   $\varphi_8(y) :\equiv \forall x.\ acc_8(n_0, x, y) = x \bullet\bullet y.$

                                    **

Proceeding similarly to §5 and §7, we will expand M_0 to an L_8-model $M \vDash T_8$ such
that $M \vDash \varphi(c-1)$ and $M \nvDash \varphi(c)$. To achieve $M \vDash \varphi(c-1)$ we will put

   $acc_8{}^M(n_0, n, c-1) := n \bullet\bullet (c-1)$

for all n in M_0. To achieve $M \nvDash \varphi(c)$ we will put

   $acc_8{}^M(n_0, n, c) := (n \bullet\bullet c)+1$

for some n in M_0. Now whether we can do this while satisfying DEF(acc$_8$) depends
on $\bullet$ and n_0. For example, for $\bullet = +$ and $n_0 = 0$, we would put

   $acc_8{}^M(0, n, c-1) := n(c-1).$

To satisfy the second conjunct of DEF(acc$_8$) we would then need, for all n in M_0,

   $n(c-1) = acc_8{}^M(0, n, c-1)$

   $= acc_8{}^M(n, n, c-2)$

   $= acc_8{}^M(2n, n, c-3)$

   $= acc_8{}^M(3n, n, c-4)$

   $\vdots$

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and

$$0 = 0(c-1)$$

$$= \text{acc}_8{}^M(0,0,c-1)$$

$$= \text{acc}_8{}^M(0,0,c)$$

$$= \text{acc}_8{}^M(0,0,c+1)$$

$$= \text{acc}_8{}^M(0,0,c+2)$$

$$\vdots$$

We will now make the construction precise.

<div align="center">**</div>

DEFINITION 8.4.

— The set $A \subseteq M_0{}^3$ is defined by

$$A := \{(a,n,m) : m \text{ is standard}\},$$

— The set $B \subseteq M_0{}^3$ is defined inductively by

$$\frac{n \text{ is in } M_0}{(n_0,n,c-1) \text{ is in } B} \qquad \frac{(a,n,m+1) \text{ is in } B}{(a \bullet n,n,m) \text{ is in } B} \qquad \frac{(a \bullet n,n,m) \text{ is in } B}{(a,n,m+1) \text{ is in } B}.$$

— The set $C \subseteq M_0{}^3$ is defined by

$$C := M_0{}^3 - (A \cup B).$$

<div align="center">**</div>

EXAMPLE 8.5.

(a) If $\bullet = +$ and $n_0 = 0$ then B consists of the triples

$$(0,0,c),$$

$$(0,0,c+1),$$

$$(0,0,c+2),$$

$$\vdots$$

together with all triples that are on one of the forms

$$(0,n,c-1),$$

$$(n,n,c-2),$$

$$(2n,n,c-3),$$

$$(3n,n,c-4),$$

$$\vdots$$

(b) If $\bullet = \times$ and $n_0 = 1$ then B consists of the triples

$$(1,1,c),$$

$$(1,1,c+1),$$

$$(1,1,c+2),$$

$$\vdots$$

together with all triples that are on one of the forms

$$(1,n,c-1),$$

$$(n,n,c-2),$$

$$(n^2,n,c-3),$$

$$(n^3,n,c-4),$$

$$\vdots$$

**

LEMMA 8.6. A, B and C partitions $M_0{}^3$.

PROOF. Clearly:

  — AUBUC = $M_0{}^3$.

  — C is disjoint from A and from B.

It thus remains to show that A and B are disjoint. If (a,n,m) is in A then m is
standard but if (a,n,m) is in B then n is non-standard. Thus A and B have no
common elements and thus they are disjoint.   □

<div align="center">**</div>

By Lemma 8.6, the following definition is well-formed.

<div align="center">**</div>

DEFINITION 8.7. $acc_8{}^M$: $M_0{}^3$ → M_0 is defined by

$$acc_8{}^M(a,n,0) := a,$$

$$acc_8{}^M(a,n,m+1) := acc_8{}^M(a\bullet n,n,m) \quad \text{if m is standard,}$$

$$acc_8{}^M(a,n,m) := n\bullet\bullet(c-1) \quad\quad \text{if (a,n,m) is in B,}$$

$$acc_8{}^M(a,n,m) := (n\bullet\bullet c)+1 \quad\quad \text{if (a,n,m) is in C.}$$

<div align="center">**</div>

LEMMA 8.8. Let X be A, B or C. For all a, n and m in M_0, (a,n,m+1) is in X
if and only if (a•n,n,m) is.

PROOF. It suffices to show this for X = A and X = B (since then it must hold for
C, by definition of C). For X = B this holds by definition. For X = A:

  (a,n,m+1) is in A iff m+1 is standard

                     iff m is standard

                     iff (a•n,n,m) is in A.   □

<div align="center">**</div>

ASSUMPTION 8.9. There is n in M_0 such that (n_0,n,c) is in C.

<div align="center">**</div>

LEMMA 8.10.

  (1) $M \vDash T_8$.

  (2) $M \vDash \varphi_8(c-1)$.

  (3) $M \nvDash \varphi_8(c)$.

PROOF.

  (1) We need to show:

      (a) $M \vDash T[\text{full}]$,

      (b) $M \vDash \forall a \forall x.\ acc_8(a,x,0) = a$,

      (c) $M \vDash \forall a \forall x \forall y.\ acc_8(a,x,y+1) = acc_8(a \bullet x,x,y)$.

  (a) holds since M is an M_0-expansion. (b) holds by Definition 8.6. For
  (c), suppose a, n and m are in M_0. We need to show

      (∗) $acc_8{}^M(a,n,m+1) = acc_8{}^M(a \bullet n,n,m)$.

  By Lemma 8.6 it suffices to consider the following cases.

    — Case (a,n,m+1) is in A: Then (∗) holds by Definition 8.7.

    — Case (a,n,m+1) is in B: Then, by Lemma 8.8, (a•n,n,m) is in B
      as well. Thus, by Definition 8.7, both sides of (∗) are equal.

    — Case (a,n,m+1) is in C: Similar to the previous case.

  (2) We need to show that

      $acc_8{}^M(n_0,n,c-1) = n \bullet \bullet (c-1)$

  for all n in M_0. Since $(n_0,n,c-1)$ is in B for all n in M_0, this holds by
  Definition 8.7.

  (3) We need to exhibit an n in M_0 such that

      $acc_8{}^M(n_0,n,c) \neq n \bullet \bullet c$.

  By Assumption 8.9 we have an n in M_0 such that (n_0,n,c) is in C, so for
  this n we have

      $acc_8(n_0,n,c) = (n \bullet \bullet c)+1$ (by Definition 8.7)

                  $\neq n \bullet \bullet c$ (by well-known theorem of T[full]).   □

**

FACT 8.11. With $\bullet = +$ and $n_0 = 0$, or with $\bullet = \times$ and $n_0 = 1$, T_8 proves $\forall y.\varphi(y)$ by necessarily non-analytic induction.

PROOF. Both cases are similar. We prove the case $\bullet = +$. We must exhibit an L_8-formula $\psi(y)$ such that conditions (1)-(5) of Definition 4.4 are satisfied. As the proof of Fact 3.2 shows,

$$\psi(y) :\equiv \forall a \forall x.\ acc_8(a,x,y) = a+x\times y$$

will work. Conditions (2)-(5) are then straightforward. Condition (1) follows from Lemmas 4.6 and 8.10. It remains to verify that Assumption 8.9 is true when $\bullet = +$ and $n_0 = 0$, which it is since for example $(0,1,c)$ is in C (see Example 8.5a).   □

**

§9 Discussion and future work

What does results like Facts 5.3, 6.5, 7.2 and 8.11 tell us about the necessity of non-analytic induction in mathematical practice? Consider, for example, Fact 5.3 and consider a working mathematician who have just introduced the function $f_2 : \mathbb{N} \to \mathbb{N}$ by Definition 2.1——that is, by the recursive definition

$$f_2(0) := 0,$$

$$f_2(n+1) := f_2(n)+2n+1.$$

Fact 5.3 tells us that if our working mathematician wants to prove——by the means provided by first-order arithmetic——that $f_2(n)$ is a perfect square for all n, then no matter what facts have been previously established, a non-analytic induction proof is necessary (or put differently: a non-analytic induction axiom is needed). As Fact 5.3 shows, even if our working mathematician would somehow have established all first-order facts of arithmetic in a language without 'f_2', a non-analytic induction axiom would still be needed. Thus results like Fact 5.3 say that some non-analytic induction proofs are necessarily non-analytic in this rather strong sense.

**

PA^-, in the language $\mathcal{L}_0$, is "the theory of the non-negative parts of discretely ordered commutative rings" (see §A). Hetzl and Wong (2017) consider the problem of finding non-analytic induction proofs from PA^-. This is motivated by the fact that for any $\mathcal{L}_0$-sentence σ, if $PA \vdash \sigma$ then there is a φ(x) such that $PA^-, IND(\varphi) \vdash \sigma$. In previous work, we stated and proved Facts 5.3 and 7.2 with PA^- instead of T[full] (in the definition of T_5 and T_7, respectively). Naturally, one might wonder whether one could always replace PA^- with T[full] in such results. This is not the case for trivial reasons: facts formulated using only $\mathcal{L}_0$ might need non-analytic induction axioms added to PA^- to be provable, but these non-analytic induction axioms are of course already in T[full]. Facts 5.3, 6.5, 7.2 and 8.11 concern results formulated in an impure language. Perhaps one could find some general criterion that says that for some class of statements in such languages, if PA^-,T proves them by necessarily non-analytic induction then so does T[full],T (where T would typically consist of axioms governing the symbols from the impure part of the language). For example one could investigate the following conjecture, which is inspired by Facts 7.2 and 8.11.

**

CONJECTURE 9.1.

— Let L be $\mathscr{L}$[full] expanded with the (n+m)-ary function symbol 'f'.

— Let T be an L-theory of arithmetic.

— Let $t_1,\ldots,t_n$ be closed $\mathscr{L}_0$-terms.

— Let $t(x_1,\ldots,x_m)$ be an $\mathscr{L}_0$-term.

— Suppose PA⁻,T proves

$$\sigma :\equiv \forall x_1 \cdots \forall x_m.\ t(x_1,\ldots,x_m) = f(t_1,\ldots,t_n,x_1,\ldots,x_m).$$

by necessarily non-analytic induction.

Then T[full],T proves σ by necessarily non-analytic induction.

**

To more systematically settle whether different facts require non-analytic induction proofs one could attempt to establish more general results—instead of hand-crafting countermodels for each particular case. This might be difficult. As a starting point one could try to develop general methods and results for comparatively simple theories, like fragments of Presburger arithmetic.

**

Another possible future line of work would be to consider other settings than arithmetic. For example, in computer science, many basic facts about functions on inductive structures seem to require non-analytic induction proofs.

**

One might also approach the problem of non-analytic induction proofs from the more proof-theoretical side, for example by studying derivations in natural deduction. Dag Prawitz's (2018) recent work might be useful.

**

§A Some definitions and results
_____

DEFINITION A.1. The $\mathscr{L}_0$-theory PA^- is axiomatized by the following.

 — 0 and 1 are distinct:

$$0 \neq 1.$$

 — Associativity of addition and multiplication:

$$(x+y)+z = x+(y+z),$$

$$(x \times y) \times z = x \times (y \times z).$$

 — Commutativity of addition and multiplication:

$$x+y = y+x,$$

$$x \times y = y \times x.$$

 — Distributivity of addition over multiplication:

$$x \times (y+z) = x \times y + x \times z.$$

 — 0 is an additive identity and a multiplicative zero:

$$x+0 = x,$$

$$x \times 0 = 0.$$

 — The order is irreflexive:

$$x \not< x.$$

 — The order is transitive:

$$x < y \wedge y < z \rightarrow x < z.$$

 — The order is total:

$$x < y \vee x = y \vee y < x.$$

 — The order is discrete:

$$x = 0 \vee x = 1 \vee 1 < x.$$

§A Some definitions and results

— Addition and multiplication respect the order:

$$x < y \rightarrow x+z < y+z,$$

$$x < y \wedge 0 \neq z \rightarrow x\times z < y\times z.$$

— Smaller elements can be subtracted from larger elements:

$$x < y \rightarrow \exists z.\ x+z = y.$$

**

FACT A.2.

— For all $\mathscr{L}_0$-structures M: M ⊨ PA⁻ if and only if M is the non-negative part of a nontrivial discretely ordered commutative ring.

— PA⁻ is a theory of $\mathscr{L}_0$-arithmetic.

— PA ≡ PA⁻+{IND(φ) : φ an $\mathscr{L}_0$-formula in one free variable}.

PROOFS. See for example Chapter 2 of Kaye's (1991) book.   □

**

§B Proofs of Lemmas 5.7 and 5.8
────────────────────────────────

LEMMA B.1. For all natural numbers m and n:

$$\beta_{m+1}(n) = \beta_m(n)+2.$$

PROOF. Induction on n.

— Base case:

$$\beta_{m+1}(0) = 2(m+1)+1 \quad \text{(by Definition 5.6)}$$

$$= (2m+1)+2$$

$$= \beta_m(0)+2 \quad \text{(by Definition 5.6)}.$$

— Induction step:

$$\beta_{m+1}(n+1) = \beta_{m+1}(n)+2n+1 \quad \text{(by Definition 5.6)}$$

$$= \beta_m(n)+2+2n+1 \quad \text{(by induction hypothesis)}$$

$$= (\beta_m(n)+2n+1)+2$$

$$= \beta_m(n+1)+2 \quad \text{(by Definition 5.6)}. \qquad \square$$

**

LEMMA 5.7. For all natural numbers m:

$$\beta_m(m) = (m+1)^2.$$

PROOF. Induction on m.

— Base case:

$$\beta_m(0) = 2\times0+1 \quad \text{(by Definition 5.6)}$$

$$= (0+1)^2.$$

— Induction step:

$$\beta_{m+1}(m+1) = \beta_m(m+1)+2 \qquad \text{(by Lemma B.1)}$$

$$= \beta_m(m)+2m+1+2 \quad \text{(by Definition 5.6)}$$

$$= \beta_m(m)+2m+3$$

$$= (m+1)^2+2m+3 \quad \text{(by induction hypothesis)}$$

$$= (m+2)^2$$

$$= ((m+1)+1)^2. \qquad\qquad\qquad\qquad\qquad\qquad □$$

**

LEMMA 5.8. For all natural numbers m:

$$\beta_m(m+1) \text{ is not a perfect square.}$$

PROOF. We have

$$\beta_m(m+1) = \beta_m(m)+2m+1 \quad \text{(by Definition 5.6)}$$

$$= (m+1)^2+2m+1 \quad \text{(by Lemma 5.7)}$$

$$= (m+2)^2-2.$$

Thus we have

$$(m+1)^2 < \beta_m(m+1) < (m+2)^2.$$

Thus $\beta_m(m+1)$ is not a perfect square.   □

**

§C Proofs of Lemmas 6.10, 6.11 and 6.13
──

LEMMA C.1. $g_2(n) \rightarrow \pi^2/6$ as $n \rightarrow \infty$.

PROOF. This is Euler's (1740) famous solution to "the Basel problem".   □

<div align="center">**</div>

LEMMA C.2. For each natural number m, $\xi_m(n) \rightarrow 2+(m+1)^{-1}$ as $n \rightarrow \infty$.

PROOF. Immediate from Lemma C.1 and Definition 6.8.   □

<div align="center">**</div>

LEMMA C.3. For each natural number n there is a natural number m such that

$$\xi_m(n) < 2.$$

PROOF. Let n be a natural number. By Lemma C.1 and since g_2 is strictly
increasing we have

(*) $g_2(n) = \pi^2/6-\varepsilon$

for some $\varepsilon > 0$. Thus for any m we have

$$\xi_m(n) = 2+(m+1)^{-1}+g_2(n)-\pi^2/6 \quad \text{(by Definition 6.8)}$$

$$= 2+(m+1)^{-1}+\pi^2/6-\varepsilon-\pi^2/6 \quad \text{(by (*))}$$

$$= 2+(m+1)^{-1}-\varepsilon.$$

Thus for m large enough, $\xi_m(n) < 2$.   □

<div align="center">**</div>

LEMMA 6.10. For all natural numbers m:

$$\gamma_m(n)/h_6(n) = \xi_m(n).$$

PROOF. Induction on n.

— Base case:

$$\gamma_m(0)/h_6(0) = (3+(m+1)^{-1}-\pi^2/6)/1 \quad \text{(by Definitions 6.1 and 6.9)}$$

$$= 2+(m+1)^{-1}+1-\pi^2/6$$

$$= 2+(m+1)^{-1}+g_2(0)-\pi^2/6 \quad \text{(by Definition 2.3)}$$

$$= \xi_m(0) \quad \text{(by Definition 6.8).}$$

— Induction step:

$$\gamma_m(n+1)/h_6(n+1)$$

$$= ((n+2)^2\gamma_m(n)+h_6(n))/((n+2)^2 h_6(n)) \quad \text{(by Definitions 6.1 and 6.9)}$$

$$= \gamma_m(n)/h_6(n)+(n+2)^{-2}$$

$$= \xi_m(n)+(n+2)^{-2} \quad \text{(by induction hypothesis)}$$

$$= 2+(m+1)^{-1}+g_2(n)-\pi^2/6+(n+2)^{-2} \quad \text{(by Definition 6.8)}$$

$$= 2+(m+1)^{-1}+g_2(n)+(n+2)^{-2}-\pi^2/6$$

$$= 2+(m+1)^{-1}+g_2(n+1)-\pi^2/6 \quad \text{(by Definition 2.3)}$$

$$= \xi_m(n+1) \quad \text{(by Definition 6.8).} \quad \square$$

<div align="center">**</div>

LEMMA 6.11. For each natural number m there is a unique natural number n such that

$$\xi_m(n) < 2$$

and

$$\xi_m(n+1) \not< 2.$$

PROOF. Let m be any natural number. By Lemma C.2 there is a natural number n' such that

$$\xi_m(n') > 2.$$

Since $\xi_m(0) < 2$ we thus have a natural number $n < n'$ such that

$$\xi_m(n) < 2$$

and

$$\xi_m(n+1) \not< 2.$$

Since ξ_m is strictly increasing n must be unique.   □

<div align="center">**</div>

LEMMA 6.13. μ is unbounded.

PROOF. We show that for any m there is k such that $\mu(k) > \mu(m)$. Let m be any natural number. By Lemma C.3 there is a natural number k such that

$$(*) \; \xi_k(\mu(m)+1) < 2.$$

Thus

$$\mu(k) \geq \mu(m)+1 \quad \text{(by Definition 6.12 and (*) and since } \xi_k \text{ is increasing)}$$

$$> \mu(m). \qquad □$$

<div align="center">**</div>

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