1 Foreword

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Disclaimer : The notes are written only for my memorization purpose after I self-studied online lecture notes (mainly Stanford CS229, CS168, CS231n, Cornell CS4780, Andrew Ng's (I am his fan) Coursera courses) and other online resources (Stack Exchange, TowardDataScience). Latex formula are generated by AI tools.

2 Confusion Matrix

Predict True, Actual True : True Positive (TP)
Predict True, Actual False : False Positive (FP)
Predict False, Actual False : False Negative (FN)
Predict False, Actual True : True Negative (TN)

Accuracy = (TP+FN)/(TP+FP+FN+TN), performance of correct classification

Precision = TP / (TP+FP) (correctly classified as positive / Everything classified as positive), example usage : Cancer detection. (We don't want to initiate cancer treatment if the person is actually healthy).

Recall = TP / (TP + FN) (correctly classified as positive / Actually positive), FP is more expensive than TN . (e.g. Fraud detection).

Note : Mathematically, Precision and Recall are inverse relationship, there is a tradeoff between recall and precision.

F1 score = 2(P*R)/(P+R), a compromised metric

3 K-Nearest Neighbour

Description : Choose the *majority* class of nearest (e.g. Eclidean Distance) K data points and classify it.
How to Choose K(hyper-paramaeter) : General rule of thumb : sqrt(number of data)/2 or by searching and comparing different k's for highest prediction accuracy.
Normalization of data in preprocessing is a must

4 K-Means Clustering

Simple Description : Identify clusters by finding the centroid of data points

Algorithm :

1. Initialize μ_1,μ_2.... μ_k randomly (k is hyper-parameter)

2. Repeated until converge :

(i) $c^{(i)} := \arg\min_j ||x^{(i)} - \mu_j||^2, j \in [1 : k]$, (i.e. $c^{(i)}$ denote which μ the $x^{(i)}$ is linked to. Link each data point to nearest μ_j. If $x^{(i)}$ is nearest to μ_s, then $c^{(i)} = j$. Thus, k partitions are created.)

(ii) $\mu_j := \frac{\sum_{i=1}^{m} 1\{c^{(i)}=j\}x^{(i)}}{\sum_{i=1}^{m} 1\{c^{(i)}=j\}}$, (i.e. For each data point in each partition from (i) , find the new centroid and assign to μ_k

Proof of convergence of the algorithm : consider

$J(c, \mu) = \sum_{i=1}^{m} ||x^{(i)} - \mu_{c^{(i)}}||^2$
Observation : J must be monotonically decreasing. It is because for step (i) It is adjusting $c^(i)$ to reduce J, for step (ii) we are adjusting μ_j to reduce J
J is non-convext, it may get to local minimum. To try several random initial values, and choose the lowest J.

5 Linear Regression(MSE approach)

Hypothesis :

$$h_\theta(x) = \sum_j \theta_j x_j = \theta^\top x$$

We want to minimize MSE (Mean Square Error)

$$J(\theta) = \frac{1}{2}\sum_i \left(h_\theta(x^{(i)}) - y^{(i)}\right)^2 = \frac{1}{2}\sum_i \left(\theta^\top x^{(i)} - y^{(i)}\right)^2$$

Gradient of J :

$$\frac{\partial J(\theta)}{\partial \theta_j} = \sum_i x_j^{(i)} \left(h_\theta(x^{(i)}) - y^{(i)}\right)$$

Each θ_j is updated for each step by gradient descent algorithm.

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

Practically :
(i) Learning rate α is hyperparameter and data dependent , larger, fewer steps

to get to min. but may miss the minimum. (Monitor the loss curve, J value vs iteration).
(ii) Batch GD is slow, may be Mini-Batch GD or Stochastic GD.
(iii) If α is small but the loss oscillate , converged and stop learning.

6 Linear Regression(MLE approach)

7 Logistic Regression

$$P(y=1|x) = h_\theta(x) = \frac{1}{1+\exp(-\theta^\top x)} \equiv \sigma(\theta^\top x)$$
$$P(y=0|x) = 1 - P(y=1|x) = 1 - h_\theta(x)$$

Loss function is

$$J(\theta) = -\sum_i \left[y^{(i)} \log(h_\theta(x^{(i)})) + (1-y^{(i)}) \log(1-h_\theta(x^{(i)})) \right]$$

Again , BGD for following gradient of J :

$$\frac{\partial J(\theta)}{\partial \theta_j} = \sum_i x^{(i)} j \left(h\theta(x^{(i)}) - y^{(i)} \right)$$

Interpretation : For a particular sample : if h return 1/0 and y return 1/0 , the term is 0. if h return 1/0 and y return 0/1 , the term is positive infinity.

8 Logistic Regression(MLE approach)

9 Softmax Regression(Multi-Class Logistic)

k classes, n x k parameters , and the hypothesis is :

$$h_\theta(x) = \begin{bmatrix} P(y=1|x;\theta) \\ P(y=2|x;\theta) \\ \vdots \\ P(y=K|x;\theta) \end{bmatrix} = \frac{1}{\sum_{j=1}^{K} \exp(\theta^{(j)\top} x)} \begin{bmatrix} \exp(\theta^{(1)\top} x) \\ \exp(\theta^{(2)\top} x) \\ \vdots \\ \exp(\theta^{(K)\top} x) \end{bmatrix} \tag{1}$$

Below Loss function is quite easy to understand : By referencing to previous hpothesis, we want to maximize the y=k associated probility if that data belongs to class k

$$J(\theta) = -\left[\sum_{i=1}^{m} \sum_{k=1}^{K} 1\{y^{(i)} = k\} \log \left(\frac{\exp(\theta^{(k)\top} x^{(i)})}{\sum_{j=1}^{K} \exp(\theta^{(j)\top} x^{(i)})} \right) \right]$$

Gradient of J is, we solve the problem by GD :

$$\nabla_{\theta^{(k)}} J(\theta) = -\sum_{i=1}^{m} \left[x^{(i)} \left(1\{y^{(i)} = k\} - P(y^{(i)} = k | x^{(i)}; \theta) \right) \right]$$

Where :

$$P(y^{(i)} = k | x^{(i)}; \theta) = \frac{\exp(\theta^{(k)\top} x^{(i)})}{\sum_{j=1}^{K} \exp(\theta^{(j)\top} x^{(i)})}$$

10 Regularization in Linear Regression

11 BGD variation : Mini BGD/SGD

BGD use all training data in a single step, which is exteremely costly.

12 Loss function in Classification(Binary) Problem - General treatment

General Hypothesis : $h_\theta(x) = x^T\theta$

Adjustment for binary classification :

$$\text{sign}(h_\theta(x)) = \text{sign}(\theta^T x) = \text{sign}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases}$$

Measure of confidence : $h_\theta(x) = x^T\theta$ gives larger value, more confident

Margin ($yx^T\theta$) : (i) if $h_\theta(x)$ classify correctly, margin is positive, otherwise negative.
(ii) Therefore our objective is to maximize the margin (we want both correct classification and be confident)

Consider the following loss function :

$$J(\theta) = \frac{1}{m} \sum_{i=1}^{m} \phi \left(y^{(i)} \theta^T x^{(i)} \right)$$

We want penalize wrong classfication and encourage correct one , we design ϕ as $\phi(z) \to 0$ as $z \to \infty$, while $\phi(z) \to \infty$ as $z \to -\infty$ where $z = yx^T\theta$, and examples are :

logistic loss : $\phi_{\text{logistic}}(z) = \log(1 + e^{-z})$,used in logistic regression

hinge loss : $\phi_{\text{hinge}}(z) = [1 - z]_+ = \max 1 - z, 0$, used in SVM

Exponential loss $\phi_{\exp}(z) = e^{-z}$, used in boosting

13 Kernel Mapping (Special case demo by Linear Regression + Polymoninal Kernel)

(I) Purpose : To x map from lower higher dimension. Useful when data are non-linearly separable(Transform to a curve)
(II) Computation complexity does not necessarily increase proportionately.
(III) Example : a mapping function $\varphi : R \to R^4$, $x \to [1, x, x^2, x^3]$, and h is $\theta^T x$ having $\theta = [\theta_1, \theta_2, \theta_3, \theta_4]$
(IV) Terms : x is called attribute, $x \to [1, x, x^2, x^3]$ called feature, φ feature map, $\varphi : R^1 \to R^4$ in this case. d =1 p = 4
(IV) Another Example : a mapping function $\varphi : R^3 \to R^{1000}$, $x \to [1, x_1, x_1^2, x_1^3, x_1x_2, x_1x_2^2....]$ (*) ,let d = 3 , p =1000. If we exhaust all possibilities, then p = 1 + d + d^2 + d^3 (**)
Recall GD stepping :

$$\theta := \theta + \alpha \sum_{i=1}^{n} (y^{(i)} - h_\theta(x^{(i)}))x^{(i)}$$

$$\theta := \theta + \alpha \sum_{i=1}^{n} (y^{(i)} - \theta^T x^{(i)})x^{(i)}$$

Putting kernel mapping to the equation :

$$\theta := \theta + \alpha \sum_{i=1}^{n} (y^{(i)} - \theta^T \phi(x^{(i)}))\phi(x^{(i)})$$

We pause here to evaluate the cost of computing each of update (Curse of Demensionality...), considering (**). If we just use the kernel direction, we suffer the curse of demensionality : Suppose d (data dimension) = 1000, then by using the mapping in (**) we have p = 10^9. $\theta^T \phi(x^{(i)})$ need O(p) (dot product) , and O(np) for summing up all data in each step.
Going back to BGD.

$$\theta := \theta + \alpha \sum_{i=1}^{n} (y^{(i)} - \theta^T \phi(x^{(i)}))\phi(x^{(i)})$$

, assuming $\theta = \sum_{i=1}^{n} \beta_i \phi(x^{(i)})$ (*) at some point, with initialization $\theta = 0 = \beta$
It becomes

$$\theta := \sum_{i=1}^{n} \beta_i \phi(x^{(i)}) + \alpha \sum_{i=1}^{n} (y^{(i)} - \theta^T \phi(x^{(i)}))\phi(x^{(i)})$$

Rearranging :

$$\theta := \sum_{i=1}^{n}(\beta_i + \alpha(y^{(i)} - \theta^T\phi(x^{(i)})))\phi(x^{(i)})$$

Therefore it is equivalent to updating β_i (instead of θ_i) by

$$\beta_i := \beta_i + \alpha(y^{(i)} - \theta^T\phi(x^{(i)}))$$

by (*) above

$$\beta_i := \beta_i + \alpha\left(y^{(i)} - \sum_{j=1}^{n}\beta_j\phi(x^{(j)})^T\phi(x^{(i)})\right)$$

Computing of LHS is fast because : (1) we can pre-compute $\phi(x^{(j)})^T\phi(x^{(i)})$ for all i,j, and (2) $\phi(x^{(j)})^T\phi(x^{(i)})$ can be represented by $< x^{(i)}, x^{(j)} >$:
$\langle\phi(x),\phi(z)\rangle = 1+\sum_{i=1}^{d} x_iz_i+\sum_{i,j\in\{1,\ldots,d\}} x_ix_jz_iz_j+\sum_{i,j,k\in\{1,\ldots,d\}} x_ix_jx_kz_iz_jz_k =$
$1 + \langle x,z\rangle + \langle x,z\rangle^2 + \langle x,z\rangle^3$ (**)

Define K where K is n x n (n is the number of training samples) matrix, with $K(x,z) = \langle\phi(x),\phi(z)\rangle$, where K_ij is $\langle\phi(x^{(i)}),\phi(x^{(j)})\rangle$

Therefore, the process is : (1) compute K_ij using (**) , for all $i,j \in \{1,\ldots,n\}$. Set $\beta :=$
0 ,
(2) Loop

$$\forall i \in \{1,\ldots,n\}, \quad \beta_i := \beta_i + \alpha\left(y^{(i)} - \sum_{j=1}^{n}\beta_jK(x^{(i)},x^{(j)})\right)$$

in vectorized notation:

$$\beta := \beta + \alpha(\tilde{y} - K\beta)$$

When doing inference :

$$\theta^T\phi(x) = \sum_{i=1}^{n}\beta_i\phi(x^{(i)})^T\phi(x) = \sum_{i=1}^{n}\beta_iK(x^{(i)},x)$$

In practice, we do computation using K (at O(d) cost) instead of directly from $\phi(x)$ is much faster. Further, We only need to know K but "just only need to know" the existence of $\phi(x)$. There is no need to be able to write down $\phi(x)$.
Consider the Kernel applied to bitmap : number of bits as d. (Great reduction!)
Intuitively, K represents similarity matrix, i.e. K is small if $\phi(x^{(j)})^T\phi(x^{(i)})$ is small
Example : Gaussian Kernel, it can support inifinitely dimensional space of mapping.

$$K(x,z) = \exp\left(-\frac{||x-z||^2}{2\sigma^2}\right)$$

Mercer Theorem : For K to be a valid Kernel iff K is PSD.
Application : To SVM, perceptron, linear regression, and other learning algorithms represented only in inner product $\langle x, z\rangle$, then Apply K(x,z)

14 Likelihood and Maximum Likelihood Estimation

15 Generative vs Discriminative Learning Algorithm for classification, discussion

D : Learn the curve that separates the classes, e.g. Logistic Regression, SVM , ANN, CNN
G : Learn the model itself and just class of data. e.g. Naive Bayes, Gradient Discriminant Analysis, GAN
An Analogy of G : Learn both English and French , and guess whether the word Bonjour is French or English.
Another Example of G (*):
Let's say a model is trained with 1000 pictures :
(i) Dog without glasses : 1
(ii) Dog with glasses : 239
(iii) Human without glasses : 500
(iv) Human with glasses : 260
Assume we have a photo with a glasses. To classify a dog or human in the picture for a generative model : Since P (H & G) = 0.26 > P(D & G) = 0.001, the model infer that it is a human.
let x be feature, y be class
Put it in another way, in D (e.g. (multi-class) logistic regression) , we learn h which is p(y|x) and infer the class with largest p(y|x). i.e. we are finding $argmax_y p(y|x)$
In D, let y be the class (Dog=1 vs Human=0) , x be feature(with glasses) , we learn p(x|y=1) (case (i)/(ii)) and p(x|y=0) (case (iii)/(iv))
Mathematically, D and G's relationship : $argmax_y p(y|x) = argmax_y \frac{p(x|y)p(y)}{p(x)} = argmax_y p(x|y)p(y) = argmax_y p(x\&y)$
(bayes rule, x is independent variable, bayes rule again, also see (*))

16 Entropy

Definition :

$$H(x) = -\sum_{i=1}^{n} p(x_i) \log_2 p(x_i)$$

where $x_1, x_2...$ are all possible events of random variable(distribution) and $p(x_1), p(x_2)...$ are the probabilities of the respective events.

Connection with uncertainty (High Entropy , High Uncertainty) :

Entropy measure the uncertainity of a distribution. Consider a random variable distribution : X=1 at 0.33 , X = 2 at 0.33, X=3 at 0.33, and another : X = 1 at 0.98 , X = 2 at 0.01, X =3 at 0.01, we say the former distribution has higher uncertainity (more difficult to guess its value).

Connection with amount of information in a *message* (not distribution) :

Average number of bits (yes/no answers) NEEDED TO PROVIDE to tell x in a message. Therefore High Entropy. Higher Uncertainty , Higher Amount of Information.

Connection with Decision Tree splitting :

Remember that DT is greedy algorithm : It is to find the split that have greatest reduction in uncertainty (Information Gain) of the distribution (after splitting). We have a certain distribution

17 SVM, Support Vector Machine

Assumptions for illustration : data in binary classes only, linearly separable (if not, then apply Kernel)
Main Idea in Training : We construct a separating hyperplane, the plane has largest distance to all data point.
How ?
Let $y \in \{-1, 1\}$
Define classifier $h_{w,b}(x) = g(w^T x + b)$, where $w^T x + b$ is the formula of hyperplane, w is the normal vector to hyperplane.
where g : $g(z) = 1$ if $z \geq 0, g(z) = -1$ if $z < 0$
where w = $[\theta_1 ... \theta_n]^T$
Define functional margin (FM): $\hat{\gamma}^{(i)} = y^{(i)}(w^T x + b)$.
If FM is positive, it classify correctly. Negative, classify incorrectly.
If the magnitude is large , classifier gives highly confident result.
Define Geometric Margin (GM), $\gamma_i = \frac{\hat{\gamma^i}}{\|w\|}$
Further define smallest distance from hyperplane to data points : $\gamma = \min_{i=1,...,m} \gamma^{(i)}$
Thus, our training objective is to maximize this smallest distance.

$$\max_{\gamma, \mathbf{w}, b} \quad \gamma$$

$$\text{s.t.} \quad y^{(i)}(\mathbf{w}^T \mathbf{x}^{(i)} + b) \geq \gamma, \quad i = 1, \ldots, m$$

$$\|\mathbf{w}\| = 1$$

The last condition ensure FM =>GM
Why use GM instead of FM in training ? We can always scale w and b to achieve greater magnitude in FM, therefore FM is meaningless for training.
Rearranging :

$$\max_{\gamma,\mathbf{w},b} \quad \hat{\gamma}/ \quad \|\mathbf{w}\|$$

$$\text{s.t.} \quad y^{(i)}(\mathbf{w}^T\mathbf{x}^{(i)} + b) \geq \hat{\gamma}, \quad i = 1, \ldots, m$$

In order to make it a convex optimization problem (another subject to study ! F!)
(1) we restrict the value of $\hat{\gamma} = 1$, by scaling w and b (can do single w/b for all $\hat{\gamma}$?)
(2) and rewriting $\frac{\hat{\gamma}}{\|\mathbf{w}\|} = \frac{1}{\|\mathbf{w}\|}$, we get w in from nominator to denominator

$$\min_{\gamma,\mathbf{w},b} \quad \frac{1}{2}\|\mathbf{w}\|^2$$

$$\text{s.t.} \quad y^{(i)}(\mathbf{w}^T\mathbf{x}^{(i)} + b) \geq 1, \quad i = 1, \ldots, m$$

Apply cvx opt. library to solve the above problem