Notes on traversable wormholes and quantum teleportation

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Abstract

Summary of some relevant papers in the literature related to seminal work of Gao, Jafferis and Wall "Traversable wormholes via double-trace deformation". List of references focuses mostly on the SYK model.

Contents

1 Review of SYK model and (Nearly) AdS_2

Summary: The Sachdev-Ye-Kitaev (SYK) model is a quantum mechanical system of interacting fermions randomly coupled. The properties of this model, especially the saturation of the chaos bound on the Lyapunov exponent, suggested the existence of a holographic dual gravitational theory. The gravitational description is given by a Nearly AdS_2 solution of two-dimensional dilaton gravity. The dilaton field diverges near the boundary and the introduction of a cutoff becomes necessary. This can be identified with the breaking of the conformal symmetry in the SYK model that emerged in the infrared. The same effective Schwarzian action can be derived in both SYK and nearly AdS_2 theories, supporting the connection between them at least near the low energy limit.

1.1 Introduction

The AdS/CFT correspondence [1, 2, 3], also referred to as gauge/gravity duality or holography, is a remarkable duality connecting a theory of gravity to a quantum field theory in one lower dimension (without gravity). The history begins with the 'area theorem' by Hawking in 1971 [4], stating that the area of the horizon of a black hole must always increase. This behavior has an obvious analogy with the second law of thermodynamics, where the entropy of a closed system must always increase. After some years, Bekenstein [5] took a step forward and established that the entropy of the black hole is proportional to the area of the horizon

$$S_{BH} = \frac{A_H}{4G_N}. \tag{1}$$

The entropy of a black hole scales like area, instead of volume as one could naively expect. Susskind interpreted these results as a holographic principle [6]: for a theory of quantum gravity, the description of a volume of the space is encoded on its boundary, in the same way as a hologram in 2d encodes the information of a 3d object. Therefore, we expect quantum gravity to have the same number of degrees of freedom of a field theory without gravity, in one lower dimension. In its original formulation, derived in the context of string theory, the conjecture stated that type IIB supergravity in $AdS_5 \times S^5$ is dual to the superconformal $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions. Following the original conjecture, many other examples of dualities have been established [7].

One might naively expect that simpler examples could have been constructed in the lowest dimensional scenario, i.e., AdS_2/CFT$_1$. However, investigations of a AdS_2/CFT$_1$ duality brings up several issues, both from the two-dimensional gravity and from the one-dimensional CFT point of view. These low dimensional theories are highly constrained, making it difficult to establish a consistent theory with non-trivial dynamics. In two-dimensional gravity, the Einstein tensor vanishes identically, and one needs to include additional fields to make gravity dynamical. From the CFT perspective, in one dimension the traceless condition of the stress tensor implies the vanishing of the Hamiltonian, and the theory becomes a theory of a constraint only.

The motivation to establish a AdS_2/CFT$_1$ relies not only on the desire to understand holography in this subtle lower dimensional scenario. Geometries of form AdS_2 times a compact manifold arise in the near horizon limit of extremal black holes in higher dimensions (e.g., charged Reisner-Nordstrom in four dimensions). Therefore, low dimensional holography is also relevant to study holography in higher dimensions.

Kitaev [8] proposed the first satisfactory realization of a AdS_2/CFT$_1$ correspondece using a quantum mechanical system with N interacting Majorana fermions as a candidate for a theory with a holographic dual. The model was based on a spin quantum Heisenberg magnet with gaussian distributed interactions previously studied by Sachdev and Ye [9] in 1992. From now on this model proposed by Kitaev will be referred to as Sachdev-Ye-Kitaev (SYK) model. The remarkable properties of the SYK model are

- **Solvability in the limit $N \gg 1$:** Analytic treatment is possible in the large N limit. From the analysis of the path integral, the large N limit can be interpreted as a classical limit;
- **Emergent conformal symmetry:** There is an emergent conformal symmetry in the low energy limit. This symmetry, however, is explicitly broken as we move away from the IR;
- **Saturation of the chaos bound:** 4-point out-of-time order correlation functions

give a Lyapunov exponent that saturates the chaos bound [10]. A similar behavior appears in many black hole systems.

The above properties, especially the saturation of the chaos bound, supported the existence of a possible holographic dual. In Section 1.2, we review the SYK model and the properties listed above. In Section 1.3, we discuss Nearly AdS_2 gravity as the gravitational theory that reproduces several aspects of the SYK model. Conclusions and several extensions are presented in Section 1.4. A very nice pedagogical review of the SYK and Nearly AdS_2 can be found in Ref. [11].

1.2 SYK model

A detailed review of the SYK model can be found in Ref. [12]. The model consists of a system with N Majorana fermions ψ_i, interacting according to the Hamiltonian

$$H = \sum_{1\leq i<j<k<l\leq N} J_{ijkl}\psi_i\psi_j\psi_k\psi_l. \tag{2}$$

The couplings J_{ijkl} are chosen randomly following a gaussian distribution with zero mean and non-zero width given by

$$\sigma = \frac{\sqrt{3!}\mathcal{J}}{N^{3/2}}. \tag{3}$$

The parameter $\mathcal{J}$ has dimensions of energy and sets the energy scale of the theory. Due to the random couplings, solving the model requires not only the computation of the usual path integral

$$Z[J_{ijkl}] = \int D\psi_i e^{-\int d\tau\left[\frac{1}{2}\sum_i \psi_i\partial\psi_i + \sum_{ijkl} J_{ijkl}\psi_i\psi_j\psi_k\psi_l\right]}, \tag{4}$$

but it has an extra integration over disorder due to the gaussian distribution of the couplings

$$\langle Z\rangle_J \sim \int dJ_{ijkl}\exp\left(-\sum_{ijkl}\frac{J^2_{ijkl}}{2\sigma}\right) Z[J_{ijkl}]. \tag{5}$$

1.2.1 Emergent conformal symmetry

In the following we check the emergence of conformal symmetry in the low energy limit by explicitly solving the path integral with the disorder average in the large N limit. The trick is to perform a change of variables defining the bilocal fermionic field

$$G(\tau_1,\tau_2) \equiv \frac{1}{N}\sum_i \psi_i(\tau_1)\psi_i(\tau_2). \tag{6}$$

The disorder averaged path integral can be rewritten, after performing a gaussian integral over J_{ijkl}, as

$$\langle Z\rangle_J \sim \int D\psi_i DGD\Sigma\Big(-\int\frac{1}{2}\sum_i\psi_i\partial_\tau\psi_i - \frac{1}{2}\int\int N\Sigma\left(G - \frac{1}{N}\sum_i\psi_i\psi_i\right) + \frac{\mathcal{J}^2 N}{2.4}\int\int G^4\Big), \tag{7}$$

where we have defined an extra bilocal field Σ that acts as a Lagrangian multiplier fixing G to the form we have defined. Integrating out the fermions, we are left with an integral for the bilocal fields G, Σ only

$$\langle Z \rangle_J \sim \int DGD\Sigma e^{-NI[G,\Sigma]},$$
$$I[G,\Sigma] = -\frac{1}{2}\log\det(\partial_\tau - \Sigma) + \frac{1}{2}\int\int\left(\Sigma G - \frac{1}{4}\mathcal{J}^2 G^4\right). \tag{8}$$

In the large N limit, the saddle point dominates and the solution will be given by the minimum of the action, which corresponds to solve the equations

$$G = [\partial_\tau - \Sigma]^{-1}, \qquad \Sigma(\tau,\tau') = \mathcal{J}^2[G(\tau,\tau')]^3. \tag{9}$$

In this sense, the large N limit simplifies the calculation because this corresponds to a classical limit. Equations of motion can be solved in the low energy limit by going to momentum space. The Fourier transform of the first equation gives

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega). \tag{10}$$

The $-i\omega$ term can be neglected in the strict low energy limit, in which $\omega \sim \tau^{-1} \ll \mathcal{J}$. The emergence of the conformal symmetry can then be checked, since the solution has the form of the two-point correlation functions in a one-dimensional CFT

$$G_c(\tau) \sim \frac{\mathrm{sgn}\ \tau}{|\tau|^{2\Delta}}, \quad \Sigma_c(\tau) \sim \frac{\mathrm{sgn}\ \tau}{|\tau|^{(2\Delta)3}} \tag{11}$$

One observation that will be relevant to make contact with Nearly AdS_2 gravity is that this emergent conformal symmetry is explicitly broken due to the $-i\omega$ term as soon as we include corrections away form the IR limit.

1.2.2 Schwarzian effective action

We have seen that conformal symmetry emerged in the low energy limit of the SYK model. It turns out that the equations of motion (9) actually have a larger reparametrization symmetry, where different solutions are all connected by reparametrizations $\tau \mapsto f(\tau)$ transforming according to

$$G(\tau_1,\tau_2) \mapsto [f'(\tau_1)f'(\tau_2)]^{\Delta} G(f(\tau_1), f(\tau_2)),$$
$$\Sigma(\tau_1,\tau_2) \mapsto [f'(\tau_1)f'(\tau_2)]^{3\Delta} \Sigma(f(\tau_1), f(\tau_2)), \tag{12}$$

with $\Delta = 1/4$. There is a degenrate family of saddle points and the conformal solution (G_c, Σ_c) spontaneously breaks the reparametrization symmetry. After the inclusion of the $-i\omega$ term, the degeneracy is lifted and we are left with a single saddle point. It turns out that the effective action for the reparametrizations $f(\tau)$ describing the leading order correction to the breaking of the conformal symmetry is given by the so called Schwarzian action [12]

$$S_{eff} \sim \int d\tau\, \mathrm{Sch}(f(\tau,\tau)),$$
$$\mathrm{Sch}(f(\tau),\tau) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\frac{f''^2}{f'^2}. \tag{13}$$

The Schwarzian action can be solved, for example, using Conformal Bootstrap techniques [13]. We will show in Section 1.3 that the same effective action can be derived in the gravity theory.

1.2.3 Saturation of the chaos bound

Kitaev [8] showed that the SYK model saturates the bound on the Lyapunov exponent [10]. This was perhaps the main piece of evidence for the existence of a holographic gravitational dual theory. This is because this saturation is very rare in quantum mechanical systems, but it occurs very often in black hole systems, in which chaos can be diagnosed by studying shockwaves propagating towards the horizon.

Let us first review the general prescription to diagnose chaos in quantum mechanical systems. Quantum chaos is typically diagnosed by evaluating the commutator

$$C(t) = -\langle [W(t), V(0)]^2 \rangle, \tag{14}$$

where W and V are arbitrary operators and $\langle . \rangle = Z^{-1}\mathrm{tr}[e^{\beta H}]$ is the thermal expectation value at temperature $T = 1/\beta$. This gives the effect of perturbations by V on later measurements of W and vice-versa. Evaluating the above expectation value is equivalent to compute the out-of-time order correlator (OTOC)

$$\langle V(0)W(t)V(0)W(t) \rangle. \tag{15}$$

If the OTOC experiences an exponential growth $\sim e^{\lambda_L t}$, then the system is chaotic and the exponent λ_L, called Lyaponov exponent, quantifies the presence of chaos in the system.

The prescription using $C(t)$ can also be interpreted from its classical counterpart. Setting $V = p$ and $W = q$ to be the momentum and position functions, the classical analog is obtained by replacing the commutator by Poisson brackets $[W, V] \to \{q, p\}_{PB}$. This gives the prescription for classical chaos

$$\frac{\partial q(t)}{\partial q(0)} = \{q(t), p(0)\}_{P.B.} \sim e^{\lambda_L t}, \tag{16}$$

that gives the dependence of the final position on small changes in the initial position. Maldacena, Shenker and Stanford proved a bound for the Lyapunov exponent [10]

$$\lambda_L \leq \frac{2\pi}{\beta}. \tag{17}$$

(in natural units where $k_B = \hbar = 1$). For the SYK model, the OTOC have been computed [8, 14] and the result was found to be

$$\langle \psi_i(0)\psi_j(\tau)\psi_i(0)\psi_j(\tau) \rangle \sim e^{\frac{2\pi}{\beta}\tau} \Rightarrow \lambda_L = \frac{2\pi}{\beta}, \tag{18}$$

where the temperature in the SYK model was introduced by the choice of reparametrization $f(\tau) = \tan \frac{\pi\tau}{\beta}$, making time periodic. Therefore, the SYK model saturates the bound, similarly to black holes. This result highly suggested that the SYK model was in some way connected to a gravitational theory.

1.3 Nearly AdS_2 gravity

Now we move to the gravitational interpretation of the SYK model. This section closely follows Ref. [15]. Anti-de Sitter space AdS_{d+1} is a hyperbolic spacetime with negative constant curvature. In two dimensions, the metric in Poincaré coordinates and Euclidean signature is given by

$$ds^2_{AdS_2} = \frac{dt^2 + dz^2}{z^2}. \tag{19}$$

The boundary in this coordinate system is located at $z = 0$. The isometry group of AdS_{d+1} is $SO(d, 2)$, which is the conformal group in d dimensions.

Besides the problem that pure Einstein-Hilbert action renders no dynamics in two dimensions, AdS_2 spacetimes cannot support finite energy excitations. This is a back-reaction problem where any excitation destroys the asymptotic AdS_2 geometry [16]. In the two-dimensional dilaton gravity theories we consider here, the backreaction problem is tractable. The asymptotics of AdS_2 gets modified by the presence of the dilaton field, and that introduces the concept of a 'Nearly' AdS_2 spacetime.

The model we consider have been previously studied by Jackiw and Teitelboim [17, 18], and recently revisited by Almheiri and Polchinski [19]. The action is a two-dimensional gravity theory non-minimally coupled to a scalar field Φ

$$I_{JT} = \frac{1}{16\pi G_N}\left[\int dx^2 \Phi\sqrt{g}(R+2) + 2\int_{bdy} \Phi_b K\right]. \tag{20}$$

The scalar field Φ is typically called dilaton, essentially because it multiplies the scalar curvature R. The boundary term is necessary to render a well defined variational problem, where K is the extrinsic curvature and Φ_b is the boundary value of Φ. This action can be interpreted as coming from a dimensional reduction of an action where the dilaton plays the role of volume of the transverse space and goes to a constant in the low energy limit.

The equation of motion for Φ gives

$$R + 2 = 0, \tag{21}$$

so the metric can be fixed as AdS_2 metric. On the other hand, variation w.r.t. the metric field gives

$$\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\nabla^2\Phi + g_{\mu\nu}\Phi = 0, \tag{22}$$

and the general solution is

$$\Phi = \frac{\alpha + \gamma t + \delta(t^2 + z^2)}{z}, \tag{23}$$

for arbitrary constants α, γ, δ. The solution for Φ diverges near the boundary as $z \to 0$. In order to handle this divergence, we introduce a cutoff at radius $z = \epsilon$. We rewrite the boundary value of the dilaton field as

$$\Phi_b = \frac{\Phi_r(u)}{\epsilon}, \tag{24}$$

where $\Phi_r(u)$ is the 'renormalized' boundary value. The parameter u parametrizes the cut boundary curve $(t(u), z = \epsilon)$. We can regard u as the physical time in the dual quantum mechanical system.

The presence of a cutoff can be directly connected to the SYK picture. We have seen that the SYK model has an emergent conformal symmetry in the IR which is explicitly broken as we start to move towards the UV. Therefore, this breaking can be identified with the introduction of the cutoff.

It turns out that the same Schwarzian action presented in Section 1.2.2, derived for the SYK model, can also be derived in the Nearly AdS_2 scenario. The effective action describes the shape of the cut boundary curve, where $t(u)$ is regarded as the dynamical field. The first term in the action (20) vanishes due to the equations of motion. The effective action comes from the extrinsic curvature term only. We have

$$\begin{aligned} K &= \frac{t'(t'^2 + z'^2 + z'z'') - zz't''}{(t'^2 + z'^2)^{3/2}} \\ &= 1 + \mathrm{Sch}(t(u), u)\epsilon^2 + \mathcal{O}(\epsilon^4), \end{aligned} \tag{25}$$

where $\mathrm{Sch}(t(u), u)$ is the Schwarzian defined in (13). The prime denotes derivatives w.r.t. u and we have used that $z = \epsilon t' + \mathcal{O}(\epsilon^3)$ near the surface $z = \epsilon$.

1.4 Some extensions

The SYK model provides a tractable way to study holography in lower dimensions. The model was the first satisfactory realization of AdS_2/CFT_1, in which the same effective action (Schwarzian) can be derived in both SYK and Nearly AdS_2 gravity. Although this duality has only been established near the infrared limit, this example provides an excellent toy model that can be extended to build more sophisticated and general dualities.

After the original work by Kitaev, several extensions have appeared. A few of such generalizations are listed below.

- q-body interaction [12]: Generalizes the 4-body interaction in the Hamiltonian (2) to an arbitrary q-body interaction. This generalization is also tractable in the limit of large q, where one can perform a $1/q$ expansion;

- Supersymmetric versions [20]: Supersymmetric extensions are desired to possibly connect/embed the SYK model into a string theory framework. In these supersymmetric versions, the couplings J_{ijkl} turn out to not be independent from each other;

- Higher dimensions: Higher dimensional versions have been considered, e.g., in Ref. [21]. Another perspective is to embbed the low dimensional theory into gravity in higher dimensions. It has been shown that the spectrum of SYK model [14] can be reproduced by a three-dimensional gravitational background [22, 23].

- $1/N$ corrections [24]: Kitaev and Suh studied $1/N$ corrections in very detail, and they were able to match the effective action correcting the Schwarzian by modifying the gravitational theory with an additional potential term for the dilaton field.

- Other examples include: addition of flavor [25], lattice SYK [26], models without disorder [27], complex fermions [28].

2 A Traversable Wormhole Teleportation Protocol in the SYK Model

This a summary of the paper [32] by Gao and Jafferis. After the construction traversable wormholes by introducing a coupling between the two asymptotic boundaries of a BTZ wormhole [Gao, Jafferis, Wall '16], several papers appeared trying to understand the dual description of a traversable wormhole. Below we mention a few of them

- "Diving into traversable wormholes" [Maldacena, Stanford, Yang '17]: Focusing on SYK model and (Neraly) AdS_2 wormhole, they provided an estimation for the bound on information transfer thorugh the wormhole.

- "Teleportation through the wormhole" [Susskind, Zhao '17]: Described a possible dual teleportation protocol but used precursors operators with much higher complexity than the double trace operator.

- "Traversable Wormholes as quantum channels" [Bao, Chatwin-Davies, Pollack, Remmen '18]: Instead of quantum teleportation, they suggested that the dual of a traversable wormhole is a quantum channel (something more general).

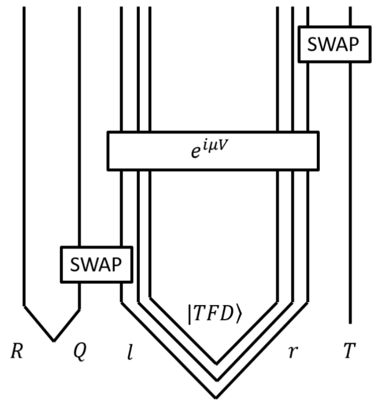

Figure 1: Quantum circuit representing the teleportation protocol. Time goes upwards.

- "Efficient decode for Hayden-Preskill protocol" [Yoshida, Kitaev '17]: They showed how we can decode a qubit that has been highly scrambled (this is a property expected from the traversable wormhole picture), but the relation to traversable wormhole was not clear.

This paper proposes a concrete teleportation protocol which involves simpler operations compared to previous work, and they argue this system have several features that are similar to the transfer of a signal through the wormhole. The setup of the teleportation protocol takes into account two copies of the SYK model (which we call l and r). The SYK model in case is the one with a q-body interaction

$$H_{l,r} = i^{q/2} \sum_{1\leq j_1<\ldots<j_q\leq N} J^{l,r}_{j_1\ldots j_q} \underbrace{\psi^{j_1}_{l,r}\ldots\psi^{j_q}_{l,r}}_{\text{q Majorana interaction}} \quad , \qquad H = H_l + H_r$$

The advantage of this model is it solvability in a $1/q$ expansion.

Each copy l and r can be viewed as a system of $\frac{N}{2}$ qubits, so that we have N qubits in total. We consider three auxiliary systems R, Q, T, where each system is a single qubit system. The system R is a reference system, and we want to send information from Q to T. Success of teleportation from Q to T using the entanglement resource of l and r is equivalent to producing maximal entanglement between R and T.

State preparation: We prepare our system in such a way that T is in the state $|0\rangle$, RQ is in the maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and l and r in TFD state $|TFD\rangle = \frac{1}{Z}\sum_n e^{-\beta E_n/2}|nn\rangle_{lr}$.

SYK teleportation protocol: The teleportation protocol consists of the following steps:

1. At time $-t'$, apply SWAP between Q and $l \to$ Qubit is inserted into the wormhole;
2. At time zero, apply interaction $e^{i\mu V}$ with $V = \frac{i}{qN}\sum_{i=1}^{N}\psi^j_l\psi^j_r$
3. At time t, apply SWAP between T and $r \to$ Qubit is extracted from the wormhole.

where the SWAP operator acts on two systems by swapping their states.

Characterization of teleportation: We will have perfect teleportation if mutual information between R and T

$$I_{RT} = S(R) + S(T) - S(RT) \tag{26}$$

is maximal. The computation of the mutual information requires the evaluation of the reduced density matrix ρ_{TR}, which in turn requires the calculation of the correlation function

$$\mathcal{K}(t,t') = \langle \{\psi_l(-t'), e^{-i\mu V}\psi_r(t)e^{i\mu V}\}\rangle \tag{27}$$

Then, the mutual information can be written as

$$I_{RT} = \frac{1}{4}\left[(\mathcal{K}-1)^2\log(\mathcal{K}-1)^2 + (\mathcal{K}+1)^2\log(\mathcal{K}+1)^2 + 2(1-\mathcal{K}^2)\log(1-\mathcal{K}^2)\right] \tag{28}$$

For this calculation we needed to consider the leading $1/N$ correction to the Schwarzian limit. The effect of the interaction is equivalent to solving the SYK model with modified boundary conditions.

SWAP operator: The SWAP operation used to insert/remove the qubit into/from the black hole is

$$S_{Ql} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \otimes \chi_l\chi_l^\dagger + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \otimes \chi_l{}^\dagger + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \otimes \chi_l + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \otimes \chi_l^\dagger\chi_l \tag{29}$$

(In the above equation χ_l is a Dirac fermion)

Encoding fermionic operators into qubits: The Majorana fermions can be encoded into spin-$\frac{1}{2}$ degrees of freedom by using the Jordan-Wigner transformation. For Majorana fermions χ the transformation is (see, e.g. [48])

$$\chi_{2n-1} = \left(\prod_{j=1}^{n-1}\sigma_j^z\right)\sigma_n^x, \qquad \chi_{2n} = \left(\prod_{j=1}^{n-1}\sigma_j^z\right)\sigma_n^y, \qquad \{\chi_i,\chi_j\} = 2\delta_{ij}. \tag{30}$$

There are other ways to represent fermionic operators as qubits. See, for example, the recent paper [35].

Results:

- **Semiclassical limit, large q**: Does not get maximal I_{RT}, but can get it by an appropriate choice of SWAP operator.
- **Partial coupling**: coupled qubits between two systems can be chosen independently from the teleported qubits.
- **Teleportation of multiple qubits**: Receive signals with same time ordering, but there is another regime in which time ordering is reversed.
- **Interference regime**: Qubit sent into the wormhole at very early times $\Rightarrow$ back-reaction and $1/N$ expansion is not valid.
- **Open question**: Understand if there is a bulk description of the partially successful teleportation at very high temperatures in the fully scrambled regime.

3 Preparing SYK in a TFD state

Eternal traversable wormhole: The original idea of Gao, Jafferis, and Wall [29] was to turn on an interaction between the two asymptotic boundaries for some amount of time, which renders the wormhole traversable for some short time interval. Later, Maldacena and Qi constructed an eternal traversable wormhole [30], which remains open for all times, by considering two copies of the SYK model with a bilinear coupling between them. Maldacena and Qi have considered the following Hamiltonian

$$H_{TFD} = H_L^{SYK} + H_R^{SYK} + H_{\text{int}}, \qquad H_{\text{int}} = i\mu \sum_j \psi_L^j \psi_R^j \tag{31}$$

where H_L^{SYK} and H_R^{SYK} are two identical SYK systems coupled with an interaction whose strength is controlled by the parameter μ. The Maldacena-Qi model is believed to be dual to a traversable wormhole for weak coupling $\mu \ll J$ (where J is the typical energy scale) and low temperatures $T \lesssim \mu$. The traversable wormhole in this case has a global AdS_2 geometry.

The thermo field double plays a crucial role in this construction and in other wormhole holographic models as well. One natural question is how to actually build this state, in particular using the SYK model. Maldacena and Qi have shown that the TFD state is approximately the ground state of the Hamiltonian H_{TFD} (see also Ref. [31] for some intuition for the ground state being approximately the TFD). More recently, Maldacena and Milekhin [33] showed that the TFD state for SYK can be prepared by starting from any excited state and then cooling the system down by coupling to an external bath.

Variational Preparation of the SYK Thermofield Double: Here we summarize the paper [38] by Vincent Paul Su, which deals with the question of simulating the thermofield double state (TFD) for the SYK model in a near term quantum device. In this paper, instead of considering an external bath, the author proposes a variational approach to build the TFD for the SYK model. It assumes that it will be approximately given by the ground state of H_{TFD} and uses a variational method whose goal is to minimize the expectation value of H_{TFD}.

Parametrized quantum circuit: The idea of the variational method is to build an Ansatz for a quantum circuit that prepares the TFD state. The sequence of gates is fixed but each gate has a free parameter that can be tuned (like the angle of a rotation operator). The algorithm uses a gradient descent to minimize the energy, and before that it uses a Jordan-Wigner transformation to express the Hamiltonian as qubits instead of Majorana fermions (a map between spinless fermionic operators and non-local Pauli operators).

Results: The algorithm is a hybrid quantum-classical algorithm because the gradient descent step is carried on a classical computer. The quantum circuit was simulated using the package Yao.jl and results (`https://github.com/vipasu/SYK-TFD`) for the ground state were obtained for $N = 8$ up to $N = 12$. At the end the author mentions qubitization as a promising way to simulated the dynamics of the system once we have prepared the TFD state, which is an essential step towards simulating holographic systems on a quantum computer.

A more general discussion about how to prepare the thermofield double state can be found in Ref. [41].

4 Making near extremal wormholes traversable

Traversable wormholes with charge might be interesting because of the AdS_2 geometry that arises in the near extremal limit (see e.g. [40]), so they could be more closely related to the SYK model, even though they live in higher dimensions. For this reason the paper [39] might be useful for qualitative comparison with results obtained by simulating two coupled SYKs. The authors studied Reissner-Nordstrom black holes in AdS_{d+1}. These black holes are special because they have finite entropy but infinitely long throat in the extremal limit. One advantage of the near extremal regime is that it is possible to evaluate the propagator for charged bulk fields (although no analytic propagator is known for the full black hole solution).

Results: It is still possible to make the wormhole traversable in the extremal limit if the coupling scales inversely with the temperature. The extremal limit then would correspond to the limit of infinite coupling. Since the distance between the two boundaries also goes to infinity, the effect of the backreaction is finite. In the near extremal regime and with coupling scaling with inverse of temperature, the timescale for travelling through the wormhole is set by the temperature of the black hole.

5 Numerical simulations

Many-Body Chaos in the Sachdev-Ye-Kitaev Model: This paper [42] provides the state-of-the-art of the numerical simulation of a single SYK model with all-to-all 4-body interaction. They built a package https://dynamite.readthedocs.io/en/latest/ that makes use of massive parallelization and the so called Krylov subspace (which makes an approximation for the action of the unitary operator $U(t) = e^{-Ht}$). They are able to compute dynamical correlators in SYK for up to $N = 60$ Majorana fermions. They also proposed a rescaling procedure for OTOCs to extract the Lyapunov exponent. In the discussion, they mention that they are also looking forward to numerical simulations to test more complex gravitational phenomena including traversable wormholes.

The Coupled SYK model at Finite Temperature: In Ref. [44], they studied two coupled SYK and obtained expected behavior from traversable wormhole analysis. Could try to relate their picture with the quantum circuit teleportation protocol in SYK recently proposed by Gao and Jafferis [32]. They studied the effect of the coupling beyond the low temperature regime. They considered the SYK with a $q/2$ body interaction.

Quantum simulation with hybrid tensor networks: [45] Bullet point in Section V Applications. Also last part of appendix D has some comments about traversable wormholes.

Variational quantum eigensolver (VQE): It is a hybrid quantum-classical algorithm to find an approximation for the ground state of a quantum system it iterates 3 steps:

1. Prepare trial wavefunction $|\psi(\theta)\rangle$ from the sucessive application of unitary quantum gates that depend on variational parameters θ.

2. Evaluate the mean energy
$$E(\theta) = \langle\psi(\theta)|H|\psi(\theta)\rangle \tag{32}$$

3. Optimize the variational parameters to minimize the mean energy by making use of a **classical** optimization algorithm.

6 Revival dynamics in a traversable wormhole

This is a summary of Ref. [43], which studied two coupled SYKs numerically to see whether 'revivals' can be detected or not. Consider the Hamiltonian

$$H = H_L^{SYK} + H_R^{SYK} + i\mu \sum_j \chi_L^j \chi_R^j \tag{33}$$

where

$$H_\alpha^{SYK} = \sum_{i<j<k<l} J_{ijkl} \chi_\alpha^i \chi_\alpha^j \chi_\alpha^k \chi_\alpha^l, \quad \alpha = L, R \tag{34}$$

is the Hamiltonian for the Left/Right SYK system. We are interested in the weak coupling regime $\mu \ll J$.

Revival dynamics: Consider the ground state $|\Psi_0\rangle$ of the full Hamiltonian H. If we create an excitation on the right side, $\chi_R^j|\Psi_0\rangle$, the excitation will be scrambled and transported to the left side where it is 'unscrambled' and after some characteristic time and the state will become close to $\chi_L^j|\Psi_0\rangle$. This is the 'revival dynamic' behavior predicted from a system dual to a traversable wormhole configuration.

Two numerical approches: The paper aims to detect numerically whether these revivals occur for the two coupled SYK system. They take two approaches

1. **Exact diagonalization:** Obtain the eigenstates and energies of the Hamiltonian H by exact diagonalization. This allows us to compute any observable or correlation function but computationally can only be done for $2N \leq 32$.

2. **Solve saddle point equations:** The second approach is to solve the saddle point equations for the large N limit written in terms of the averaged fermion propagator

$$G_{\alpha\beta}(\tau_1\tau_2) = \frac{1}{N}\sum_j \langle \mathcal{T} \chi_\alpha^j(\tau_1)\chi_\beta^j(\tau_2)\rangle \tag{35}$$

   where $\mathcal{T}$ is the imaginary-time ordering and $\langle ...\rangle$ is the thermal average. The calculation is simplified due to time-translation symmetry and Left-Right mirror symmetry.

Holographic prediction: Maldacena and Qi predicted two conformal towers of states at low energies

$$E_n^{\text{conf}} = \epsilon(\delta + n), \quad E_n^{bg} = \epsilon\sqrt{2(1-\Delta)}\left(n + \frac{1}{2}\right) \tag{36}$$

In order to study dynamics in this setup, the authors switch to the real-time representation of the SD equations and employ a weighted-iteration scheme. The numerical simulation requires both small and large frequency regulators among other things that make the numerics challenging.

Capture revivals: Diagnose a 'revival' by looking at the transmission amplitudes between the two systems

$$T_{\alpha\beta}(t) = 2|G^{>}_{\alpha\beta}(t)|, \qquad G^{>}_{\alpha\beta}(t) = \frac{\theta(t)}{N}\sum_j \langle \chi^j_\alpha(t)\chi^j_\beta(0)\rangle \tag{37}$$

The transmission $T^2_{\alpha\beta}$ then gives the probability to recover $\chi^j_\alpha(t)$ after inserting $\chi^j_\beta(0)$ initially, averaged over all modes j.

Results: In the large N limit, they observed a sharp revival and oscillations at frequency $\sim \mu^{2/3}$. For small N, no tower of states are visible in the exact diagonalization approach. In this case the revivals come from a different mechanism so that the revival amplitude decreases with N.

Open questions: It is still not clear how the dynamics change with increasing N. Here, it was not possible to study the intermediate regime because diagonalization could only be applied for small N while saddle point equations are only valid for large N. The authors suggest the use of Krylov space as a possible alternative approach. They also mention that it would be interesting to related their results with the circuit-based teleportation protocol of Gao and Jafferis. The main difference is the time dependence in the interaction term and the use of a SWAP operator after rewriting the SYK system in terms of qubits.

7 A Sparse Model of Quantum Holography

Motivation: The SYK model has been a successful toy model of low dimensional quantum black holes. The sparse SYK model [51] (see also [46]) provides a wider class of systems exhibiting maximal chaos, and this new class of models may serve as effective theories for dense connected system with the advantage of allowing for a more efficient computer simulation (both classical and quantum). Here, we will refer to the original SYK model proposed by Kitaev as all-to-all SYK or dense SYK. The low temperature limit of the sparse SYK features emergent gravitational features as in the all-to-all SYK. The main property is the saturation of the MSS chaos bound $t_* \geq \frac{\beta}{2\pi}\log N$. Recall that scrambling is the process in which initially simple information becomes thoroughly mixed across the degrees of freedom of the system that it cannot be read-out by any few-body measurement, and t_* is the scrambling time, defined as the timescale for information to spread across the entire system.

Hamiltonian: The sparse Hamiltonian for the sparse SYK model with N Majorana fermions is defined as

$$\begin{aligned} H &= i^{q/2}\sum_{a_1<\ldots<a_q} J_{a_1\ldots a_q} x_{a_1\ldots a_q}\chi_{a_1}\cdots\chi_{a_q} \\ &= \frac{i^{q/2}}{q!}\sum_{a_1\neq\ldots\neq a_q} J_{a_1\ldots a_q} x_{a_1\ldots a_q}\chi_{a_1}\cdots\chi_{a_q}, \end{aligned} \tag{38}$$

where the coupling $J_{a_1\ldots a_q}$ is drawn from a normal distribution with zero mean and variance

$$\langle J^2_{a_1\ldots a_q}\rangle = \frac{(q-1)!}{J}^2 N^{q-1}. \tag{39}$$

The parameters $x_{a_1\ldots a_q}$ are either 0 or 1 and they can be defined in different ways leading to different sparse models.

Sparseness: One version of the sparse SYK model consists of taking $x_{ijkl} = 1$ with probability p and $x_{ijkl} = 0$ with probability $1 - p$. The system becomes sparser as $p \to 0$ and we recover the all-to-all dense SYK when $p = 1$. Formally, the SYK Hamiltonian can be defined over random hypergraphs. Generically, the Hamiltonian has the form

$$H = \sum_{i}^{L} H_{\{v_i\}}, \tag{40}$$

where $H_{\{v_i\}}$ acts on a set of vertices $\{v_i\}$. Each vertex of the hypergraph is associated with a single Majorana fermion, so there are N vertices. Each fermion interacts with a bounded number of other fermions. A hyperedge is present if $x_{a_1 \dots a_q} = 1$, so that the sites $a_1, \dots, a_q$ are connected. The number of hyperedges equals the number of terms in the Hamiltonian. In the all-to-all SYK there are $L = \binom{N}{q} \simeq \frac{N^q}{q!}$ (for large N) terms.

Expanders: We want to define the sparseness in the Hamiltonian in such a way that the associated hypergraph is an expander, i.e., a sparse graph in which the number of neighboring vertices outside subgraph is proportional to the size of the system. Intuitively, expanders are sparse graphs with strong connectivity properties. As an example, random regular graphs are known (rigourously) to be expanders. For a random regular graph, each vertex appears in exactly d hyperedges and the number of terms in the Hamiltonian is

$$L = \frac{dN}{2}, \tag{41}$$

where d is degree (i.e., the number of edges connected to a vertex) for every vertex. We can also remove the regularity condition and obtain similar results by considering sparse random graphs, in which we set the edges independently with edge probability $p = \frac{2k}{N}$. In this way, there are kN egdes on average and the degrees obey a Poisson distribution $P(2k)$.

Some results: In Ref. [51], the path integral analysis combined with numerics for up to 52 fermions implies fast scrambling and maximal chaos at low temperature provided k is large enough. In Ref. [46], the authors wanted to find the minimal value of p such that the system still display features of the dense SYK. For example, exact diagonalization of the Hamiltonian was carried out for $N = 24$ and 5000 disorder realizations. They took $p \sim \frac{k}{N^\alpha}$ and found that around $\alpha = 3$ and $k \sim 1$ is the maximum degree of sparseness. In Appendix A they build an algorithm to build regular hypergraphs.

Disorder average partition function: Need to use replica.

Parisi model: In Ref. [46], the authors pointed out a strongly similarity between the sparse SYK model and the so call Parisi model. See also "Chaos on the hypercube" 2005.13017.

Self-averaging property: If a certain property is satisfied both for a single realization and for the disorder averaged theory, this property is said to be self-averaging. Partition function of all-to-all SYK is self-averaging, but it is not for the sparse version. However, it was argued in the paper that average Green's function and energy density are self-averaging in the sparse SYK.

8 Other References

Expanding the Black Hole Interior: Partially Entangled Thermal States in SYK: In Ref. [47], they constructed a model that decreases the entanglement of the TFD state associated to SYK models. Geometrically this gives a scenario in which a causal shadow appears, so one could study the effect of this causal shadow on the traversability of the wormhole.

Experimental probes of traversable wormholes: [49] Comments on the difference between the signal traversing the wormhole and the signal propagating instantaneously through the interaction. This is for an eternal traversable wormhole.

Quantum Gravity in the Lab: Teleportation by Size and Traversable Wormholes: [50] Definitions of teleportation by size, size winding, and obtained bounds on fidelity.

Regenesis and quantum traversable wormholes: Ref. [34] has an interesting Discussion section from a general point of view.

Diagnosing quantum chaos in many-body systems using entanglement as a resource: In Appendix F of Ref. [31] there are some details about exact diagonalization of Maldacena-Qi Hamiltonian.

Traversable wormhole and Hawking-Page transition in coupled SYK models: Ref. [36] follows the idea of revivals but for complex fermions.

Tunneling through an Eternal Traversable Wormhole: Ref. [37] is a recent paper with numerical studies on the complex version of Maldacena-Qi model.

9 Ideas on traversable wormholes and sparse SYK

Here we summarize some possible directions for a project relating traversable wormholes and the sparse SYK model.

General motivation: The SYK model was proposed by Kitaev in 2015 as a simple (perhaps the simplest) quantum mechanical model with a holographic gravitational dual. It has the advantage that is solvable in the large N limit, and 1/N corrections are also tractable (these corrections are mapped to quantum corrections to the classical gravity limit). There is also some amount of work in the numerical simulation of this system. The simulation is computationally hard to perform for larger values of N, but there is some progress in that direction to simulate up to $N = 60$ Majorana fermions. The sparse SYK then appears as a model that keeps some of the analytical solvability while making the numerical simulation much more tractable.

Understand properties of sparse SYK: The authors of the sparse model claim that the sparse version of the SYK retains many properties of the original SYK model proposed by Kitaev (all-to-all 4-body interaction). The first natural step is to understand what are the properties observed in the sparse SYK model, focusing on the properties that suggest the existence of a gravitational dual (like emergent conformal symmetry and saturation of chaos bound). Exact diagonalization can be done numerically for sufficiently small values of N. A simple test to gain intuition about the effect of sparseness is to compute the exact diagonalization for the all-to-all (dense) SYK and compare to the sparse version.

Preparing the sparse SYK TDF state: In the eternal traversable wormhole scenario, the ground state for two coupled SYKs is approximately the TFD. It is not obvious that the same happens when we introduce sparsity in the SYK model. The study of the ground state can be done using exact diagonalization for small N (see e.g. 2006.06019).

Two coupled sparse SYKs: Then, we could try to build a quantum mechanical system that mimics a traversable wormhole. Originally, this was done by taking two copies of SYK and preparing the combined system in a thermofield double state. To make it traversable, we can add an interaction between the two copies. There are different ways of choosing this interaction: in the original version, an interaction was turned on only for a certain amount of time, while later an interaction that is time independent was proposed to build an 'eternal traversable wormhole'. The natural question here is what is the maximum sparseness for which the two coupled sparse SYKs display features consistent with traversable wormhole.

Teleportation protocol: In a recent paper, Gao and Jafferis studied two coupled SYK systems with an interaction that is turned on at time $t = 0$. They used the q-body variation of the SYK, the reason is because analytical expressions can be obtained in a $1/q$ expansion. If we use the sparse SYK model instead, we would lose some of the analytic treatment, but perhaps the numerics would be viable due to sparsity property. It would be interesting to see which aspects of wormhole traversability change once we use the sparse SYK model. For example, in the Gao, Jafferis and Wall wormhole the message had to be sent through the wormhole at an early time before we turn on the interaction between the two boundaries, and then the message is recovered at the other boundary after we wait some time (the time it takes for a particle to travel through the wormhole)

Bound on information transfer: A natural question in the traversable wormhole scenario is the amount of information that can be transferred through it. From the bulk perspective this be done by studying the effect of backreaction of the signal that we want to send. If the backreaction gets too strong the geometry will be deformed in such a way that it makes the opening of the wormhole smaller. From the quantum teleportation point of view, the bound on information transfer can be quantified from the amount of entanglement that is available. In typical teleportation protocols, this is understood as entanglement being used as a resource for the success of the teleportation.

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