Physics coded in the ray tracing program

Description below taken from

Rice W.K.M, 1997, "A ray tracing study of VLF phenomena", PhD thesis, University of Natal

A spherical polar coordinate system has been used (r, θ, ϕ) with azimuthal symmetry (no ϕ dependence) and with θ measured from the pole. This reduces the ray tracing to effectively 2 dimensions. The equations also use χ, the angle between the wave normal direction and the radial direction, instead of ψ, the angle between the wave normal direction and the magnetic field direction. The equations are as follows

$$\begin{aligned}
\frac{dr}{dt} &= \frac{1}{\mu^2}\left(\mu\cos\chi + \frac{\partial\mu}{\partial\chi}\sin\chi\right) \qquad (2.11)\\
\frac{d\theta}{dt} &= \frac{1}{r\mu^2}\left(\mu\sin\chi - \frac{\partial\mu}{\partial\chi}\cos\chi\right)\\
\frac{d\chi}{dt} &= \frac{1}{r\mu^2}\left(\frac{\partial\mu}{\partial\theta}\cos\chi - \left[r\frac{\partial\mu}{\partial r} + \mu\right]\sin\chi\right)
\end{aligned}$$

These equations can be simplified since χ and ψ differ by a constant and therefore $\frac{\partial\mu}{\partial\chi} = \frac{\partial\mu}{\partial\psi}$. As can be seen from equation (2.2), $\frac{\partial\mu}{\partial\psi}$ is much easier to evaluate than $\frac{\partial\mu}{\partial\chi}$.

2.2 Phase Refractive Index

The phase refractive index is found by solving Maxwell's equations for an anisotropic, homogeneous medium in which collisions between neutrals and ions are neglected. This assumption is valid above the D and E regions of the ionosphere. Although this study considers propagation through the D and E regions, it is assumed that the time spent in these regions is small enough for the effect of collisions to be ignored.

The dispersion relation for the medium has the following form

$$A\mu^4 - B\mu^2 + C = 0 \tag{2.1}$$

where μ is the phase refractive index and

$$\begin{aligned} A &= S\sin^2\psi + P\cos^2\psi \\ B &= RL\sin^2\psi + PS\left(1+\cos^2\psi\right) \\ C &= PRL \end{aligned} \tag{2.2}$$

ψ is the angle between the wave normal and the magnetic field direction and R, L, P and S depend on the wave frequency (ω) and are defined in terms of the electron and ion plasmafrequencies (π_k) and gyrofrequencies (Ω_k) in the following way

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$$\begin{aligned} S &\equiv \frac{1}{2}(R+L) \qquad D \equiv \frac{1}{2}(R-L) \\ R &\equiv 1-\sum_k \frac{\pi_k^2}{\omega^2}\left(\frac{\omega}{\omega+\varepsilon_k\Omega_k}\right) \\ L &\equiv 1-\sum_k \frac{\pi_k^2}{\omega^2}\left(\frac{\omega}{\omega-\varepsilon_k\Omega_k}\right) \\ P &\equiv 1-\sum_k \frac{\pi_k^2}{\omega^2} \end{aligned} \tag{2.3}$$

The plasmafrequency (π_k) and gyrofrequency (Ω_k) of the k^{th} species (in rad.s^{-1}), are $\left(\frac{n_k e^2}{\epsilon_o m_k}\right)^{\frac{1}{2}}$ and $\frac{eB}{m_k}$ respectively, where e is the electron charge, ϵ_o is the dielectric constant of free space, m_k is the mass of the k^{th} species, n_k is the density of the k^{th} species and B is the magnetic field strength. Both n_k and B are determined using the density and magnetic field models that will be discussed in the next chapter. The term ε_k in equations (2.3) is -1 for the electrons and $+1$ for the ions. All the above frequencies are angular frequencies (measured in rad.s^{-1}) and can be converted to a frequency in Hertz using $f = \frac{\omega}{2\pi}$. The numerical values for the wave frequency, the gyrofrequencies and the plasma frequencies that will be cited later in the text will be in kHz or MHz.

Equation (2.1) can be solved to give

$$\mu^2 = \frac{B \pm F}{2A} \tag{2.4}$$

where

$$F^2 = (RL - PS)^2 \sin^4 \psi + 4P^2 D^2 \cos^2 \psi \tag{2.5}$$

For each value of the wave normal angle (ψ), equation (2.4) has two solutions corresponding to two different polarisations. For the longitudinal case ($\psi = 0$), $\mu^2 = R$ or $\mu^2 = L$ and for the transverse case ($\psi = \frac{\pi}{2}$), $\mu^2 = \frac{RL}{S}$ or $\mu^2 = P$. This study will consider mainly the mode that would have $\mu^2 = R$ if it were propagating longitudinally. This is known as the electron whistler mode. Waves propagating in the mode that would have $\mu^2 = L$ if they were propagating longitudinally are ion-cyclotron whistlers. A very good review on 'ion' cyclotron whistlers is given in Jones [1972].

The wave polarisation is given by

$$\rho = \frac{E_x}{E_y} = \frac{\mu^2 - S}{D} \tag{2.6}$$

For the longitudinal case the wave corresponding to $\mu^2 = R$ has a polarisation $\rho = 1$ and the wave with $\mu^2 = L$ has a polarisation $\rho = -1$. Both of these waves are therefore circularly polarised: the R, or electron whistler, in the right hand sense and the L, or 'ion' whistler, in the left hand sense. For cases other than the longitudinal case, the polarisation becomes elliptical for both modes.

2.3.1 Ray Travel Time

The time of flight of a pulse of energy along a ray path is

$$\frac{1}{c}\int \mu_g \cos\alpha ds = \int \left(\frac{\mu_g}{\mu}\right) dt \tag{2.12}$$

where μ_g is the group refractive index and α is the angle between the ray direction and the wave normal direction. The right hand side of the above equation was derived using that $\frac{ds}{dt} = ray\ velocity = \frac{c}{\mu\cos\alpha}$.

The group refractive index (in the direction of the phase refractive index) is given by

$$\mu_g = \frac{\partial\left(\mu\omega\right)}{\partial\omega} = \mu + \omega\frac{\partial\mu}{\partial\omega} \tag{2.13}$$

Using equation (2.12) a new variable P, called the 'equivalent path', is defined such that

$$\frac{dP}{dt} = \frac{\mu_g}{\mu} = 1 + \frac{\omega}{\mu}\frac{\partial\mu}{\partial\omega} \tag{2.14}$$

The wave travel time is then given by $\frac{P}{c}$ where c is the free space speed of light.

2.4 Derivatives of μ

The calculation of the ray path requires the derivatives of the phase refractive index (μ) with respect to r, θ, ψ and ω. It must be remembered that the derivative w.r.t. ψ is the same as the derivative w.r.t. χ. μ depends explicitly on ψ and ω and implicitly on r and θ. The derivative of μ w.r.t. ψ is found by differentiating equation (2.4) to get

$$\frac{\partial\mu}{\partial\psi} = \frac{1}{2\mu}\left(\frac{\frac{\partial B}{\partial\psi} \pm \frac{\partial F}{\partial\psi}}{2A} - 2\frac{\partial A}{\partial\psi}\frac{B \pm F}{2A^2}\right) \tag{2.15}$$

where B, F and A are defined in equations (2.2) and (2.5). The derivative of μ w.r.t. ω has the same form as equation (2.15). In this case B, F and A do not depend explicitly on ω but are defined in terms of R, L, P, S and D (equations (2.3)) which do depend explicitly on ω.

Determining $\frac{\partial\mu}{\partial r}$ and $\frac{\partial\mu}{\partial\theta}$ was done numerically from first principles. If x is either r or θ then

$$\frac{\partial\mu}{\partial x} = \frac{\mu\left(x + \frac{\delta x}{2}\right) - \mu\left(x - \frac{\delta x}{2}\right)}{\delta x} \tag{2.16}$$

where all variables other than x are kept constant. To test this numerical differentiation, the derivative of μ w.r.t. ψ computed analytically was compared with the result obtained using the above numerical technique. The results obtained numerically were very close to those obtained analytically.

3.3.1 Centered Dipole Field

The centered dipole model assumes that the dipole field is centered at the earth's centre. It also assumes axial symmetry and hence there are only two field components, one in the r direction and one in the θ direction with θ measured with respect to the geomagnetic equator. The components of the field at a point r,θ are

$$B_r = -2B_o\frac{r_e^3}{r^3}\sin\theta \tag{3.13}$$

$$B_\theta = B_o\frac{r_e^3}{r^3}\cos\theta$$

where r_e is the earth's radius and B_o is the magnitude of the field on the earth's surface at the geomagnetic equator. The resultant field strength is then given by

$$B = B_o\frac{r_e^3}{r^3}\sqrt{4 - 3\cos^2\theta} \tag{3.14}$$

The equation of a field line, in polar co-ordinates, is

$$\begin{aligned} r &= r_o \cos^2\theta \\ \varphi &= const \end{aligned} \tag{3.15}$$

where r_o is the distance from the centre of the earth at which the field line intersects the equatorial plane. Field lines are generally described by a dimensionless parameter, called the $L - value$. This is the maximum distance, in earth radii, of the field line from the earth's centre and is given by

$$L = \frac{r_o}{r_e} \tag{3.16}$$

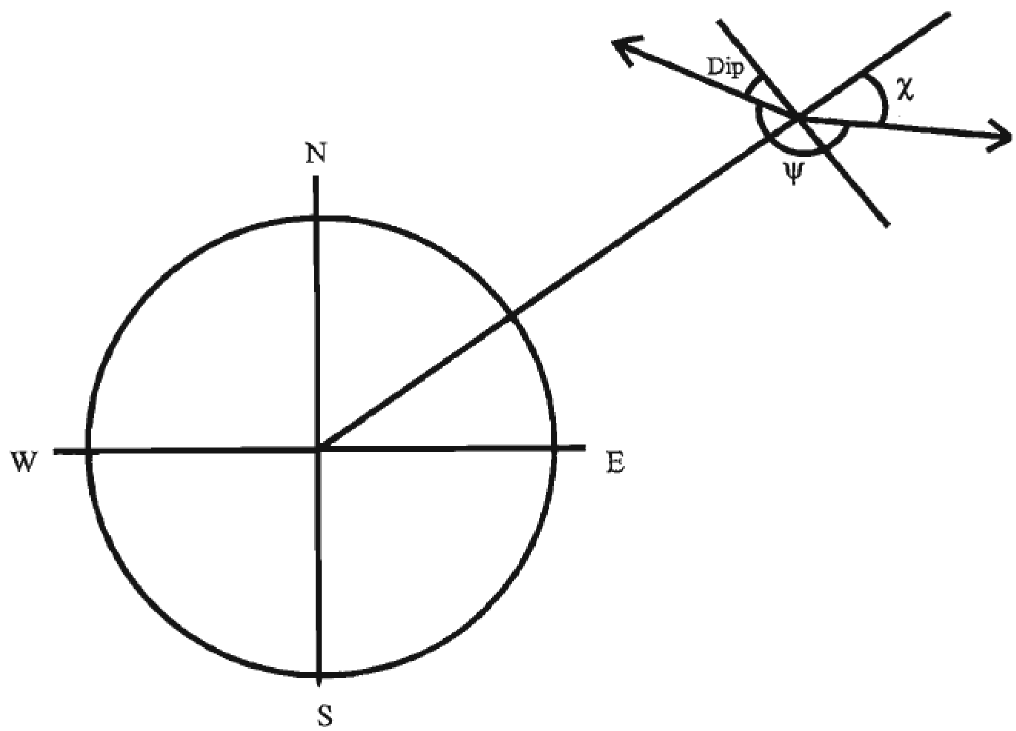

Figure 3.9: Geometry of the wave normal and magnetic dip angle.

Using equation (3.15) and (3.16), equation (3.14) can be rewritten as

$$B = \frac{B_o}{L^3}\frac{(4 - 3\cos^2\theta)}{\cos^6\theta} \tag{3.17}$$

To determine the phase refractive index, the angle between the wave normal and the magnetic field direction is needed. The wave normal, χ, used in the ray tracing equations is the measured relative to the radial direction. The magnetic field direction is given in terms of the dip angle. This is the angle between the magnetic field direction and the horizontal ($r = constant$) and is given by $\arctan(2\tan\theta)$. These angles are illustrated in figure 3.9.

As is shown in the figure, the angle (ψ) between the wave normal and the magnetic field is $\frac{3\pi}{2} - Dip - \chi$.

$$\frac{dr}{dt} = \frac{1}{\mu^2}\left(\mu\cos\chi + \frac{\partial\mu}{\partial\chi}\sin\chi\right)$$
$$\frac{d\theta}{dt} = \frac{1}{r\mu^2}\left(\mu\sin\chi - \frac{\partial\mu}{\partial\chi}\cos\chi\right)$$
$$\frac{d\chi}{dt} = \frac{1}{r\mu^2}\left(\frac{\partial\mu}{\partial\theta}\cos\chi - \left[r\frac{\partial\mu}{\partial r} + \mu\right]\sin\chi\right)$$

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$2\tan\Psi = \tan\theta$

$$N = 180{,}000 \exp\left[-4.183119\ (r - 1.0471)\right]$$

$\tan\Psi = \frac{1}{2}\tan\theta$

$$B = B_o\frac{r_e^3}{r^3}\sqrt{4 - 3\cos^2\theta}$$

$\tan dip = 2\tan\theta$

Dip = $\arctan(2\tan\theta)$.

$X = R\cos\theta$
$Y = R\sin\theta$

Psi = $\frac{3\pi}{2} - Dip - \chi$

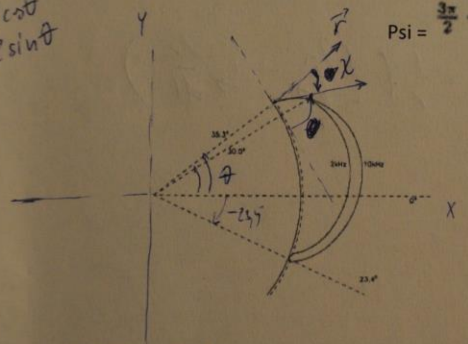
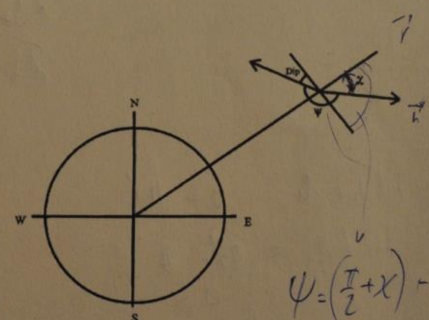


Figure 4.10: Ray paths of 2 kHz and 10 kHz signals starting at a latitude of −23.4° and 32.3° and reaching a satellite at an altitude of 1400 km and latitude of 30°.



$\psi = \left(\frac{\pi}{2} + \chi\right) - (\pi - dip)$

$= \frac{3}{2}\pi - \chi - dip$