Rotation Averaging and Strong Duality

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Abstract

In this paper we explore the role of duality principles within the problem of rotation averaging, a fundamental task in a wide range of computer vision applications. In its conventional form, rotation averaging is stated as a minimization over multiple rotation constraints. As these constraints are non-convex, this problem is generally considered challenging to solve globally. We show how to circumvent this difficulty through the use of Lagrangian duality. While such an approach is well-known it is normally not guaranteed to provide a tight relaxation. Based on spectral graph theory, we analytically prove that in many cases there is no duality gap unless the noise levels are severe. This allows us to obtain certifiably global solutions to a class of important non-convex problems in polynomial time.

We also propose an efficient, scalable algorithm that out-performs general purpose numerical solvers and is able to handle the large problem instances commonly occurring in structure from motion settings. The potential of this proposed method is demonstrated on a number of different problems, consisting of both synthetic and real-world data.

Rotation Averaging

Given estimates of the relative rotations between the cameras for many pairs of images, how can we assign camera rotations for each image in a way that is most consistent with the data?

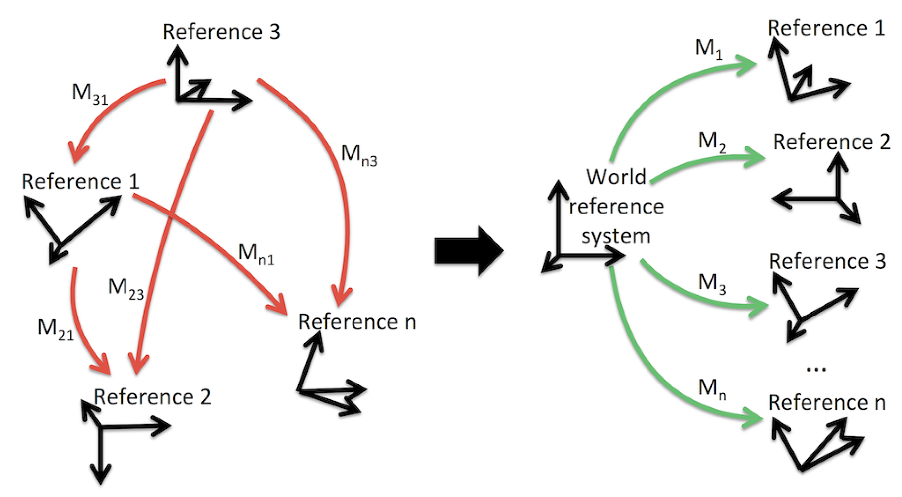

Figure 1: Determine absolute camera rotations from pairwise relative measurements.

Under ideal conditions:

$$R_i R_{ij} = R_j, \quad \forall (i,j) \in \mathcal{N}$$

In the presence of noise, typically solved in a least-metric sense:

$$\underset{R_1,\ldots,R_n}{\arg\min} \sum_{(i,j)\in\mathcal{N}} d(R_i \tilde{R}_{ij}, R_j)^p$$

Here we use the *chordal distance*

$$(P) \quad \underset{R_i \in \mathrm{SO}(3)}{\arg\min} \sum_{(i,j)\in\mathcal{N}} \|R_i \tilde{R}_{ij} - R_j\|_F^2,$$

Which constitutes our (non-convex) *primal problem.*

Dual Problem

$$\begin{array}{ll} \min & -\mathrm{tr}\left(R\tilde{R}R^T\right) \\ \text{s.t.} & R_i^T R_i = I \end{array} \quad (P)$$

$$R^T = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}, \ \tilde{R} = \begin{bmatrix} 0 & a_{12}\tilde{R}_{12} & \ldots & a_{1n}\tilde{R}_{1n} \\ a_{21}\tilde{R}_{21} & 0 & \ldots & a_{2n}\tilde{R}_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1}\tilde{R}_{n1} & a_{n2}\tilde{R}_{n2} & \ldots & 0 \end{bmatrix}, \ \Lambda = \begin{bmatrix} \Lambda_1 & 0 & 0 & \ldots \\ 0 & \Lambda_2 & 0 & \ldots \\ 0 & 0 & \Lambda_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Lagrangian:

$$L(R,\Lambda) = \\ -\mathrm{tr}\left(R\tilde{R}R^T\right) - \mathrm{tr}\left(\Lambda(I - R^T R)\right) = \mathrm{tr}\left(R(\Lambda - \tilde{R})R^T\right) - \mathrm{tr}(\Lambda)$$

$$\max_{\Lambda}\min_{R} L(R,\Lambda) = \max_{\Lambda}\min_{R} \mathrm{tr}\left(R(\Lambda-\tilde{R})R^T\right) - \mathrm{tr}(\Lambda) \Rightarrow$$
$$\Rightarrow \boxed{\max_{\Lambda - \tilde{R} \succeq 0} -\mathrm{tr}(\Lambda) \quad (D)}$$

Weak & Strong Duality:

$$\min_{R\in\mathbf{O}(3)^n} -\mathrm{tr}\left(R\tilde{R}R^T\right) \quad (P) \quad \geq \quad \max_{\Lambda-\tilde{R}\succeq 0} -\mathrm{tr}(\Lambda) \quad (D)$$

- (P) non-convex problem, difficult to solve.
- (D) convex problem, easy to solve.
- Weak Duality, (P) ≥ (D) always holds.
- **Strong Duality**, (P) = (D) sometimes hold.

Observation: For Rotation Averaging, Strong Duality typically holds $(P = D)$ if noise levels are not too severe.

Main Result

Sufficient Conditions for Strong Duality

Theorem 1: Strong Duality

et R_i^*, $i = 1, \ldots, n$ denote a stationary point to the primal problem (P) for a connected camera graph G with Laplacian L_G. Let α_{ij} denote the angular residuals, i.e., $\alpha_{ij} = \angle(R_i^*\tilde{R}_{ij}, R_j^*)$. Then R_i^*, $i = 1, \ldots, n$ will be globally optimal and strong duality will hold for (P) if

$$|\alpha_{ij}| \leq \alpha_{\max} \quad \forall (i,j) \in E,$$

where

$$\boxed{\alpha_{\max} = 2\arcsin\left(\sqrt{\frac{1}{4} + \frac{\lambda_2(L_G)}{2d_{\max}}} - \frac{1}{2}\right)},$$

and $d_{\max}$ is the maximal vertex degree[a].

[a]For fully connected camera graphs: $42.9° \leq a_{\max} \leq 60°$.

Outline of Proof

1. Necessary conditions for local optima :

(Stationarity)	$(\Lambda^* - \tilde{R})R^{*T} = 0$
(Primal feasibility)	$R^* \in \mathrm{O}(3)^n$.

From (4a) we get

$$\Lambda_i^* R_i^{*T} = \sum_{j\neq i} \tilde{R}_{ij} R_j^{*T} \iff \boxed{\Lambda_i^* = \sum_{j\neq i} \tilde{R}_{ij} R_j^{*T} R_i^*}$$

2.

Lemma: 3.2

If a stationary point R^* with corresponding Lagrangian multiplier Λ^* fulfills $\Lambda^* - \tilde{R} \succeq 0$ then:

1. There is no duality gap between (P) and (D).
2. R^* is a global minimum for (P).

Proof:

1. Λ^* is feasible in (D). From (5) $\left(\Lambda_i^* = \sum_{j\neq i} \tilde{R}_{ij} R_j^{*T} R_i^*\right)$ we obtain $-\mathrm{tr}(\Lambda^*) = -\mathrm{tr}\left(R^*\tilde{R}R^{*T}\right) \Rightarrow D^* = P^*$.
2. By definition of psd, $x^T(\Lambda^* - \tilde{R})x \geq 0$,

$$0 \leq \mathrm{tr}\left(R(\Lambda^* - \tilde{R})R^T\right) = \mathrm{tr}(\Lambda^*) - \mathrm{tr}\left(R\tilde{R}R^T\right) = \mathrm{tr}\left(R^*\Lambda^*R^{*T}\right) - \mathrm{tr}\left(R\tilde{R}R^T\right)$$

3. $$\Lambda^* - \tilde{R} \succeq 0 \iff D_{R^*}(\Lambda^* - \tilde{R})D_{R^*}^T + \mu NN^T \succeq 0$$

$$D_{R^*}(\Lambda^* - \tilde{R})D_{R^*}^T = \\ = \begin{bmatrix} \sum_{j\neq 1} a_{1j}\mathcal{E}_{1j} & -a_{12}\mathcal{E}_{12} & -a_{13}\mathcal{E}_{13} & \ldots \\ -a_{12}\mathcal{E}_{12}^T & \sum_{j\neq 2} a_{2j}\mathcal{E}_{2j} & -a_{23}\mathcal{E}_{23} & \ldots \\ -a_{13}\mathcal{E}_{13}^T & -a_{23}\mathcal{E}_{23}^T & \sum_{j\neq 3} a_{3j}\mathcal{E}_{3j} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \succeq 0$$

with $\mathcal{R}_{ij} = R_i^*\tilde{R}_{ij}R_j^{*T}$

4. $$\Delta = D_{R^*}(\Lambda^* - \tilde{R})D_{R^*}^T - L_G \otimes I_3$$

Lemma: 4.1-4.3

$$|\lambda| \leq \sum_{j=1}^{n} \|\Delta_{ij}\| \quad \text{for some } i = 1,\ldots,n. \qquad \text{(Lemma 4.1)}$$

$$\|\Delta_{ii}\| \leq 2d_i \sin^2\left(\frac{\alpha_{\max}}{2}\right) \quad \forall i = 1,\ldots n, \qquad \text{(Lemma 4.2)}$$

$$\|\Delta_{ij}\| \leq 2a_{ij}\sin\left(\frac{\alpha_{\max}}{2}\right). \qquad \text{(Lemma 4.3)}$$

5. $$D_{R^*}(\Lambda^* - \tilde{R})D_{R^*}^T + \mu NN^T = L_G \otimes I_3 + \mu NN^T + \Delta$$

$$\lambda_1(D_{R^*}(\Lambda^* - \tilde{R})D_{R^*}^T + \mu NN^T) \geq \lambda_2(L_G) - |\lambda_{\max}(\Delta)|$$

Lemma: 4.4

The matrix $\Lambda^* - \tilde{R}$ is positive semidefinite if

$$\lambda_2(L_G) - 2d_{\max}\sin\left(\frac{\alpha_{\max}}{2}\right)\left(1 + \sin\left(\frac{\alpha_{\max}}{2}\right)\right) \geq 0.$$

Completing the squares and solving for $\alpha_{\max}$ gives Theorem 4.1.

Solving the Dual Problem

$$\min_{R\in\mathbf{O}(3)^n} -\mathrm{tr}\left(R\tilde{R}R^T\right) \quad (P) \quad \geq \quad \max_{\Lambda-\tilde{R}\succeq 0} -\mathrm{tr}(\Lambda) \quad (D)$$

$$\|$$

$$\min_{X_{ii}=1,\ X\succeq 0} \mathrm{tr}\left(\tilde{R}X\right) \quad (DD)$$

$$\begin{array}{ll} \min & \mathrm{tr}\left(\tilde{R}X\right) \\ \text{s.t.} & \begin{bmatrix} I & x_{12} & x_{13} & \ldots & x_{1n} \\ x_{21} & I & x_{23} & \ldots & x_{2n} \\ x_{31} & x_{32} & I & \ldots & x_{3n} \\ \vdots & & & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \ldots & I \end{bmatrix} \succeq 0, \end{array} \quad x_1 = -X_1\tilde{R}_1\left[\left(\tilde{R}_1^T X_1 \tilde{R}_1\right)^{\frac{1}{2}}\right]^{\dagger}$$

Experimental Results

Figure 2: Images and reconstructions of the datasets in Table 1.

Dataset	n	*time[s]* Alg. 1	*time[s]* SeDuMi	$\lvert\alpha_{ij}\rvert$	$\alpha_{\max}$
Gustavus	57	3.25	8.28	6.33°	8.89°
Sphinx	70	3.87	14.40	6.14°	12.13°
Alcatraz	133	12.73	117.19	7.68°	43.15°
Pumpkin	209	9.23	688.65	8.63°	3.59°
Buddha	322	16.71	1765.72	7.29°	14.01°

Table 1: The average run time and largest resulting angular residual ($|\alpha_{ij}|$) and bound ($\alpha_{\max}$) on five different real-world datasets.