Written Tasks
Mandatory Assignment 1 - MAT-MEK4270

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1.2.3 Exact solution

The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{1}$$

has the exact solution

$$u_e(t, x, y) = e^{i(k_x x + k_y y - \omega t)} \tag{2}$$

where $i = \sqrt{-1}$ is the imaginary unit, $k_\alpha = m_\alpha \pi,\ \alpha = (x, y)$ denotes the wave number in x and y, respectively, with m_α as an arbitrary integer. Here, ω, represents the wave dispersion coefficient.

To show that (2) satisfies (1), we can start calculating the different second derivatives of u_e as

$$\begin{aligned} \frac{\partial^2}{\partial t^2}(u_e) = \frac{\partial^2}{\partial t^2}(e^{i(k_x x + k_y y - \omega t)}) = \frac{\partial}{\partial t}(-i\omega\ e^{i(k_x x + k_y y - \omega t)}) = \\ = -\omega^2\ e^{i(k_x x + k_y y - \omega t)}) = -\omega^2\ u_e \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2}(u_e) = \frac{\partial^2}{\partial x^2}(e^{i(k_x x + k_y y - \omega t)}) = \frac{\partial}{\partial x}(ik_x\ e^{i(k_x x + k_y y - \omega t)}) = \\ = -k_x^2\ e^{i(k_x x + k_y y - \omega t)}) = -k_x^2\ u_e \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{\partial^2}{\partial y^2}(u_e) = \frac{\partial^2}{\partial y^2}(e^{i(k_x x + k_y y - \omega t)}) = \frac{\partial}{\partial y}(ik_y\ e^{i(k_x x + k_y y - \omega t)}) = \\ = -k_y^2\ e^{i(k_x x + k_y y - \omega t)}) = -k_y^2\ u_e \end{aligned} \tag{5}$$

where we use that $i^2 = -1$. Substituting these three into (1), we get

$$\frac{\partial^2 u_e}{\partial t^2} = c^2 \nabla^2 u_e\ \Rightarrow\ -\omega^2\ u_e = c^2(-k_x^2\ u_e - k_y^2\ u_e) = -c^2(k_x^2 + k_y^2)u_e \tag{6}$$

The general definition of the dispersion coefficient, ω, is

$$\omega = |\boldsymbol{k}|c, \text{ for } \boldsymbol{k} = k_x \boldsymbol{i} + k_y \boldsymbol{j}$$

Using this in the LHS of Eq.(6), gives

$$\begin{aligned} -\omega^2\ u_e = -(|\boldsymbol{k}|c)^2\ u_e = -c^2(|k_x \boldsymbol{i} + k_y \boldsymbol{j}|^2 = \\ = -c^2(k_x^2 + k_y^2)u_e \end{aligned} \tag{7}$$

making the LHS equal the RHS of Eq. (6) s.t. Eq. (2) satisfies Eq. (1).

1.2.4 Dispersion coefficient

If we assume $m_x = m_y$, s.t. $k_x = k_y = k$, we can write a discrete version of (2) as

$$u_{rs}^n = e^{i(kh(r+s)-\tilde{\omega}n\Delta t)} \tag{8}$$

where $x_r = rh,\ y_s = sh$, and $t = n\Delta t$ is the discrete versions of $x,\ y,\ t$, with $h, \Delta t$ is the step sizes in space and time. Similar to (2), $\tilde{\omega}$ is the numerical approximation of the exact wave dispersion coefficient, ω.

Using the property of addition for exponents, we can rewrite (8) as

$$u_{rs}^n = e^{i(kh(r+s)} \left(e^{-i(\tilde{\omega}\Delta t)}\right)^n \tag{9}$$

If we define $A = e^{-i(\tilde{\omega}\Delta t)}$, and $E(r,s) = e^{i(kh(r+s)}$, we can write this on a more compact for as

$$u_{rs}^n = A^n\, E(r,s) \tag{10}$$

A discrete version of (1) based on a central finite differences scheme reads as

$$\frac{u_{rs}^{n+1} - 2u_{rs}^n + u_{rs}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{r+1,s}^n - 2u_{r,s}^n + u_{r-1,s}^n}{h^2} + \frac{u_{r,s+1}^n - 2u_{r,s}^n + u_{r,s-1}^n}{h^2}\right) \tag{11}$$

Substituting (10) into the numerical stencil in (11), we get

$$\begin{aligned}\frac{(A^{n+1} - 2A^n + A^{n-1})E(r,s)}{\Delta t^2} = \frac{c^2 A^n}{h^2}\Big[&E(r+1,s) - 2E(r,s) + E(r-1,s)\dots \\ &\dots + E(r,s+1) - 2E(r,s) + E(r,s-1)\Big]\end{aligned} \tag{12}$$

In order to isolate A, we can multiply by some common elements, namely $\frac{\Delta t^2}{A^n E(r,s)}$, which yields

$$A - 2 + A^{-1} = \left(\frac{c\Delta t}{h}\right)^2 \left(\frac{E(r+1,s)}{E(r,s)} + \frac{E(r,s-1)}{E(r,s)} + \frac{E(r,s+1)}{E(r,s)} + \frac{E(r,s-1)}{E(r,s)} - 4\right) \tag{13}$$

as $\frac{A^{n+1}}{A^n} = A$ and $\frac{A^{n-1}}{A^n} = A^{-1}$.

This can be further simplified by recognizing

$$\frac{E(r+1,s)}{E(r,s)} = \frac{e^{ikh(r+1+s)}}{e^{ikh(r+s)}} = \frac{e^{ikh(r+s)}e^{ikh}}{e^{ikh(r+s)}} = e^{ikh} \tag{14}$$

$$\frac{E(r,s-1)}{E(r,s)} = \frac{e^{ikh(r+s-1)}}{e^{ikh(r+s)}} = \frac{e^{ikh(r+s)}e^{-ikh}}{e^{ikh(r+s)}} = e^{-ikh} \tag{15}$$

With similar relations for the instances with $E(r-1,s)$ and $E(r,s+1)$, we can write (13) as

$$A - 2 + A^{-1} = \left(\frac{c\Delta t}{h}\right)^2 \left(2e^{ikh} + 2e^{-ikh} - 4\right) \tag{16}$$

Lets denote $\frac{c\Delta t}{h} = C$. As $2\cos(x) = e^{ix} + e^{-ix}$, we can write

$$A - 2 + A^{-1} = C^2(4\cos(kh) - 4) \ \Rightarrow\ A + A^{-1} = 2 + 4C^2(\cos(kh) - 1) = \beta \tag{17}$$

For stability of the discrete solution we require $|A| < 1$. The equation in (17) implies that $|A| = 1$ if

$$-2 \leqslant \beta \leqslant 2 \tag{18}$$

so we get

$$-2 \leqslant 2 + 4C^2(\cos(kh) - 1) \leqslant 2 \;\Rightarrow\; -1 \leqslant C^2(\cos(kh) - 1) \leqslant 0 \tag{19}$$

Recognizing that the term $\cos(kh)$ will get its smallest, i.e. "worst outcome" for our inequality, for $\cos(kh) = -1$, for $kh = (2k + 1)\pi$, we get

$$-1 \leqslant -2C^2 \leqslant 0 \;\Rightarrow 2C^2 \leqslant 1 \;\Rightarrow C \leqslant \frac{1}{\sqrt{2}} \tag{20}$$

Going back to Eq. (17), substituting back for A^n, and rearranging, gives

$$e^{-i(\tilde{\omega}\Delta t)} + e^{i(\tilde{\omega}\Delta t)} = 4C^2\cos(kh) - 4C^2 + 2 \tag{21}$$

Substituting for $C^2 = 1/\sqrt{2}$, and writing $e^{-i(\tilde{\omega}\Delta t)} + e^{i(\tilde{\omega}\Delta t)} = 2\cos(\tilde{\omega}\Delta t)$, as before, we get

$$2\cos(\tilde{\omega}\Delta t) = 2\cos(kh) \tag{22}$$

Isolating $\tilde{\omega}$,

$$\tilde{\omega} = \frac{kh}{\Delta t} \tag{23}$$

With $\omega = kc$, and

$$C = \frac{c\Delta t}{h} \;\Rightarrow\; h = \frac{c\Delta t}{C} \tag{24}$$

this then becomes

$$\tilde{\omega} = \frac{k}{\Delta t}\frac{c\Delta t}{C} = \frac{\omega}{C} \tag{25}$$

This relation only becomes equal to each other for $C = 1$. We found/was given $C = 1/\sqrt{2}$, and I struggle to see how to do this differently to get rid of the Courant number here.