Analytic, geometric, and topological methods for Sobolev mappings to manifolds

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Contents

Abstract

This is a set of lecture notes written in support of a minicourse that I gave at ICMAT (Madrid) in November 2024. Hopefully, they provide more details on some technical parts of the minicourse that would only have been sketched at the blackboard. In addition, they contain some extra material, and most importantly, an extensive (although not always exhaustive) list of references. On the other hand, some explanations given verbally in the minicourse may not appear here, especially of course if they were initially not planned, for instance to answer a question or adapt to a remark.

In this minicourse, we will explore the theory of Sobolev mappings with values into a compact manifold. The study of such mappings is motivated by applications coming from geometry, physics, computer graphics, and numerical methods, and their study per se raises many beautiful and challenging problems. The objective is to give an overview of some of these problems, and to explain a few tools that have been developed to tackle them. The ideas will systematically be illustrated on insightful model cases, avoiding too much technicality. The only prerequisite is a relative comfort with functional analysis, especially classical Sobolev spaces. All concepts of topology and geometry that will be used in the course shall be duly reminded.

These notes may remain quite sketchy. For more detailed introductions or reviews of this topic, I refer the reader for instance to the monograph by H. Brezis and P. Mironescu [BM21] focusing mostly on the case of maps with values into the circle, with some excursions to more general targets; the lecture notes by J. Van Schaftingen [VS19]; and the lecture notes by P. Mironescu [Mir23]. Some parts of these notes may also bear similarities with my master thesis [Det22] (in French).

Concerning the sketchy aspect of these notes, sometimes I may have omitted to explain how to deal with some technical issues. However, I have tried to draw attention on them as often as possible, to warn the reader about possible common traps. Sometimes, I have left exercises between braces about filling gaps that have been left on purpose for the sake of conciseness.

That being said, I hope that these notes will be useful to any participant to the minicourse, and more generally to anyone looking for a soft introduction to this beautiful topic of Sobolev mappings to manifolds.

Lecture 1

The main problems concerning Sobolev mappings to manifolds

In this session, we will review the main problems that are raised by the study of Sobolev mappings with values into manifolds, such as density, extension of traces, and lifting. Other problems could be mentioned briefly if time allows. This will also be the opportunity to revise some fundamental concepts in topology that are of crucial importance to understand those problems.

1.1 Classical Sobolev spaces

Although this set of lectures assumes a basic knowledge of classical Sobolev spaces, let us provide a brief reminder of their definition. This is especially useful for fractional Sobolev spaces, since they admit several equivalent definitions. In addition, this provides a convenient starting point for the first problem that will be considered concerning Sobolev mappings. We provide essentially no reference for the material in this section, since most of it is classical and can be found in virtually any textbook about Sobolev spaces.

The motivation from studying Sobolev spaces comes from problems from partial differential equations or calculus of variations for instance, where it is useful to measure the size of a function and its derivatives *via* an integral norm. More precisely, it is natural to work with the following norm, where $k \in \mathbb{N}_*$, $1 \leq p < +\infty$, and Ω is a sufficiently smooth open subset of $\mathbb{R}^m$ (one may think Ω as being a ball or a cube):

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{j=0}^{k} \int_{\Omega} |\mathrm{D}^j u|^p\right)^{\frac{1}{p}}.$$

This defines a norm for instance on the space $C^\infty(\overline{\Omega})$ of functions that are smooth on Ω *up to the boundary*. Here, our precise definition of $C^\infty(\overline{\Omega})$ is that this space consists of the restrictions to Ω of functions in $C^\infty_{\mathrm{c}}(\mathbb{R}^m)$. We note importantly that, with this definition, we have $C^\infty(\overline{\mathbb{R}^m}) \neq C^\infty(\mathbb{R}^m)$.

However, the main problem here is that this norm does not endow $C^\infty(\overline{\Omega})$ with the structure of a complete space. This motivates the following definition.

Definition 1.1. *The space $W^{k,p}(\Omega)$ is the completion of $C^\infty(\overline{\Omega})$ with respect to the $W^{k,p}$ norm.*

This definition is not the usual definition of Sobolev spaces. In a first instance, it makes $W^{k,p}$ a set of *equivalence classes* of Cauchy sequences of smooth functions. However, using the completeness of L^p, one may show that to each equivalence class in $W^{k,p}(\Omega)$ corresponds a unique function $u \in L^p(\Omega)$, obtained as the common limit in L^p of all Cauchy sequences in the equivalence class. Moreover, working at the level of the derivatives, one may show that (assuming $k = 1$ for convenience) to each $u \in W^{1,p}(\Omega)$ corresponds a unique $g \in L^p(\Omega; \mathrm{Lin}(\mathbb{R}^m; \mathbb{R}))$, obtained as the common limit in L^p of the derivatives of all Cauchy sequences in the equivalence class, satisfying the integration by parts formula

$$\int_\Omega g\varphi = -\int_\Omega u \mathrm{D}\varphi \quad \text{for every } \varphi \in C^\infty_{\mathrm{c}}(\Omega).$$

(This is obtained by writing the corresponding formula for smooth maps and passing to the limit.) The function g is denoted $\mathrm{D}u$, and is actually the derivative in the sense of distributions of u. This way, we recover the usual definition of Sobolev spaces. (Actually, the above reasoning only shows that our definition yields a function space *included* in the classically defined Sobolev space. The converse inclusion follows from the *strong density theorem*.)

We now turn to the definition of Sobolev spaces of *fractional* order. Given $0 < \sigma < 1$, we define the *Gagliardo seminorm* of a measurable function $u \colon \Omega \to \mathbb{R}$ as

$$|u|_{W^{\sigma,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{m+\sigma p}} \, \mathrm{d}x \, \mathrm{d}y\right)^{\frac{1}{p}}.$$

The Sobolev space $W^{\sigma,p}(\Omega)$ is then defined as the space of all $L^p(\Omega)$ functions whose Gagliardo seminorm is finite, endowed with the norm defined through

$$\|u\|^p_{W^{\sigma,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + |u|^p_{W^{\sigma,p}(\Omega)}.$$

One may wonder why bother with fractional Sobolev spaces. Aside from providing a suitable scale of spaces lying "between" L^p and $W^{1,p}$, which is formalized through interpolation theory, they are also the correct framework for *trace theory*.

Indeed, assume that one is willing to make sense of the restriction of a Sobolev function on the boundary of the domain Ω, which is natural both in PDE and calculus of variations to account for boundary conditions. Since L^p functions are only defined almost everywhere and $\partial\Omega$ is a negligible set, it is not clear *a priori* how to make sense of $u_{|\partial\Omega}$. However, the additional regularity of $W^{1,p}$ saves the game: the boundary operator $u \in C^\infty(\overline{\Omega}) \mapsto u_{|\partial\Omega}$ can be shown to have an extension $W^{1,p}(\Omega) \to L^p(\partial\Omega)$, which is called the *trace operator*. However, this operator is not surjective: otherwise stated, not every L^p function on $\partial\Omega$ is the trace of a $W^{1,p}(\Omega)$ function. The right space to work with in the respect is the space $W^{1-1/p,p}(\partial\Omega)$. Indeed, it has been shown by E. Gagliardo [Gag57] that, if $p > 1$, then $\operatorname{tr}\colon W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega)$ is surjective, and moreover, it has a continuous linear right inverse, called the extension operator. If $p = 1$, then it was also shown by E. Gagliardo that the trace operator is surjective from $W^{1,1}(\Omega) \to L^1(\partial\Omega)$, and that it has a continuous right inverse; see also the proof by P. Mironescu [Mir15]. However, there is no continuous *linear* right inverse, as has been shown by J. Peetre [Pee79].

Just as we defined intermediate spaces between L^p and $W^{1,p}$, we may define intermediate spaces between $W^{k,p}$ and $W^{k+1,p}$. Given $0 < s < +\infty$ noninteger, we let $s = k + \sigma$ with $k \in \mathbb{N}$ and $0 < \sigma < 1$, and we define $W^{s,p}(\Omega)$ as the set of all $W^{k,p}(\Omega)$ functions u such that $\mathrm{D}^k u \in W^{\sigma,p}(\Omega)$, endowed with the norm defined by

$$\|u\|^p_{W^{s,p}(\Omega)} = \|u\|^p_{W^{k,p}(\Omega)} + |\mathrm{D}^k u|^p_{W^{\sigma,p}(\Omega)}\,,$$

where it is implicitly understood that $W^{k,p}$ is replaced by L^p in case $k = 0$.

1.2 Sobolev mappings to manifolds: definition and motivation

In this section, we introduce the main object of study of this minicourse, namely *Sobolev spaces of mappings with values into a manifold*. But before actually giving a definition, let us first proceed with some motivation.

Although Sobolev spaces of real-valued (or vector-valued) functions are the natural functional analytic framework for the study of numerous problems in partial differential equations or calculus of variations for instance, when dealing with questions coming from practical applications such as physics, it may be natural to impose geometric constraints to the values allowed for our maps.

The most classical example comes from the study of condensed matter physics, which has as an emblematic special case the modeling of liquid crystals. A field of liquid crystals can viewed as a liquid suspension of rod-like particles, having one preferred optic direction; see Figure 1.1. A naive idea to model such a field, located in a container $\Omega \subset \mathbb{R}^3$, would be to rely on a vector field $u\colon \Omega \to \mathbb{R}^3$, where the vector $u(x)$ would give the direction of the crystal at point x. However, this is not suitable for our purposes, since the only information we care about is the direction of the crystals (loosely speaking, all rods have the same length), while this modeling would also carry a length information. Therefore, it is natural to instead make use of maps $u\colon \Omega \to \mathbb{S}^2$, where $\mathbb{S}^2 \subset \mathbb{R}^3$ is the unit sphere, so that each point $x \in \Omega$ is associated with a *unit* vector $u(x)$, which therefore carries only a direction information. If one wishes to go even further and take into account the fact that the crystals are not oriented (that is, flipping the direction yields the same configuration, crystals are not arrows but rods with indistinguishable extremities), one may be willing to use instead maps $u\colon \Omega \to \mathbb{RP}^2$, where $\mathbb{RP}^2$ is the two-dimensional *projective plane*, obtained from the sphere by identifying all pairs of antipodal points, which exactly amounts to forget about orientation. More models from condensed matter physics include, but not only, supraconductivity, related to Ginzburg–Landau models, involving maps into the circle $\mathbb{S}^1$; biaxial liquid crystals, with two distinguished directions, involving maps into $\mathbb{S}^3/H$, where H is the group of quaternions; and also several phases of superfluid helium. We refer the reader to [BC07] and the references therein for a nice review of such applications.

Another application from physics is related with Cosserat materials in elasticity, involving maps with values into $\mathbb{R}^3 \times SO(3)$; see e.g. [ET58].

Mappings to manifolds are also related to problems in computer graphics, for instance when considering how to mesh a surface or a domain in order to apply a finite elements method. This comes from the fact that the attitudes of a square or a cube may be described by maps with values into $\mathbb{S}^1$ or $SO(3)$. We refer the reader to [HTWB11] and the *Hextreme* project, and to Figure 1.2 for an illustration.

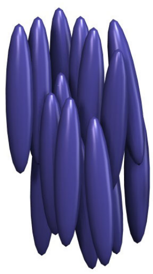

Figure 1.1: A field of nematic liquid crystals

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In all these examples, the target is a Riemannian manifold $\mathscr{N}$. In the sequel, except explicitly stated, we will assume $\mathscr{N}$ to be compact; this covers most of the previous examples, with the notable exception of Cosserat materials (in which the target is nevertheless a product of a compact manifold with a Euclidean space). It is not a loss of generality to assume moreover that $\mathscr{N} \subset \mathbb{R}^\nu$ for some ν sufficiently large, in view of the Nash isometric embedding theorem [Nas54, Nas56]. The Sobolev space of mappings with values into $\mathscr{N}$ is then defined as

$$W^{s,p}(\Omega;\mathscr{N}) = \{u \in W^{s,p}(\Omega;\mathbb{R}^\nu)\colon u(x) \in \mathscr{N} \text{ for almost every } x \in \Omega\}.$$

Figure 1.2: Meshing the earth (see the *Hextreme* project: www.hextreme.eu)

With this definition, the space $W^{s,p}(\Omega;\mathcal{N})$ is simply a subset of a classical Sobolev space of vector-valued functions, the manifold acting as a constraint. However, as we will see soon, the spaces of mappings may have a striking *qualitatively* different behavior compared with their classical counterparts.

We conclude with two comments. First, we note that we could as well have allowed for the domain to be a manifold as well. For instance, we could consider mappings from a sphere to a sphere. However, for the sake of simplicity, and to avoid additional technicality, we restrict ourselves to mappings defined on a sufficiently smooth bounded open subset of $\mathbb{R}^m$. Second, our definition is *extrinsic*, since it relies on the manifold living in some ambient space. Although it may be readily checked that, in the compact case, the definition does not depend on the choice of the embedding, up to identification *via* the transition map, it is a natural question to ask if an intrinsic definition is possible. This question has been studied by A. Convent and J. Van Schaftingen; see e.g. [Con17] and the references therein. This question is of importance especially since, in the *non-compact* case, the Sobolev space of mappings *may* depend on the choice of the embedding. However, we shall not explore these considerations here.

In the next three sections, we explore what can be described as the three main problems concerning Sobolev mappings to manifolds, namely the *density problem*, the *extension of traces problem*, and the *lifting problem*. (There are certainly plenty of other interesting questions in this topic, such as the *homotopy problem*, but we focus on these three. Actually, the next three lectures will especially focus on the first problem, the density problem, exploring several of its aspects.) For each problem, our main objective is to give a motivation for its study along with a strong intuition about why it is nontrivial. We will also take the first problem as an example to illustrate the fact that there is a crucial difference between the range $sp \geq m$, where Sobolev spaces of mappings essentially behave like their classical counterparts, and the range $sp < m$, where striking new phenomena occur and where the aforementioned problems become highly nontrivial.

1.3 The density problem

In Section 1.2, we defined the Sobolev space of mappings as a subset of a classical Sobolev space of vector-valued functions. However, it could as well have been natural to define it *via* the completion of smooth maps under the Sobolev norm, as we did in Section 1.1. That is, we could have defined the space

$$H^{s,p}_{\mathrm{S}}(\Omega;\mathscr{N}) = \overline{C^{\infty}(\overline{\Omega};\mathscr{N})}^{W^{s,p}}.$$

The strong density theorem says that, for classical Sobolev spaces, we have $H^{s,p}_{\mathrm{S}}(\Omega;\mathbb{R}) = W^{s,p}(\Omega;\mathbb{R})$. However, it is not clear that the same holds true for spaces of mappings. Indeed, given $u \in W^{s,p}(\Omega;\mathscr{N})$, the standard regularization by convolution procedure (applied component by component) merely yields a sequence of smooth maps $(u_n)_{n\in\mathbb{N}}$ in $C^{\infty}(\overline{\Omega};\mathbb{R}^{\nu})$ that converges strongly to u, but there is no reason for which the convolution should preserve the constraint that the maps are valued into $\mathscr{N}$.

It was first observed by R. Schoen and K. Uhlenbeck [SU83, Section 4, Example] that smooth maps *need not* be dense in the corresponding Sobolev space of mappings. More precisely, they constructed the following example, that we present in detail. The notions of topology used in the proof will be explained shortly after the example.

Example 1.2. We show that, if $2 \leq p < 3$, then $H^{1,p}_{\mathrm{S}}(\mathbb{B}^3;\mathbb{S}^2) \neq W^{1,p}(\mathbb{B}^3;\mathbb{S}^2)$. Actually, our argument only applies to $2 < p < 3$, but it can be adapted to cover the case $p = 2$ with tools that will be explained later on.

We define $u_0 \in W^{1,p}(\mathbb{B}^3;\mathbb{S}^2)$, sometimes called the *hedgehog map*, by

$$u_0(x) = \frac{x}{|x|},$$

and we claim that there is no sequence $(u_n)_{n\in\mathbb{N}_*}$ in $C^{\infty}(\overline{\mathbb{B}^3};\mathbb{S}^2)$ that converges strongly to u_0 in $W^{1,p}$. Assume by contradiction that this is the case. By a genericity argument, up to extraction of a subsequence, for almost every $r \in (0,1)$, we have $u_{n|\partial B^3_r} \to u_{0|\partial B^3_r}$ in $W^{1,p}$. But, since $p > 2$, the Morrey–Sobolev embedding implies that $u_{n|\partial B^3_r} \to u_{0|\partial B^3_r}$ uniformly. However, the map $u_{n|\partial B^3_r}$ being by definition the restriction to ∂B^3_r of a continuous map on the whole ball $\overline{B^3_r}$, it is homotopic to a constant. On the other hand, by construction, the map $u_{0|\partial B^3_r}$ is a degree 1 map, and hence not homotopic to a constant. Since homotopy classes are stable through uniform convergence, this is a contradiction, and finishes the proof.

Since it is the first time that we rely on a genericity argument, let us provide a complete proof of our claim. Although in principle, it is just a Fubini–Tonelli-type argument, we will see that a complete proof requires a bit more care. In the sequel, we will leave the task of justifying rigorously the genericity arguments to the reader.

We want to prove that, up to extraction of a subsequence, $u_{n|\partial B^3_r} \to u_{0|\partial B^3_r}$ in $W^{1,p}$ for almost every $r \in (0,1)$. For the sake of concision, we only prove the convergence of the derivatives, the convergence of the functions themselves being similar. The extraction of a subsequence serves to assume that

$$\sum_{n\in\mathbb{N}_*} \int_{\mathbb{B}^3} |\mathrm{D}u_n - \mathrm{D}u_0|^p < +\infty.$$

The monotone convergence theorem and Tonelli's theorem imply that

$$\int_0^1 \left(\int_{\partial B^3_r} \sum_{n\in\mathbb{N}_*} |\mathrm{D}u_n - \mathrm{D}u_0|^p \right) \mathrm{d}r < +\infty.$$

This implies that, for almost every $r \in (0,1)$, we have

$$\int_{\partial B^3_r} \sum_{n\in\mathbb{N}_*} |\mathrm{D}u_n - \mathrm{D}u_0|^p < +\infty.$$

Another application of the monotone convergence theorem proves that then

$$\sum_{n\in\mathbb{N}_*} \int_{\partial B^3_r} |\mathrm{D}u_n - \mathrm{D}u_0|^p < +\infty,$$

from which the result follows by the fact that a summable sequence converges to 0.

As the proof may suggest, this genericity argument may be viewed, in a more abstract fashion, as an instance of the partial converse of the dominated convergence theorem applied to $L^p((0,1); W^{1,p}(\partial B^3_r))$. □

We observe that, in the previous example, the obstruction to strong convergence is of purely *topological* nature. It is due to the presence of a point singularity in the map u_0, sufficienlty mild so that u_0 is $W^{1,p}$, but sufficiently strong so that, around the singularity, u_0 realizes copies of the identity map on $\mathbb{S}^2$. The key feature behind this obstruction is the fact that $\mathrm{id}_{\mathbb{S}^2}$ is not homotopic to a constant.

We briefly recall the basic concepts of homotopy theory that will be useful to us in this discussion. Given topological spaces X and Y (in our applications, $Y = \mathcal{N}$ is a compact Riemannian manifold and X is most of the time a sphere), we say that two continuous maps $f, g\colon X \to Y$ are *homotopic* whenever there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H_0 = f$ and $H_1 = g$, where $H_t = H(\cdot, t)$. In this case, we write $f \sim g$. It is readily checked that being homotopic is an equivalence relation.

Intuitively speaking, this means that we can deform continuously f into g. The *k-th order homotopy group of Y*, denoted by $\pi_k(Y)$, is the quotient of the set of all continuous mappings $f\colon \mathbb{S}^k \to Y$ by this equivalence relation. It is called a homotopy *group* because it can be endowed with the structure of a group, but we will only use the underlying set in this minicourse. In particular, $\pi_k(Y) \simeq \{0\}$ if and only if every map $f\colon \mathbb{S}^k \to Y$ is homotopic to a constant map. This is also equivalent to the fact that every continuous map $f\colon \mathbb{S}^k \to Y$ has a continuous extension $g\colon \overline{\mathbb{B}^{k+1}} \to Y$. Indeed, to a homotopy H between a constant map f and corresponds an extension g of f defined in polar coordinates by $g(r,\sigma) = H_r(\sigma)$, and vice-versa.

In our example above, we used the fact that homotopy classes are stable through uniform convergence. This is a direct consequence of the following proposition.

Proposition 1.3. *There exists $\delta > 0$ depending only on $\mathcal{N}$ such that, if $f, g\colon X \to \mathcal{N}$ are such that $\|f - g\|_{L^\infty} \leq \delta$, then $f \sim g$.*

Proof. Let $\iota > 0$ be such that there exists a smooth retraction $\Pi\colon \mathcal{N}_\iota \to \mathcal{N}$, see Proposition 1.4 below. Let $0 < \delta < \iota$. A homotopy between f and g can be obtained by letting

$$H_t = \Pi((1-t)f + tg). \qquad \square$$

Proposition 1.3 relies on the existence of a smooth retraction onto $\mathcal{N}$, defined on a uniform neighborhood of $\mathcal{N}$, whose existence is granted by Proposition 1.4 below. Such a result is classical in differential geometry, but a proof seems difficult to find in the literature. We refer the reader e.g. to [Foo84].

Proposition 1.4. *If $\mathcal{N}$ is a smooth compact Riemannian manifold embedded into $\mathbb{R}^\nu$, then there exists $\iota > 0$ and a smooth map $\Pi\colon \mathcal{N}_\iota = \mathcal{N} + B_\iota \to \mathcal{N}$ such that $\Pi_{|\mathcal{N}} = \mathrm{id}_{\mathcal{N}}$.*

Due to the use of Proposition 1.4 above, our proof of Proposition 1.3 is limited to compact manifolds. Actually, it may indeed fail in general metric spaces. (Can you find a counterexample?)

With this background, we see that the key topological feature used in Example 1.2 is the fact that $\mathrm{id}_{\mathbb{S}^2}$ is not homotopic to a constant. That is, if $\mathcal{N}$ is a target manifold such that $\pi_k(\mathcal{N}) \not\simeq \{0\}$, taking $f\colon \mathbb{S}^k \to \mathcal{N}$ that is not homotopic to a constant, we obtain a similar obstruction by considering the map $u_0\colon \mathbb{B}^{k+1} \to \mathcal{N}$ defined by

$$u_0(x) = f\Big(\frac{x}{|x|}\Big).$$

This yields the following theorem, due to F. Bethuel and Zheng X. [BZ88] for $W^{1,p}$ and M. Escobedo [Esc88] in the general case.

Theorem 1.5. *Assume that $sp < m$. Then, $H^{s,p}_{\mathrm{S}}(\Omega;\mathscr{N}) = W^{s,p}(\Omega;\mathscr{N})$ implies that $\pi_{\lfloor sp \rfloor}(\mathscr{N}) \simeq \{0\}$, where $\lfloor sp \rfloor$ is the integer part of sp.*

The condition $sp < m$ is necessary to be able to construct a counterexample of the type $x/|x|$ in the space $W^{s,p}$. Actually, when $sp \geq m$, the spaces of Sobolev mappings behave essentially like their classical counterparts involving real-valued functions. In particular, strong density always holds, as was first observed by by R. Schoen and K. Uhlenbeck [SU83], and later on clarified by the work of H. Brezis and L. Nirenberg [BN95] in connection with functions of vanishing mean oscillation (VMO).

Theorem 1.6. *If $sp \geq m$, then $H^{s,p}_{\mathrm{S}}(\Omega;\mathscr{N}) = W^{s,p}(\Omega;\mathscr{N})$ regardless of Ω and $\mathscr{N}$.*

The proof relies on a beautiful averaging argument allowing to estimate the distance to the target when performing a regularization by convolution process, and we present it for the sake of illustration. In the presentation of the next two main problems, we shall omit the explanation of why the case $sp \geq m$ also boils down to the situation for classical Sobolev spaces. Most often, the key ideas are the same as here for strong density. The proof makes use of the fact that $W^{s,p} \hookrightarrow \mathrm{VMO}$ when $sp \geq m$. This is actually the only reason for which this condition is needed.

Proof. Let $\rho \in C^\infty_{\mathrm{c}}(\mathbb{B}^m)$ be a standard mollifying kernel, and ρ_t the associated family of mollifiers, defined by

$$\rho_t(x) = \frac{1}{t^m}\rho(x/t).$$

Given $u \in W^{s,p}(\Omega;\mathscr{N})$, we define $u_t = \rho_t * u \in C^\infty(\overline{\Omega};\mathbb{R}^\nu)$, leaving aside the technical fact that this expression does not make sense in a neighborhood of $\partial\Omega$, which is easily fixed by an extension by dilation argument. (That is, we assume here that u is defined on a slightly larger open set containing $\overline{\Omega}$.)

Of course, u_t being a kind of average of u, it need not be $\mathscr{N}$-valued. We wish to estimate how far from $\mathscr{N}$ it can go. Since u is valued into $\mathscr{N}$, we have $\operatorname{dist}(u_t(x),\mathscr{N}) \leq |u_t(x) - u(y)|$ for every $x \in \Omega$ and almost every $y \in \Omega$. We take the average over all $y \in B_t(x)$:

$$\operatorname{dist}(u_t(x),\mathscr{N}) \leq \fint_{B_t(x)} |u_t(x) - u(y)| \,\mathrm{d}y.$$

Using the definition of the convolution product, we readily find

$$\fint_{B_t(x)} |u_t(x) - u(y)| \, \mathrm{d}y \leq \fint_{B_t(x)} \left(\int_{\mathbb{R}^m} |u(z) - u(y)| \rho_t(x - z) \, \mathrm{d}z \right) \mathrm{d}y.$$

Since ρ_t is supported in B_t and satisfies $\rho_t \lesssim t^{-m}$, we find

$$\fint_{B_t(x)} \left(\int_{\mathbb{R}^m} |u(z) - u(y)| \rho_t(x - z) \, \mathrm{d}z \right) \mathrm{d}y \lesssim \fint_{B_t(x)} \fint_{B_t(x)} |u(z) - u(y)| \, \mathrm{d}z \, \mathrm{d}x.$$

Since $u \in \mathrm{VMO}$, we conclude that the right-hand-side above converges to 0 as $t \to 0$, uniformly with respect to x. Therefore, for $t > 0$ sufficiently small, we have $u_t \in \mathscr{N}_\iota$, where $\iota > 0$ is such that a smooth retraction $\Pi \colon \mathscr{N}_\iota \to \mathscr{N}$ exists. We conclude by letting $v_t = \Pi \circ u_t \in C^\infty(\overline{\Omega}; \mathscr{N})$, which satisfies $v_t \to \Pi \circ u = u$ in $W^{s,p}$ as $t \to 0$. □

There is a subtle but important gap in the above proof: we need to be sure that it is legitimate to deduce that $v_t \to \Pi \circ u = u$ from the fact that $u_t \to u$. For $W^{1,p}$ for instance, this is rather straightforward. However, the general case, especially $s > 1$ noninteger, is far from being trivial. This is called the *continuity of the composition operator*. We refer the reader to the work of H. Brezis and P. Mironescu [BM01], who established the continuity of the composition operator $W^{s,p} \cap W^{1,sp} \to W^{s,p}$. Here, we only need the continuity of the composition operator $W^{s,p} \cap L^\infty \to W^{s,p}$, which is less difficult; see e.g. the historical section in [BM01] for an extensive list of references. We also refer to the more elementary proof by V. Maz'ya and T. Shaposhnikova [MS02]; see also [BPVS13]. We do not pretend to give a detailed history of this problem nor an exhaustive list of references in these notes.

The strong density problem was the second one among the three presented in this first lecture to receive a complete solution, that is, a characterization of those Ω, $\mathscr{N}$, s, and p such that $H^{s,p}_{\mathrm{S}}(\Omega; \mathscr{N}) = W^{s,p}(\Omega; \mathscr{N})$. Since strong density will be the main focus of the next lecture, unlike we do for the next two problems, we skip the review of the state of the art and references, and postpone it to Lecture 2.

1.4 The extension problem

As a motivation for the introduction of fractional spaces in Section 1.1, we mentioned Gagliardo's theorem, according to which there is a well-defined trace operator defined on $W^{1,p}(\Omega)$, which extends the usual restriction operator on the boundary, and whose image is exactly $W^{1-1/p,p}(\partial\Omega)$. As for the question of strong density, it is natural to wonder whether or not every $W^{1-1/p,p}(\partial\Omega; \mathscr{N})$ map is actually the trace of a $W^{1,p}$ map on Ω which takes its values into $\mathscr{N}$. As a first remark, let us note that the trace of an $\mathscr{N}$-valued map is indeed an $\mathscr{N}$-valued map. (This is not as trivial as it may seem, since being $\mathscr{N}$-valued is only an a.e. constraint. Can you give a rigorous proof?)

As we explained, this question is of importance in problems of PDE and calculus of variations, when dealing with prescribed values on the boundary. For instance, when studying minimizing harmonic maps, which minimize the Dirichlet energy under a sphere constraint, it is related to the question of the nonemptyness of the set of possible competitors for the minimization problem.

Just as for strong density, it turns out that obstructions to the extension of traces *may* arise, and they again come from the topology of the target, *via* a similar mechanism. This was first observed by R. Hardt and Lin F. [HL87], and we present their example in detail.

Example 1.7. Let us consider again $u_0 \colon \mathbb{B}^2 \to \mathbb{S}^1$ defined by

$$u_0(x) = \frac{x}{|x|}.$$

We already know that $u_0 \in L^\infty \cap W^{1,q}$ for every $1 \leq q < 2$. Therefore, the fractional Gagliardo–Nirenberg inequality, see e.g. [BM18] and the references therein, implies that $u_0 \in W^{\theta, q/\theta}$ for any $\theta \in (0,1)$ as soon as $1 < q < 2$ (the case $q = 1$ has to be excluded). Taking $\theta = 1 - \frac{1}{p}$ and $q = p - 1$ with $2 < p < 3$ shows that $u_0 \in W^{1-1/p,p}$. A direct proof can be found in [VS19, Lemma 3.7], covering the limiting case $p = 2$ (which actually follows from the previous discussion and the fact that we are on a bounded set).

Let us prove that there is no map $U \in W^{1,p}(\Omega; \mathbb{S}^1)$ such that $\operatorname{tr}_{\mathbb{B}^2} U = u$, where $\Omega = \mathbb{B}^2 \times (0,1)$. Assume by contradiction that such a map exists. By a genericity argument, for almost every $r \in (0,1)$, we have $U_{|S_r} \in W^{1,p}(S_r; \mathbb{S}^1)$ and $\operatorname{tr}_{B_r} U_{|S_r} = u_{|\partial B_r}$, where $S_r = B^3_r \cap \Omega$. (Can you give a rigorous proof of this statement?) But then, the Morrey–Sobolev embedding implies that U_{S_r} is continuous up to the boundary. Therefore, $u_{|\partial B_r}$ is homotopic to a constant, since it can be extended inside ∂B_r, relying on the fact that B_r and S_r are homeomorphic. This is a contradiction, and completes the proof. □

Once again, the obstruction in this example arises from the fact that the $(\lfloor p \rfloor - 1)$-th homotopy group of the target is nontrivial. This leads to the following necessary condition for the extension of traces, whose general form is due to F. Bethuel and F. Demengel [BD95].

Theorem 1.8. *Assume that $2 \leq p < m$. If every map in $W^{1-1/p,p}(\partial\Omega; \mathscr{N})$ is the trace of a map in $W^{1,p}(\Omega; \mathscr{N})$, then $\pi_{\lfloor p \rfloor - 1}(\mathscr{N}) \simeq \{0\}$.*

The extension problem was the last of the three problems covered in this first lecture to receive a complete solution. Let us list here a brief (non exhaustive) history of the problem.

- In [HL87], R. Hardt and Lin F. have shown the topological obstruction presented in Example 1.7. They also proved that every $W^{1-1/p}$ mapping is the trace of a $W^{1,p}$ mapping if $\pi_1(\mathscr{N}) \simeq \cdots \simeq \pi_{\lfloor p \rfloor - 1}(\mathscr{N}) \simeq \{0\}$, relying on the *method of projection*, adapting a technique devised by H. Federer and W. Fleming for the study of currents in geometric measure theory.
- In [BD95], F. Bethuel and F. Demengel have obtained the general case of the topological obstruction, Theorem 1.8. They also showed that *global* topological obstructions, due to the interplay between the topology of the domain and the target, may arise.
- In [Bet14], F. Bethuel showed that the first $\lfloor p \rfloor - 1$ homotopy groups of $\mathscr{N}$ *must* be *finite* in order to ensure that any mapping on the boundary is the trace of some mapping inside the domain. If one of these groups is infinite, then an *analytical* obstruction may occur, raised by the presence of a family of (smooth) maps whose minimal extension energy grows superlinearly with respect to their Sobolev energy.
- In [MVS21b], P. Mironescu and J. Van Schaftingen showed that an additional analytical obstruction may appear if p is an integer and $\pi_{p-1}(\mathscr{N}) \not\simeq \{0\}$, in addition to the already known topological obstruction. They also extended Hardt and Lin's positive result to the case where π_1 is allowed to be nontrivial (but finite), by a lifting argument.
- In [VS24], J. Van Schaftingen gave the complete answer to the extension problem, by showing that the already known obstructions are the only ones: if $\pi_{\lfloor p \rfloor - 1}(\mathscr{N}) \simeq \{0\}$ and if $\pi_1(\mathscr{N}), \dots, \pi_{\lfloor p \rfloor - 2}(\mathscr{N})$ are finite, then the extension of traces is always possible.

1.5 The lifting problem

The last problem we explore in this first lecture is the *lifting problem*. To motivate its study, assume that we wish to prove strong density of smooth maps in the space of $W^{1,p}$ mappings with values into the circle. In this special case, we can take profit of the fact that maps into the circle have a phase: a map $u\colon \mathbb{B}^m \to \mathbb{S}^1$ can be written as $u = \mathrm{e}^{\mathrm{i}\theta}$ for some function $\theta\colon \mathbb{B}^m \to \mathbb{R}$. Then, it suffices to apply the classical density theorem to approximate θ by a sequence of smooth functions θ_n, and defining $u_n = \mathrm{e}^{\mathrm{i}\theta_n}$ would provide the desired approximation of u by sphere-valued maps.

However, this sketchy argument is incomplete: we need to prove that Sobolev maps with values into the circle have a Sobolev phase.

For readers familiar with topology, this problem calls for a more general one: given a manifold $\mathscr{N}$ and a *covering* $\pi \colon \widetilde{\mathscr{N}} \to \mathscr{N}$, does any $\mathscr{N}$-valued Sobolev map have an $\widetilde{\mathscr{N}}$-valued Sobolev *lifting*?

To state the problem precisely, let us recall some basic notions of covering theory. We restrict ourselves to the material required in this lecture, for a much more complete exposition of this concept, we refer the reader e.g. to [Hat02]. We start by defining the notion of a *Riemannian covering*.

Definition 1.9. *We say that $\pi \colon \widetilde{\mathscr{N}} \to \mathscr{N}$ is a Riemannian covering whenever, for every $x \in \mathscr{N}$, there exists an open neighborhood $U \subset \mathscr{N}$ of x such that $\pi^{-1}(U)$ is a disjoint union of open sets on which π restricts to an isometry.*

Examples of Riemannian coverings include the circle, which is covered by $\mathbb{R}$ *via* the exponential map (and more general, the torus $\mathbb{T}^n$ is covered by $\mathbb{R}^n$, simply taking the product), and the projective plane $\mathbb{RP}^2$, which is covered by $\mathbb{S}^2$ *via* the natural quotient map (this example is of great importance for the study of liquid crystals, see e.g. [BZ11]).

An important feature of Riemannian coverings is that continuous maps into the base space $\mathscr{N}$ can be *lifted* to continuous maps into the covering.

Theorem 1.10. *If $\pi \colon \widetilde{\mathscr{N}} \to \mathscr{N}$ is a Riemannian covering, then any continuous map $u \colon \mathbb{B}^m \to \mathscr{N}$ admits a continuous lifting $\tilde{u} \colon \mathbb{B}^m \to \widetilde{\mathscr{N}}$, i.e., such that $u = \pi \circ \tilde{u}$.*

The situation is conveniently illustrated *via* the following commutative diagram.

$$\begin{tikzcd} & \widetilde{\mathscr{N}} \arrow[d, "\pi"] \\ \mathbb{B}^m \arrow[ur, dashed, "\exists \tilde{u}"] \arrow[r, "u"] & \mathscr{N} \end{tikzcd}$$

Hence, a natural question in the context of Sobolev mappings is the following: Does every $u \in W^{s,p}(\mathbb{B}^m; \mathscr{N})$ have a lifting $\tilde{u} \in W^{s,p}(\mathbb{B}^m; \widetilde{\mathscr{N}})$?

It is the right moment to summon our old friend, the Swiss-knife counterexample.

Example 1.11. We claim that, for $1 \leq p < 2$, the map $u_0 \in W^{1,p}(\mathbb{B}^2; \mathbb{S}^1)$ defined by

$$u_0(x) = \frac{x}{|x|}$$

has no lifting $\tilde{u} \in W^{1,p}(\mathbb{B}^2; \mathbb{R})$. Once again, we argue by contradiction: assume it has a lifting $\tilde{u}_0 \in W^{1,p}(\mathbb{B}^2; \mathbb{R})$. By a genericity argument, for a.e. $r \in (0,1)$, $u_{0|\partial B_r} \in W^{1,p}$ and has $\tilde{u}_{0|\partial B_r}$ as a lifting. This combined with the Morrey–Sobolev embedding contradicts the fact that $\mathrm{id}_{\mathbb{S}^1} \colon \mathbb{S}^1 \to \mathbb{S}^1$ has no continuous lifting, and proves our claim. $\square$

The previous example is due to J. Bourgain, H. Brezis, and P. Mironescu [BBM00]. It can be generalized to the following necessary condition for the existence of a lifting, due to F. Bethuel and D. Chiron [BC07].

Theorem 1.12. *If $0 < s < +\infty$ and $1 \le p < +\infty$ are such that $1 \le sp < 2$, then there exists a map $u \in W^{s,p}(\mathbb{B}^m; \mathcal{N})$ that has no lifting $\tilde{u} \in W^{s,p}(\mathbb{B}^m; \widetilde{\mathcal{N}})$. (Assuming $m \ge 2$ and that the covering is non-trivial.)*

The lifting problem was the first of the three problems covered in this first lecture to receive a complete solution. Let us list a brief (non-exhaustive) history of the contributions to its solution.

- In [BZ88], F. Bethuel and Zheng X. proved that every $W^{1,p}$ map to the circle has a lifting when $p \ge 2$.
- In [BBM00], J. Bourgain, H. Brezis, and P. Mironescu gave a complete solution to the lifting problem for maps with values into the circle.
- In [BC07], F. Bethuel and D. Chiron extended the results by Bourgain, Brezis, and Mironescu to an arbitrary covering, almost completely solving the lifting problem. More precisely, they proved that the answer to the lifting problem is
  - positive if $m = 1$;
  - positive if $s \ge 1$ and $sp \ge 2$ (the case $s > 1$ requires a very mild assumption on the covering, see [Det22, Section 8.2]);
  - negative if $1 \le sp < 2$ and $m \ge 2$;
  - negative if $0 < s < 1$ and $1 \le sp < m$ when $\widetilde{\mathcal{N}}$ is not compact;
  - positive if $0 < s < 1$ and $sp \ge m$;
  - positive if $sp < 1$.
- In [MVS21a], P. Mironescu and J. Van Schaftingen solved the remaining open case, by showing that the answer to the lifting problem is positive when $2 \le sp < m$ and $\widetilde{\mathcal{N}}$ is compact.

Lecture 2

Analytic and topological tools for strong density

This session will be devoted to the presentation of the strong density theorem and the building blocks of its proof, that display a fascinating interplay between analysis and topology. We shall not attempt to present the proof of the general case in full detail, but instead explain the tools on basic, and hopefully insightful, cases, in order to give the intuition about how they work.

2.1 The strong density theorem

Being faced with the possibility of topological obstructions to strong density as in Example 1.2 and more generally Theorem 1.5, two natural questions arise: (i) when does strong density occur; and (ii) can we find a suitable class of "almost smooth maps" that would always be dense. These two questions have received a lot of attention since the observation by R. Schoen and K. Uhlenbeck [SU83] which has been explained in Example 1.2. In this section, we are going to explain a few tools and ideas that have been introduced to tackle them. We adopt a kind of inverted presentation: we state and essentially prove a very specific case of the strong density theorem due to F. Bethuel [Bet91] in his seminal contribution, which hopeful avoids much technicality, and only at the end of the lecture we mention what happens in the general case and give an extensive list of references.

The presentation I adopt here follows the one from my master thesis [Det22]. It has been slightly adapted to reflect more the scheme of proof which has been introduced by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15] to handle the higher order case of $W^{k,p}$ spaces (see [Det23] for the adaptation of this approach to the fractional setting). In particular, the sections have been named according to the tools that are used in [BPVS15]. It may happen that our use of them in the special case treated here may not be totally faithful with respect to [BPVS15]. In this case, we give a brief explanation at the end of the section about how to adapt the tool to the general setting. However, there is hopefully at least a common idea between the version presented here and the general setting.

More specifically, we shall restrict to $s = 1$, $m - 1 < p < m$, and the domain Ω is the unit m-dimensional cube Q^m. (The domain could equivalently be a ball, it is easy to go from one to another by a bi-Lipschitz transformation, but we stick to a cube for technical reasons.) Two observations are crucial to understand what a positive result should look like: (i) the obstruction to strong density comes from the nontriviality of $\pi_{m-1}(\mathscr{N})$; and (ii) the typical obstruction is a map which is smooth everywhere except at point singularities. It turns out that this is the only obstruction to strong density.

Theorem 2.1. *Assume that $m-1<p<m$. The class $C^\infty(\overline{Q^m};\mathscr{N})$ is dense in $W^{1,p}(Q^m;\mathscr{N})$ if and only if $\pi_{m-1}(\mathscr{N})\simeq\{0\}$. Moreover, the class of $W^{1,p}(Q^m;\mathscr{N})$ that are smooth outside of a finite set of points is always dense in $W^{1,p}(Q^m;\mathscr{N})$.*

Before delving into the proof, let us give a few words about it. As already explained, we cannot rely on a basic convolution procedure to regularize a Sobolev mapping on the whole Q^m while preserving the geometric constraint. On the other hand, since $p>m-1$, we know that a $W^{1,p}$ is continuous on $(m-1)$-dimensional sets. The idea is therefore to find a suitable $(m-1)$-dimensional grid, and to take advantage of the nice behavior of the mapping we wish to approximate on this grid.

Another important idea is the following. Since convolution is a kind of averaging, its enemy is oscillation. If a map does not oscillate too much, then a convolution procedure at small scale should not disrupt the geometric constraint too much (and we know we can deal with small perturbations by projecting back onto the target). A key observation is that, being allowed a fixed amount of energy, a Sobolev map cannot wildly oscillate everywhere. We shall therefore distinguish between the regions where our map does not oscillate too much, and where we can actually rely on convolution, and the regions where it oscillates wildly and we have to proceed otherwise. This is enforced by the *method of good and bad cubes*, devised by F. Bethuel in his seminal 1991 contribution, that we explain in the next section.

2.2 Good and bad cubes, and the opening construction

In the rest of the lecture, we take $u\in W^{1,p}(Q^m;\mathscr{N})$, where $m-1<p<m$. By a classical reflection or dilation argument, we may always assume that u is defined on a slightly larger set than Q^m, so that we can perform constructions that rely on values of u slightly outside of Q^m. For convenience, here we shall assume that $u\in W^{1,p}(2Q^m;\mathscr{N})$, where $2Q^m$ is the cube with same center as Q^m and double sidelength.

As we explained, we wish to find a suitable grid on which u is well-behaved. For this purpose, we introduce some notation. From now on, we fix $\eta>0$. We denote $K^m_\eta=Q_\eta+\eta\mathbb{Z}^m$ the standard decomposition of $\mathbb{R}^m$ by cubes of sidelength η. (We note that K^m_η is a set of cubes.) For $a\in\mathbb{R}^m$, we write $K^m_{\eta,a}=K^m_\eta+a$ the set of all translates of the cubes in K^m_η by a. We also write $K^{m-1}_{\eta,a}$ to denote the set of all faces of cubes in

$K^m_{\eta,a}$. Finally, given a family of cubes, denoted by a roman letter, the corresponding calligraphic letter will denote the set obtained by taking the union of all cubes in the family. For instance, $\mathscr{K}^{m-1}_{\eta,a}$ is an $(m-1)$-dimensional set in $\mathbb{R}^m$.

We now prove the following result concerning the existence of a suitable grid on which u is well-behaved.

Proposition 2.2. *There exists $a \in \mathbb{R}^m$ such that $u \in W^{1,p}(\mathscr{K}^{m-1}_{\eta,a} \cap 2Q^m; \mathscr{N})$ with*

$$\int_{\mathscr{K}^{m-1}_{\eta,a}\cap 2Q^m} |\mathrm{D}u|^p \lesssim \frac{1}{\eta}\int_{2Q^m} |\mathrm{D}u|^p.$$

Proof. The claims follows readily from a genericity and averaging argument. Indeed, by Fubini–Tonelli's theorem, we estimate

$$\int_{Q_\eta}\int_{\mathscr{K}^{m-1}_{\eta,a}\cap 2Q^m} |\mathrm{D}u|^p = \eta^{m-1}\int_{2Q^m} |\mathrm{D}u|^p.$$

This implies that

$$\fint_{Q_\eta}\int_{\mathscr{K}^{m-1}_{\eta,a}\cap 2Q^m} |\mathrm{D}u|^p = \frac{1}{\eta}\int_{2Q^m} |\mathrm{D}u|^p,$$

which ensures the existence of a subset of Q_η of positive measure on which

$$\int_{\mathscr{K}^{m-1}_{\eta,a}\cap 2Q^m} |\mathrm{D}u|^p \lesssim \frac{1}{\eta}\int_{2Q^m} |\mathrm{D}u|^p.$$

We note that there are a few technicalities to check, that require to possibly exclude some extra set of zero measure of Q_η. (Can you find them and make the argument fully rigorous?) □

As we explained, we shall now define the *good and bad cubes*. We let $\kappa > 0$ to be chosen later on, and we say that $\sigma \in K^m_{\eta,a}$ that is entirely contained in $2Q^m$ is a *good cube* whenever

$$\frac{1}{\eta^{m-p}}\int_\sigma |\mathrm{D}u|^p \leq \kappa^p \quad \text{and} \quad \frac{1}{\eta^{m-p-1}}\int_{\partial\sigma} |\mathrm{D}u|^p \leq \kappa^p. \tag{2.1}$$

Otherwise, we call σ a *bad cube*. We let $B_{\eta,a}$ be the set of bad cubes, and $G_{\eta,a}$ be the set of good cubes. (Although a is fixed from now, we keep it in the notation to minimize possible confusion between the set of bad cubes and the ball.) We note that the above quantities are rescaled energies, respectively on the cubes and their boundaries.

An important feature of bad cubes is that there are not too many of them.

Proposition 2.3. *The number of bad cubes satisfies the estimate*

$$\operatorname{card} B_{\eta,a} \lesssim \eta^{p-m}\kappa^{-p}\int_{2Q^m}|\mathrm{D}u|^p.$$

Proof. By definition of bad cubes, we have

$$\operatorname{card} B_{\eta,a} = \sum_{\sigma\in B_{\eta,a}} 1 \leq \sum_{\sigma\in B_{\eta,a}} \eta^{p-m}\kappa^{-p}\int_{\sigma}|\mathrm{D}u|^p + \eta^{p+1-m}\kappa^{-p}\int_{\partial\sigma}|\mathrm{D}u|^p.$$

The second term above is estimated using Proposition 2.2:

$$\sum_{\sigma\in B_{\eta,a}}\int_{\partial\sigma}|\mathrm{D}u|^p \lesssim \int_{\mathscr{K}^{m-1}_{\eta,a}}|\mathrm{D}u|^p \lesssim \frac{1}{\eta}\int_{2Q^m}|\mathrm{D}u|^p.$$

This concludes the proof. □

Thanks to Proposition 2.2, we have obtained a grid on which u is well-behaved. However, it shall be useful to us to slightly modify u in order to extend this nice behavior to a neighborhood of $\mathscr{K}^{m-1}_{\eta,a}$. We accomplish this by a prototypical variant of the *opening* construction. More specifically, on every $\sigma \in K^m_{\eta,a}$ that is entirely contained in $2Q^m$, we define the map u^{op}_η by

$$u^{\mathrm{op}}_\eta(x) = \begin{cases} u(c_\sigma + \lambda^{-1}(x-c_\sigma)) & \text{if } x \in \lambda\sigma,\\ u\left(c_\sigma + \frac{x-c_\sigma}{|x-c_\sigma|}\right) & \text{otherwise,}\end{cases}$$

where c_σ is the center of σ, and where $0 < \lambda < 1$ has to be chosen very close to 1. An important feature of u^{op}_η is that, thanks to the choice of $K^m_{\eta,a}$ and Morrey–Sobolev inequality, it is continuous on a neighborhood of width $1-\lambda$ of $\mathscr{K}^{m-1}_{\eta,a}$, and its oscillation over there can be controlled by the $W^{1,p}$ norm of u on $\mathscr{K}^{m-1}_{\eta,a}$.

The important feature about the opening construction is to modify the map u into a map u^{op} that depends on less variables than u on a small region (here, u^{op} depends only on $m-1$ variables around $\partial\sigma$, as it is a radial map on this region). However, there is an important difference between our presentation here and the standard opening construction. Indeed, here, we first constructed a low-dimensional region on which u is well-behaved, and then used the values of u on this region to construct u^{op} on a small neighborhood. On the contrary, the usual opening procedure is performed on a region which is fixed *a priori*, and an averaging argument picks the values to be used.

The opening construction was first devised by H. Brezis and Li Y. [BL01] in order to study the topology of the spaces of Sobolev mappings to manifolds. Their original construction starts from a map u defined on a cube, and produces an opened map u^{op} that is constant in a small region near the center of the cube, and coincides with u outside of a slightly larger region. This illustrates well our previous comment: in this construction, the aforementioned regions are *fixed*, and the averaging argument serves to pick a suitable value to fill in the central region.

This construction was then pursued by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15] in order to study the strong density problem in higher order Sobolev spaces (see [Det23] for the adaptation to the fractional setting). There, the goal of the construction is to open a map u around the $\lfloor sp \rfloor$-dimensional skeleton of a *fixed* decomposition of Q^m into small cubes of sidelength η, in order to provide a map u^{op} which depends only on $\lfloor sp \rfloor$ variables locally around this skeleton. (The value $\lfloor sp \rfloor$ is the largest dimension on which we have the Morrey–Sobolev embedding of $W^{s,p}$ into L^∞ — or into VMO in the limiting case.) The construction is iterative: the map is first opened around the vertices of the skeleton, then around the edges, and so on until one reaches the required dimension. An illustration can be found on Figure 2.1 for $m = 2$ and $\lfloor sp \rfloor = 1$.

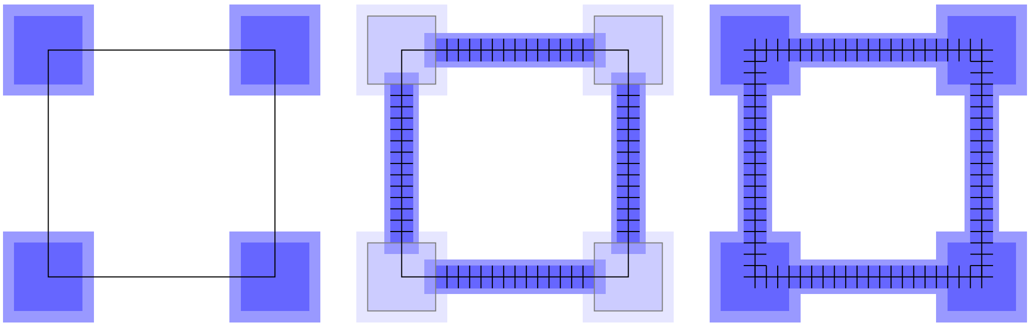

Figure 2.1: Opening for $m = 2$ and $\ell = 1$

In our presentation here, the difference is not only that the averaging argument is used to choose the skeleton, which is not fixed, but also that the region on which the opening construction is performed can be chosen as small as we please while keeping the required estimates. This simplifies a lot the construction, which is useful here for pedagogical purposes, but does not generalize quite well to the general setting. As a consequence of this, here we may perform the opening construction around the whole $(m-1)$-skeleton $\mathscr{K}^{m-1}_{\eta,a}$, since we may then choose the size of the opening region to converge to zero. In the general construction, since the opened region should be a fixed proportion of each cube on which it is performed, the opening procedure is only performed near the bad cubes.

2.3 A weak convergence result on bad cubes: the thickening procedure

Since we do not have much control on the behavior of u on bad cubes, it is unclear whether we can perform an accurate approximation on this region. Good news is that, since there are not too much bad cubes, it is actually sufficient to approximate it very roughly in this region. The main result in this section is the following proposition, which features the so-called *thickening procedure*. The idea is to fill in cubes with values of u only on their boundary, where it is well-behaved.

Proposition 2.4. *Let $v \in W^{1,p}(Q^m; \mathscr{N})$. For every $0 < \lambda < 1$, there exists $\lambda < \lambda_0 < 1$ and a map $w \in W^{1,p}(\lambda_0 Q^m; \mathscr{N})$ that is continuous outside of a finite number of points and such that $\operatorname{tr}_{\partial\lambda_0 Q^m} w = \operatorname{tr}_{\partial\lambda_0 Q^m} v$ and*

$$\int_{\lambda_0 Q^m} |\mathrm{D}w|^p \lesssim \int_{Q^m} |\mathrm{D}v|^p.$$

Moreover, the hidden constants do not depend on λ.

The key ingredient to prove Proposition 2.4 is the following thickening procedure on *one* cube.

Lemma 2.5. *Let $Q(r)$ denote the cube centered at 0 with inradius equal to r. Given $v \in W^{1,p}(\partial Q(r))$, we define $v^{\mathrm{th}}(x) = v\big(\frac{rx}{|x|_\infty}\big)$. Then, $v^{\mathrm{th}} \in W^{1,p}(\partial Q(r))$ and it satisfies the estimate*

$$\int_{Q(r)} |\mathrm{D}v^{\mathrm{th}}|^p \lesssim r \int_{\partial Q(r)} |\mathrm{D}v|^p.$$

Proof. The proof is a simple polar integration computation. Indeed, we estimate

$$\mathrm{D}v^{\mathrm{th}}(x) \lesssim \frac{r}{|x|_\infty} \mathrm{D}v\big(\frac{rx}{|x|_\infty}\big),$$

and the polar integration formula yields

$$\begin{aligned}\int_{Q(r)} |\mathrm{D}v^{\mathrm{th}}|^p &= \int_0^r \int_{\partial Q(\rho)} \frac{r^p}{|x|_\infty^p} \Big|\mathrm{D}v\Big(\frac{rx}{|x|_\infty}\Big)\Big| \, \mathrm{d}x \, \mathrm{d}\rho \\ &= r^{p-m+1} \int_0^r \rho^{m-1-p} \Big(\int_{\partial Q(r)} |\mathrm{D}v|^p\Big) \mathrm{d}\rho \lesssim r \int_{\partial Q(r)} |\mathrm{D}v|^p.\end{aligned}$$

We note that only the condition $p < m$ is used, to ensure the convergence of the above integral. This lemma does not rely on $p > m - 1$.

We importantly mention that we have only proved the required estimate on the derivative of v. However, it remains to prove the actual Sobolev regularity. In particular, for a complete argument, one should prove the weak differentiability of the map that we have constructed. (Can you fill in the missing details to finish a fully rigorous proof of the lemma?) □

With this basic tool at hand, we may now prove Proposition 2.4. The idea is to perform the previous homogeneous extension construction on each cube of a suitable grid.

Proof of Proposition 2.4. Given $\eta > 0$ as in Proposition 2.2, we obtain a number $a \in \mathbb{R}^m$ such that $v \in W^{1,p}(\mathscr{K}^{m-1}_{\eta,a} \cap Q^m; \mathscr{N})$ and

$$\int_{\mathscr{K}^{m-1}_{\eta,a} \cap Q^m} |\mathrm{D}v|^p \lesssim \frac{1}{\eta} \int_{Q^m} |\mathrm{D}v|^p.$$

We apply the construction from Lemma 2.5 to each cube $\sigma \in K^m_{\eta,a}$ that lies completely inside Q^m, and this yields a map $w \in W^{1,p}$ on the union of all these cubes and with values into $\mathscr{N}$ that satisfies

$$\int_{\sigma} |\mathrm{D}w|^p \lesssim \eta \int_{\partial\sigma} |\mathrm{D}v|^p.$$

Summing over all such cubes σ yields the required estimate.

It is clear from its construction and the Morrey–Sobolev embedding that the map w is continuous outside of a finite union of points. Choosing $\eta > 0$ sufficiently small, we may ensure that w is defined on a cube containing λQ^m. Possibly excluding a set of zero measure of potential values of a allows to ensure the coincidence of traces. □

We wish to mention that, letting $\eta \to 0$, the above construction actually yields almost everywhere convergence in addition to the uniform bound on the energy of the derivatives. (Can you prove it?) We shall come back on this later on, in connection with *weak density* questions.

We now explain how to conclude the approximation on bad cubes. We apply Proposition 2.4 to the map u^{op}_η on each bad cube σ (after scaling), and we replace u^{op}_η by the corresponding map w on $\lambda_0\sigma$. The resulting map is Sobolev and continuous on the whole cube σ except on a finite number of points (continuity in $\lambda_0\sigma$ except some points follows from Proposition 2.4, continuity outside follows from the opening construction and the Morrey–Sobolev embedding, and Sobolev regularity follows from the fact that the traces match). The convergence follows from the energy bound provided by Proposition 2.4 and Lebesgue's lemma, since the volume of the bad cubes converges to 0 as $\eta \to 0$.

In this section, what we have called *thickening* is nothing else but a standard homogeneous extension procedure. The main feature of this construction is to fill in a cube with the well-behaved values on the boundary. A variation of the construction, replacing $\frac{x}{|x|}$ by $\lambda(x)x$ with λ a smooth function chosen so that $\frac{\lambda(x)}{|x|}$ is radially increasing and $\lambda = 1$ near the boundary of the cube, allows instead to fill in the cube by the values on a thick *neighborhood* of its boundary, while leaving the map unchanged near the boundary. In addition to allowing for extension to higher order spaces (the homogeneous extension procedure exhibiting obvious issues for gluing constructions on neighboring cubes, that are incompatible with higher order Sobolev regularity), it also allows to use only *one* thickening step in each cube, instead of having to make a subdivision of each bad cube into a grid of smaller cubes on which homogeneous extension is performed. This is where combination with the opening procedure turns useful, to provide the required neighborhood of the skeleton on which the map to which the thickening procedure is applied is well-behaved.

If $p < m - 1$ (or more generally if $sp < m - 1$), it is required to iterate the thickening procedure until reaching a skeleton of suitable dimension. Recall that we want to use the values of u on a skeleton of dimension $\lfloor sp \rfloor$, in order to have at hand the Morrey–Sobolev embedding (or the embedding into VMO in the limiting case). For instance, if $m - 2 \leq sp < m - 1$, one has first to fill in the faces of the cube by values taken from the $(m-2)$-skeleton, and then fill in the cube with these values. The first step creates point singularities at the middle of each face of the cube. The second step propagates these singularities inside the cube, hence resulting in the singular set being a finite union of lines meeting at the center. More generally, for arbitrary values of sp, the singular set is of dimension $m - \lfloor sp \rfloor - 1$, it is the so-called *dual skeleton* of the $\lfloor sp \rfloor$-skeleton of the cube that is used in the proof. An illustration is provided on Figure 2.2 below.

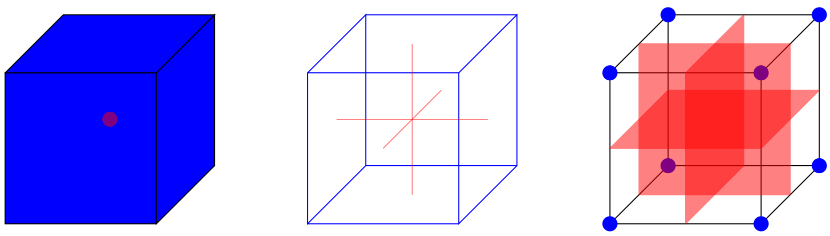

Figure 2.2: Skeletons and their dual skeletons

2.4 Approximation on good cubes: a projection technique and adaptive smoothing

We now turn to approximation on good cubes. Since this region accounts for most of the cube on which we are working, we need to construct a more precise approximation there. But luckily, we have the energy estimates available on good cubes at our disposal to help us in this task.

The idea is in two steps. First, we prove that, on each good cube, u takes its value on a given small ball except on a set of small measure. Therefore, projecting radially onto this ball, we do not mess up too much with the values of u. Then, once this projection has been performed, we regularize by convolution. Since this procedure preserves convex constraints, we know that we remain in this ball, which allows to conclude by the classical reprojection onto the manifold trick.

The first result is the following. We let $\delta > 0$ be sufficiently small so that $\delta < \iota$, where ι is the radius of a neighborhood of $\mathscr{N}$ on which there is a smooth retraction onto $\mathscr{N}$.

Proposition 2.6. *For every sufficiently small $\eta > 0$, if $\kappa > 0$ is chosen sufficiently small in the definition of good cubes, there exists a map $w_\eta \in W^{1,p}(\mathscr{G}^m_{\eta,a}; \mathscr{N})$ defined on the set of all good cubes, such that*

1. *for every $\sigma \in G_{\eta,a}$, there exists a point $y_\sigma \in \mathscr{N}$ such that $w_\eta \in B_\delta(y_\sigma)$;*
2. *$w_\eta = u^{\mathrm{op}}_\eta$ on $\sigma \setminus \lambda\sigma$ for every $\sigma \in G_{\eta,a}$;*
3. *there exists a measurable set $A_\eta \subset \mathscr{G}^m_{\eta,a}$ such that $|A_\eta| \lesssim \kappa^p \delta^{-p}$ and*

$$\int_{\mathscr{G}^m_{\eta,a}} |u^{\mathrm{op}}_\eta - w_\eta|^p + |\mathrm{D}u^{\mathrm{op}}_\eta - \mathrm{D}w_\eta|^p \lesssim \int_{A_\eta} |\mathrm{D}u|^p.$$

Proof. By the Morrey–Sobolev inequality and the second condition in the definition of good cubes (2.1), we may choose κ sufficiently small, depending on δ, so that the oscillation of u^{op}_η on every $\sigma \setminus \lambda\sigma$ with $\sigma \in G_{\eta,a}$ is less than $\frac{\delta}{2}$. Choose y_σ to be any point in the image of $\sigma \setminus \lambda\sigma$ by u^{op}_η.

We let P_σ be the projection onto the ball $B_\delta(y_\sigma)$, defined by

$$P_\sigma(y) = \begin{cases} y & \text{if } y \in B_\delta(y_\sigma), \\ y_\sigma + \delta\frac{y - y_\sigma}{|y - y_\sigma|} & \text{else.} \end{cases}$$

We define $w_\eta = P_\sigma \circ u^{\mathrm{op}}_\eta$ on every $\sigma \in G_{\eta,a}$, and we show that it has the required properties.

Property 1 is obvious from the definition. Similarly, property 2 follows from the choices of κ and y_σ. It only remains to prove 3.

Let us define $\mathcal{U}_{\eta,\sigma} = \{x \in \sigma : u_\eta^{\mathrm{op}}(x) \notin B_\delta(y_\sigma)\}$. By the construction of w_η, we have $u_\eta^{\mathrm{op}} = w_\eta$ and $\mathrm{D}u_\eta^{\mathrm{op}} = \mathrm{D}w_\eta$ almost everywhere on $\sigma \setminus \mathcal{U}_{\eta,\sigma}$. We therefore estimate

$$\int_\sigma |\mathrm{D}u_\eta^{\mathrm{op}} - \mathrm{D}w_\eta|^p \lesssim \int_{\mathcal{U}_{\eta,\sigma}} |\mathrm{D}u_\eta^{\mathrm{op}}|^p.$$

Moreover, since u_η^{op} and w_η coincide on $\partial\sigma$, the Poincaré inequality ensures that

$$\int_\sigma |u - w_\eta|^p \lesssim \eta^p \int_\sigma |\mathrm{D}u_\eta^{\mathrm{op}} - \mathrm{D}w_\eta|^p \lesssim \eta^p \int_{\mathcal{U}_{\eta,\sigma}} |\mathrm{D}u_\eta^{\mathrm{op}}|^p.$$

Summing over good cubes,

$$\int_{\mathcal{G}^m_{\eta,a}} |u_\eta^{\mathrm{op}} - w_\eta|^p + |\mathrm{D}u_\eta^{\mathrm{op}} - \mathrm{D}w_\eta|^p \lesssim \int_{A_\eta} |\mathrm{D}u_\eta^{\mathrm{op}}|^p,$$

with

$$A_\eta = \bigcup_{\sigma \in G_{\eta,a}} \mathcal{U}_{\eta,\sigma}.$$

We conclude by estimating the measure of A_η.

This relies on a nice truncation argument. We consider a truncation function $\varphi \colon \mathbb{R} \to [0,1]$ such that $\varphi \in C^\infty(\mathbb{R})$, $\varphi(x) = 0$ if $x \leq \frac{\delta}{2}$, $\varphi(x) = 1$ if $x \geq \delta$, and $|\varphi'| \lesssim \delta^{-1}$. We observe that $\varphi(|u_\eta^{\mathrm{op}} - y_\sigma|)^p = 1$ on $\mathcal{U}_{\eta,\sigma}$, and $\mathcal{U}_{\eta,\sigma} \subset \lambda\sigma$. Hence,

$$|\mathcal{U}_{\eta,\sigma}| \leq \int_{\lambda\sigma} \varphi(|u_\eta^{\mathrm{op}} - y_\sigma|)^p.$$

Since $\varphi(|u - y_\sigma|) = 0$ on $\partial(\lambda\sigma)$, the Poincaré inequality implies that

$$\int_{\lambda\sigma} \varphi(|u_\eta^{\mathrm{op}} - y_\sigma|)^p \lesssim \eta^p \int_{\lambda\sigma} |\mathrm{D}(\varphi(|u_\eta^{\mathrm{op}} - y_\sigma|))|^p \lesssim \eta^p \delta^{-p} \int_{\lambda\sigma} |\mathrm{D}u_\eta^{\mathrm{op}}|^p.$$

Now we use the first condition in the definition of good cubes (2.1) and a change of variable to find that

$$|\mathcal{U}_{\eta,\sigma}| \lesssim \kappa^p \delta^{-p} \eta^m = \kappa^p \delta^{-p} |\sigma|.$$

It suffices to sum over all good cubes to finish the proof. □

We now explain how to finish the approximation procedure on good cubes. For this, as we already announced, we rely on regularization by convolution, which is possible in this setting since we know that it will not leave the ball $B_\delta(y_\sigma)$. However, to connect the constructions on two neighboring cubes, we need to preserve the values of the map on the boundary of each cube. For this purpose, we rely on *adaptive smoothing*, a tool introduced by R. Schoen and K. Uhlenbeck [SU82] in the context of minimizing harmonic maps, and pursued by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15] for strong density questions.

Given a map $\psi \in C^\infty(Q^m)$ and ρ a convolution kernel, we define

$$\rho_\psi * u(x) = \int_{\mathbb{B}^m} \rho(z) u(x + \psi(x)z)\,\mathrm{d}z.$$

We have to assume that $\psi(x) \leq \operatorname{dist}(x, \partial Q^m)$ in order to ensure that $x + \psi(x)z \in Q^m$ whenever x is. This is exactly like the standard convolution, except that the regularization parameter now depends on the point where the convolution is evaluated. As such, this convolution enjoys the same convergence properties as the fixed-parameter convolution. Let us exemplify this by showing the L^p estimate for derivatives. We first compute

$$\mathrm{D}(\rho_\psi * u)(x) = \int_{\mathbb{B}^m} \rho(z) \mathrm{D}u(x + \psi(x)z)[\mathrm{id} + \mathrm{D}\psi(x) \otimes z]\,\mathrm{d}z.$$

Therefore,

$$\begin{aligned}
&|\mathrm{D}\rho_\psi * u(x) - \mathrm{D}u(x)| \\
&\quad \leq \int_{\mathbb{B}^m} \rho(z)|\mathrm{D}u(x + \psi(x)z) - \mathrm{D}u(x)|\,\mathrm{d}z + \|\mathrm{D}\psi\|_{L^\infty} \int_{\mathbb{B}^m} \rho(z)|\mathrm{D}u(x + \psi(x)z)|\,\mathrm{d}z.
\end{aligned}$$

For the first term, we rely on Minkowsky's integral inequality to obtain

$$\begin{aligned}
\int_{Q^m} \left(\int_{\mathbb{B}^m} \rho(z)|\mathrm{D}u(x+\psi(x)z) - \mathrm{D}u(x)|\,\mathrm{d}z \right)^p \mathrm{d}x &\leq \int_{\mathbb{B}^m} \rho(z) \|\mathrm{D}u(\cdot + \psi z) - \mathrm{D}u\|_{L^p(Q^m)}\,\mathrm{d}z \\
&\leq \sup_{z \in \mathbb{B}^m} \|\tau_{\psi z}\mathrm{D}u - \mathrm{D}u\|_{L^p(Q^m)},
\end{aligned}$$

where $\tau_{\psi z} v(x) = v(x + \psi(x)z)$. For the second term, we use Minkowsky's integral inequality as well to find

$$\int_{Q^m} \left(\int_{\mathbb{B}^m} \rho(z)|\mathrm{D}u(x + \psi(x)z)|\,\mathrm{d}z \right)^p \mathrm{d}x \leq \sup_{z \in \mathbb{B}^m} \|\tau_{\psi z}\mathrm{D}u\|_{L^p(Q^m)}.$$

We use the change of variable $y = x + \psi(x)z$, which is well-defined provided that $\|D\psi\|_{L^\infty} < 1$, and we find

$$\|\tau_{\psi z} Du\|_{L^p(Q^m)} \le \frac{1}{1 - \|D\psi\|_{L^\infty}} \|Du\|_{L^p(Q^m)}.$$

Hence, if we replace ψ by $t\psi$ and let $t \to 0$, using the continuity of translations in L^p, we find

$$D(\rho_{t\psi} * u) \to Du \quad \text{in } L^p.$$

By similar computations, we have $\rho_{t\psi} * u \to u$ in L^p, and even in L^∞ in case u is continuous.

The approximation on good cubes is the concluded by proceeding to an adaptive smoothing of w_η on each good cube σ, with ψ vanishing on $\partial\sigma$. By the above estimates, the resulting smooth maps converge in $W^{1,p}$ to w_η, it takes values into $B_\delta(y_\sigma)$ as convolution preserves convex constraints, and since w_η is continuous near $\partial\sigma$, boundary values match as a consequence of the vanishing of the convolution parameter on $\partial\sigma$. It then suffices to compose with a smooth retraction into $\mathcal{N}$.

This proves the part of Theorem 1.6 involving the density of Sobolev maps that are continuous outside a finite number of points. (To be precise, the map we have constructed is only continuous and Sobolev, but need not be smooth everywhere. Can you approximate it by smooth maps?)

In the limiting case $p = m - 1$ (or more generally $sp \in \mathbb{N}$), the argument is more subtle, since the Morrey–Sobolev embedding fails. Therefore, it is no longer possible to ensure that u takes values on a small ball on the boundary of the good cubes. This is where, in the approach by Bousquet, Ponce, and Van Schaftingen, the adaptative convolution is used at its full potential. Indeed, on the good cubes, one should choose the convolution parameter of order η in order to be able to exploit the energy estimate on these cubes, relying on an averaging estimate as in the proof of Theorem 1.6 combined with the Poincaré–Wirtinger inequality, in order to estimate the distance between the smoothened map and the target manifold. Near the $\lfloor sp \rfloor$-skeleton on the bad cubes, where the map has been opened, one should on the contrary be able to choose the convolution parameter to be very small to exploit the embedding $W^{s,p} \to \mathrm{VMO}$, using the fact that the opened map only depends on $\lfloor sp \rfloor$ variables in this region. A careful argument is then required to explain how to handle the transition region between both these regimes, that we do not explain here.

2.5 Removing the singularities: shrinking

To finish the proof of Theorem 1.6, we only need to explain how to exploit the assumption $\pi_{m-1}(\mathscr{N})$ to remove the pointwise singularities created by the above procedure. Since the construction is local, it suffices to deal with the case of a Sobolev mapping v on $Q_1 = (-1,1)^m$ that is continuous except at the origin. Therefore, the following proposition will suffice to conclude the argument.

Proposition 2.7. *Let $v \in W^{1,p}(Q_1;\mathscr{N})$ be continuous in $Q_1 \setminus \{0\}$. If $\pi_{m-1}(\mathscr{N}) \simeq \{0\}$, then for every $\eta > 0$, there exists a map $w_\eta \in C^\infty(Q^m;\mathscr{N})$ such that $\|v - w_\eta\|_{W^{1,p}(Q_1)} \leq \eta$ and that coincides with v outside of an arbitrarily small neighborhood of 0.*

Proof. Fix $\delta > 0$, and let w_δ^{ext} be a continuous Sobolev extension of $v_{|\partial Q_\delta}$ to Q_δ. (The existence of a continuous extension relies on the assumption that $\pi_{m-1}(\mathscr{N})$ is trivial. Can you prove rigorously that the extension can be taken to be Sobolev? This requires an extra regularization argument.) The problem here is that we have no control on the Sobolev energy of w_δ^{ext}: even though it is defined on a very small set, its derivative could have an arbitrarily high norm. We shall correct this by the shrinking procedure.

We define $w_{\delta,t}^{\mathrm{sh}}$ by

$$w_{\delta,t}^{\mathrm{sh}}(x) = \begin{cases} v(x) & \text{if } x \notin Q_\delta, \\ v(\frac{\delta x}{|x|_\infty}) & \text{if } x \in Q_\delta \setminus Q_t, \\ w_\delta^{\mathrm{ext}}(\frac{x}{t}) & \text{if } x \in Q_{t\delta}. \end{cases}$$

We wish to estimate the Sobolev distance between v and $w_{\delta,t}^{\mathrm{sh}}$. We exemplify the computation by showing the estimate for the derivatives (the L^p estimate being much simple, as the maps are uniformly bounded and differ only on a small set). We have

$$\int_{Q_1} |\mathrm{D}v - \mathrm{D}w_{\delta,t}^{\mathrm{sh}}|^p \lesssim \int_{Q_\delta} |\mathrm{D}v|^p + \int_{Q_\delta \setminus Q_{t\delta}} |\mathrm{D}w_{\delta,t}^{\mathrm{sh}}|^p + \int_{Q_{t\delta}} |\mathrm{D}w_{\delta,t}^{\mathrm{sh}}|^p.$$

From Lemma 2.5, we have

$$\int_{Q_\delta \setminus Q_{t\delta}} |\mathrm{D}w_{\delta,t}^{\mathrm{sh}}|^p \lesssim \delta \int_{\partial Q_\delta} |\mathrm{D}v|^p.$$

From a simple change of variable, we find

$$\int_{Q_{t\delta}} |\mathrm{D}w_{\delta,t}^{\mathrm{sh}}|^p \lesssim t^{m-p} \int_{Q_\delta} |\mathrm{D}w_\delta^{\mathrm{ext}}|^p.$$

Here it is important that $p < m$: letting $t \to 0$, we may make the right-hand-side as small as we please. Choosing δ appropriately (how exactly?), and then t sufficiently small, we obtain the required map through $w_\eta = w^{\mathrm{sh}}_{\delta,t}$. □

Just as thickening, the shrinking construction can be appropriately modified to be compatible with the extra rigidity of higher order Sobolev spaces; see [BPVS15, Det23].

When $sp < m - 1$, there is an additional difficulty, since the extension procedure is no longer local. Indeed, one has to remove a singular set of dimension $m - \lfloor sp \rfloor - 1$, which is the dual skeleton of some $\lfloor sp \rfloor$-skeleton of Q^m. When the domain is a cube (or more generally, when the domain is $(\lfloor sp \rfloor - 1)$-connected), the condition $\pi_{\lfloor sp \rfloor}(\mathscr{N})$ is still sufficient to be able to extend a continuous map defined on the $\lfloor sp \rfloor$-skeleton to the whole cube, which permits to remove the singular set. However, when the domain may have non trivial topology, things may become more involved, as was observed by Hang F. and Lin F. [HL03a]. We give a precise statement in the next section, which explains the current state of the art concerning the strong density problem.

As a final remark, let us mention that the above proof shows that, regardless of the topology of the target, a *given* map $u \in W^{1,p}(Q^m; \mathscr{N})$ may be approximated by a sequence of smooth mappings if and only if its restriction to a *generic* square is homotopic to a constant. By generic, we mean that the above property holds for $\partial Q_r(a)$ for almost every $a \in Q^m$ and almost every $r \in (0, 1)$ such that $Q_r(a) \subset Q^m$.

2.6 The complete answer to the strong density problem

In the course of this lecture, we have given a proof of the strong density theorem in the case $s = 1$, $m - 1 < p < m$, and where the domain is a cube. We have tried to give some hints about what happens in the general case. In particular, as should be suggested by the proof, the appropriate class of almost smooth maps consists of those maps that are smooth outside of the dual skeleton of an $\lfloor sp \rfloor$-dimensional skeleton of the domain. More precisely, we define the class $\mathscr{R}_i(\Omega; \mathscr{N})$ as the set of those maps $u\colon \Omega \to \mathscr{N}$ that are smooth outside of a finite union $\mathscr{S}_u$ of i-dimensional affine spaces and such that

$$|\mathrm{D}^j u(x)| \leq C_j \frac{1}{\operatorname{dist}(x, \mathscr{S}_u)^j} \quad \text{for every } x \in \Omega \setminus \mathscr{S}_u \text{ and every } j \in \mathbb{N}_*.$$

The above condition ensures that the maps in the class $\mathscr{R}_i$ indeed belong to the appropriate Sobolev spaces. We note in particular that the class of maps that are smooth outside of a finite number of singularities discussed in this lecture corresponds to the class $\mathscr{R}_0$. To be precise, due to the required estimate on the derivatives, the class $\mathscr{R}_0$

is slightly smaller, but it is readily seen that the maps constructed in the above proof indeed belong to $\mathscr{R}_0$, as a consequence of the properties of homogeneous extension.

With this definition, the *strong density theorem* then reads as follows.

Theorem 2.8. *If $sp < m$, then $\mathscr{R}_{m-\lfloor sp \rfloor-1}(Q^m;\mathscr{N})$ is dense in $W^{s,p}(Q^m;\mathscr{N})$, and $C^\infty(Q^m;\mathscr{N})$ is dense in $W^{s,p}(Q^m;\mathscr{N})$ if and only if $\pi_{\lfloor sp \rfloor}(\mathscr{N}) \simeq \{0\}$.*

The case $s = 1$ of the above theorem was obtained in F. Bethuel's seminal 1991 contribution [Bet91], with some partial results by F. Bethuel and Zheng X. [BZ88]. The approach by Bethuel relies on the method of good and bad cubes that was explained in this lecture. This methodology was later on adapted to higher order Sobolev spaces ($s \in \mathbb{N}_*$) by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15], complemented by a whole set of new tools suited to the extra rigidity of higher order spaces and that were also briefly explained in this lecture.

In parallel, H. Brezis and P. Mironescu [BM15] gave an orthogonal proof covering the range $0 < s < 1$. Their approach is well-suited for fractional spaces, and has the extra advantage of being conceptually much simpler, but cannot be adapted to the case where the is at least one full derivative involved, as proved by the authors themselves [BM15, Lemma 4.9]. Let us give the key idea behind. As we saw in Section 2.3, choosing a suitable grid *via* an averaging argument, the homogeneous extension procedure produces a sequence of almost smooth maps converging weakly (we shall come back to this notion in Lecture 4) to our target map $u \in W^{1,p}(Q^m;\mathscr{N})$. The crucial observation by Brezis and Mironescu is that, when $0 < s < 1$, a miracle occurs: this construction actually yields strong convergence. Therefore, to obtain the strong density of almost smooth maps, it is not needed to distinguish good and bad cubes and relying on different methods to approximate the target map u depending on the type of cube, it suffices to perform homogeneous extension *everywhere*. The density of smooth maps under the topological assumption $\pi_{\lfloor sp \rfloor}(\mathscr{N}) \simeq \{0\}$ then follows from the singularity removal procedure (shrinking), although a nontrivial technical work is needed to adapt the required estimates and constructions to the fractional setting.

The missing case $s > 1$ noninteger was then obtained in [Det23], using the method of good and bad cubes by Bethuel, the additional tools introduced by Bousquet, Ponce, and Van Schaftingen, and new estimates suited to the fractional case. We mention that the approach also covers the case $0 < s < 1$ that was handled by a different approach in [BM15], although it is conceptually more difficult.

Partial contributions in this direction include the work by M. Escobedo [Esc88], who obtained strong density in the whole range $0 < s < +\infty$ for sphere-valued mappings; the approach by P. Hajłasz [Haj94], who gave a simpler proof of Bethuel's result for $s = 1$ and when the target is assumed to be $\lfloor p \rfloor$-connected, relying on a method of *almost projection*, which was then pursued by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS13] for $s \geq 1$; the contributions by F. Bethuel [Bet95], T. Rivière [Riv00], and D. Mucci [Muc09] providing partial results in the range $0 < s < 1$ exploiting the extension of traces to bring back to $W^{1,p}$ up to adding an extra dimension; and the work by P. Bousquet [Bou07], and by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS14], who proved the strong density of almost smooth maps respectively for $\mathbb{S}^1$-valued maps in the full range $0 < s < +\infty$, and when the target is $(\lfloor sp \rfloor - 1)$-connected in the range $0 < s < 1$ using the *method of singular projection* devised by R. Hardt and Lin F. [HL87], with roots in the work by H. Federer and W. Fleming [FF60] (see also [Det24] for the adaptation of the method of singular projection to the full range $0 < s < +\infty$ and an $(\lfloor sp \rfloor - 1)$-connected target).

We conclude this survey section by a short discussion about the case of a general domain, which was understood by Hang F. and Lin F. [HL03a]. The density of the class of almost smooth maps remains valid, and only requires technical adaptations to handle the geometry of the domain. The density of smooth maps in the slightly subcritical case $m - 1 \leq sp < m$ also remains valid, since the singularities are isolated, and may then be removed individually. However, in the general case, new obstructions may appear. Indeed, as we briefly explained at the end of Section 2.5, the removal of singularities then requires a global topological construction: we need to be able to extend to Ω an $\mathscr{N}$-valued map defined on an $\lfloor sp \rfloor$-skeleton. If Ω has nontrivial topology, new obstructions may arise from the interplay between the topology of Ω and $\mathscr{N}$. In terms that would require to be defined more precisely, the strong density theorem for an arbitrary domain reads as follows.

Theorem 2.9. *If $sp < m$, then $\mathscr{R}_{m-\lfloor sp \rfloor-1}(\Omega;\mathscr{N})$ is dense in $W^{s,p}(\Omega;\mathscr{N})$, and $C^\infty(\overline{\Omega};\mathscr{N})$ is dense in $W^{s,p}(\Omega;\mathscr{N})$ if and only if any continuous map from a generic $\lfloor sp \rfloor$-skeleton of Ω to $\mathscr{N}$ may be extended to a continuous $\mathscr{N}$-valued map defined on the whole Ω.*

We mention that the latter topological assumption is valid in particular if Ω if $(\lfloor sp \rfloor - 1)$-connected and if $\pi_{\lfloor sp \rfloor}(\mathscr{N}) \simeq \{0\}$. Moreover, the condition $\pi_{\lfloor sp \rfloor}(\mathscr{N}) \simeq \{0\}$ remains necessary for strong density regardless of the topology of the domain. However, in [HL03a], Hang F. and Lin F. have constructed a Sobolev map between projective planes that cannot be approximated by a sequence of smooth maps, although the relevant homotopy group of the target is trivial. An interesting feature of this map is that it has a pointwise singularity at some point, exactly as the hedgehog map from Example 1.2, but this singularity cannot be localized: the map may be approximated strongly by a sequence of mappings that are smooth everywhere except at a point singularity that can be placed wherever we please. One should compare this with the hedgehog map, which cannot be approximated by a sequence of smooth maps in a neighborhood of the origin, regardless of how small it is chosen.

Lecture 3

The singular set of a Sobolev mapping, or when differential geometry comes into play

After having discovered that Sobolev mappings are not always strongly approximable by smooth mappings, we will turn to the question of detecting which mappings can nevertheless by strongly approached. This will show a beautiful connection with one more area of mathematics, namely differential geometry, through the construction of objects such as the Jacobian.

For the sake of simplicity, this lecture will focus on the case $W^{1,k}(\mathbb{B}^{k+1};\mathbb{S}^k)$, where $k \in \mathbb{N}_*$. In particular, since $\pi_k(\mathbb{S}^k) = \mathbb{Z}$, strong density of smooth maps fails. At the end, we shall give some hints about how to extend the ideas presented here to a more general setting.

3.1 From the singular set to the Jacobian

We have seen in Lecture 2 that the class $\mathscr{R}_0(\mathbb{B}^{k+1};\mathscr{N})$ is always dense in $W^{1,k}(\mathbb{B}^{k+1};\mathbb{S}^k)$. As strong density fails, we wish to find a hopefully tractable and enlightening way to determine whether a *given* map $u \in W^{1,k}(\mathbb{B}^{k+1};\mathbb{S}^k)$ can be approximated by smooth maps. Given a map $u \in \mathscr{R}_0(\mathbb{B}^{k+1};\mathscr{N})$, we know what its singular set looks like: it is the set $\mathcal{S}_u \subset \mathbb{B}^{k+1}$ of points where it fails to be continuous.

There is a legitimate reason not to be fully convinced with this definition: we have seen in Lecture 2 that, if the restriction of u on a sphere enclosing one and only one singularity is homotopically trivial, then the shrinking procedure allows to remove this singularity. This suggests that we should not take into account the singularities around which u has trivial topology. Since homotopy classes of maps into the sphere are totally classified by the degree, let us suggest the following definition for the singular set:

$$S_u = \sum_{a \in \mathcal{S}_u} \deg(u, a)\delta_a. \tag{3.1}$$

Here, $\deg(u, a)$ is the degree of the restriction of u to any sphere containing a and no other singularity. (Can you prove that it does not depend on the choice of such sphere?) This way, S_u is now a distribution, concentrated on a finite set of points. In particular, this give zero weight to "fake" singularities, around which u has zero degree and that can be removed by the shrinking procedure.

The drawback of this definition is that it is not obvious how to extend it to an arbitrary $W^{1,k}$ mapping. Indeed, such maps may be wildly discontinuous, even everywhere on the domain. We rely on concepts from differential geometry to define a new object, the *Jacobian*, which will turn to coincide with the singular set defined in (3.1) for maps in the class $\mathscr{R}_0$.

We denote by $\omega_{\mathbb{S}^k}$ the volume form on $\mathbb{S}^k$. We recall that the volume form on a manifold of dimension k is a k-differential form that vanishes nowhere. The following considerations do not depend on the choice of the volume form, so let us work with the standard volume form defined by

$$\omega_{\mathbb{S}^k} = \sum_{j=1}^{k+1}(-1)^{j-1}x_j \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_{k+1},$$

where the hat denotes the omission of the factor with corresponding index. An amusing fact is that it happens that the differential of this form, viewed as a form on the ambient space $\mathbb{R}^{k+1}$, happens to be the standard volume form on $\mathbb{R}^{k+1}$:

$$\mathrm{d}_{\mathbb{R}^{k+1}}\omega_{\mathbb{S}^k} = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{k+1}.$$

This should not be confused with the fact that $\mathrm{d}_{\mathbb{S}^k}\omega_{\mathbb{S}^k} = 0$, since there is no nontrivial differential form of order more than k on a k-dimensional manifold.

Given a map $u \colon \Omega \to \mathbb{S}^k$, we may *pull $\omega_{\mathbb{S}^k}$ back* to Ω along u, defining

$$u^\sharp\omega_{\mathbb{S}^k} = \sum_{j=1}^{k+1}(-1)^{j-1}u_j \mathrm{d}u_1 \wedge \cdots \wedge \widehat{\mathrm{d}u_j} \wedge \cdots \wedge \mathrm{d}u_{k+1}.$$

This expression obviously makes sense for smooth maps. It is also well-defined for $u \in W^{1,k}$ by Hölder's inequality, in which case $u^\sharp\omega_{\mathbb{S}^k}$ is an L^1 map.

An important fact about the pullback of the volume form is that it detects the degree: for every $f \in C^\infty(\mathbb{S}^k;\mathbb{S}^k)$,

$$\deg f = \frac{1}{|\mathbb{S}^k|}\int_{\mathbb{S}^k} f^\sharp\omega_{\mathbb{S}^k}. \tag{3.2}$$

We define the *Jacobian* of u as

$$\langle Ju, \alpha\rangle = -\int_{\mathbb{B}^{k+1}} \mathrm{d}\alpha \wedge u^\sharp\omega_{\mathbb{S}^k} \quad \text{for every } \alpha \in C^\infty_c(\mathbb{B}^{k+1}). \tag{3.3}$$

This way, Ju is a distribution, and actually $Ju = \mathrm{d}(u^\sharp\omega_{\mathbb{S}^k})$ in the sense of distributions.

We shall not attempt to draw the (very rich) history of this fascinating object. It can be tracked down to the pioneering work of C. G. Jacobi, and the distributional definition is due to J. Ball. We refer the reader to the survey by H. Brezis, J. Mawhin, and P. Mironescu [BMM24] and the numerous references therein.

We prove the following formula, which justifies the introduction of such an object.

Proposition 3.1. *For every $u \in \mathscr{R}_0(\mathbb{B}^{k+1};\mathbb{S}^k)$,*

$$Ju = |\mathbb{S}^k| S_u.$$

Proof. Let $\mathcal{S}_u$ be the singular set of u. By the Leibniz formula, we have

$$\mathrm{d}(\alpha u^\sharp\omega_{\mathbb{S}^k}) = \mathrm{d}\alpha \wedge u^\sharp\omega_{\mathbb{S}^k} \quad \text{on } \mathbb{B}^{k+1}\setminus\mathcal{S}_u \text{ for every } \alpha \in C^\infty_c(\mathbb{B}^{k+1}),$$

where we have used the fact that $\mathrm{d}(u^\sharp\omega_{\mathbb{S}^k}) = u^\sharp(\mathrm{d}\omega_{\mathbb{S}^k}) = 0$ on $\mathbb{B}^{k+1}\setminus\mathcal{S}_u$. Therefore, we deduce from Stokes' formula that

$$\begin{aligned}\langle Ju, \alpha\rangle &= \lim_{\varepsilon\to 0} -\int_{\mathbb{B}^{k+1}\setminus\bigcup_{a\in\mathcal{S}_u} B_\varepsilon(a)} \mathrm{d}\alpha\wedge u^\sharp\omega_{\mathbb{S}^k} = \lim_{\varepsilon\to 0} -\int_{\mathbb{B}^{k+1}\setminus\bigcup_{a\in\mathcal{S}_u} B_\varepsilon(a)} \mathrm{d}(\alpha u^\sharp\omega_{\mathbb{S}^k}) \\ &= \lim_{\varepsilon\to 0}\sum_{a\in\mathcal{S}_u}\int_{\partial B_\varepsilon(a)} \alpha u^\sharp\omega_{\mathbb{S}^k} = \sum_{a\in\mathcal{S}_u}\alpha(a)\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(a)} u^\sharp\omega_{\mathbb{S}^k} = |\mathbb{S}^k|\sum_{a\in\mathcal{S}_u}\alpha(a)\deg(u,a),\end{aligned}$$

where the last equality follows from (3.2). Since this holds for every $\alpha \in C^\infty_c(\mathbb{B}^{k+1})$, the conclusion follows. ◻

We now note that the Jacobian is actually continuous with respect to u.

Proposition 3.2. *The map*

$$u \in W^{1,k}(\mathbb{B}^{k+1};\mathbb{S}^k) \mapsto Ju \in \mathscr{D}'(\mathbb{B}^{k+1})$$

is continuous.

The proof is straightforward. Indeed, it suffices to show that, if $u_n \to u$ in $W^{1,k}$, then

$$\langle Ju_n, \alpha\rangle \to \langle Ju, \alpha\rangle \quad \text{for every } \alpha \in C^\infty_c(\mathbb{B}^k).$$

But this follows from the fact that $u^\sharp\omega_{\mathbb{S}^k}$ is made of products of u_j, which is L^∞, times k derivatives of u.

Since the extension by continuity from a dense set is unique, this essentially tells us that Ju is the right object to define the singular set of an arbitrary map $u \in W^{1,k}(\mathbb{B}^{k+1};\mathbb{S}^k)$.

We conclude with the following proposition, whose proof is omitted, about the structure of the Jacobian.

Proposition 3.3. *For every $u \in W^{1,k}(\mathbb{B}^{k+1}; \mathbb{S}^k)$, there exist families of points $\{P_j\}_{j\in\mathbb{N}}$ and $\{N_j\}_{j\in\mathbb{N}}$ in $\overline{\mathbb{B}^{k+1}}$ such that*

$$Ju = |\mathbb{S}^k| \sum_{j\in\mathbb{N}} \delta_{P_j} - \delta_{N_j}$$

and

$$\sum_{j\in\mathbb{N}} |P_j - N_j| \lesssim \int_{\mathbb{B}^{k+1}} |\mathrm{D}u|^k.$$

This proposition tells us that the Jacobian of an arbitrary Sobolev mapping is still made of points, although there may now be (countably) infinitely many of them. It was announced in [Bre03], with roots in the work by J. Bourgain, H. Brezis, and P. Mironescu [BBM04].

3.2 An analytical characterization of the closure of smooth maps

In view of Section 3.1, it is natural to expect that a Sobolev mapping to the sphere can be approximated by smooth maps if and only if its Jacobian vanishes. The following theorem states that it is indeed the case.

Theorem 3.4. *For every $k \in \mathbb{N}_*$,*

$$\overline{C^\infty(\overline{\mathbb{B}^{k+1}}; \mathbb{S}^k)}^{W^{1,k}} = \{u \in W^{1,k}(\mathbb{B}^{k+1}; \mathbb{S}^k) \colon Ju = 0\}.$$

This result is due to F. Bethuel [Bet90] for $k = 2$ and to F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein [BCDH91] in the general case.

Proof. The fact that any strong limit of smooth mappings has zero Jacobian follows readily from Proposition 3.1, which implies in particular that $Ju = 0$ is u is a smooth mapping, combined with the continuity of the Jacobian from Proposition 3.2. We prove the converse inclusion by showing that $Ju = 0$ implies that the restriction of u to the boundary of a generic cube is homotopic to a constant.

We prove that, for a generic cube $\sigma \subset Q^{k+1}$, there exists a sequence of smooth test functions $(\alpha_n)_{n\in\mathbb{N}}$ such that

$$\langle Ju, \alpha_n \rangle \to \int_{\partial\sigma} u^\sharp \omega_{\mathbb{S}^k} = |\mathbb{S}^k| \deg(u, \partial\sigma).$$

We define

$$\alpha_n(x) = \min\Big\{1, \frac{n}{2}\operatorname{dist}_\infty(x, \mathbb{R}^{k+1} \setminus (1+n^{-1})\sigma)\Big\}.$$

Then, α_n is only Lipschitz, but it is easy to see, *via* a smoothing procedure, that $\langle Ju, \alpha\rangle$ is well-defined and vanishes also on Lipschitz compactly supported functions.

We exemplify our main computation by taking $k = 1$ and $\sigma = Q_1 = [-1, 1]^2$ (assuming u is defined on a larger domain). We have

$$\langle Ju, \alpha_n\rangle = -\int_{(1+n^{-1})\sigma\setminus(1-n^{-1})\sigma} \mathrm{d}\alpha_n \wedge u^\sharp\omega_{\mathbb{S}^1}.$$

We restrict our attention to the region $(1 - n^{-1}, 1 + n^{-1}) \times (-1 + n^{-1}, 1 - n^{-1})$. On this region, $\mathrm{d}\alpha_n = -\frac{n}{2}\mathrm{d}x_1$. Therefore, letting β_2 being the part of $u^\sharp\omega_{\mathbb{S}^1}$ that contains only a $\mathrm{d}x_2$, we have

$$-\int_{(1-n^{-1},1+n^{-1})\times(-1+n^{-1},1-n^{-1})} \mathrm{d}\alpha_n \wedge u^\sharp\omega_{\mathbb{S}^1} = \frac{n}{2}\int_{1-n^{-1}}^{1+n^{-1}} \left(\int_{-1+n^{-1}}^{1-n^{-1}} \beta_2(x_1, \cdot)\right) \mathrm{d}x_1.$$

Assuming that we are at a Lebesgue point, we find

$$\frac{n}{2}\int_{1-n^{-1}}^{1+n^{-1}} \left(\int_{-1+n^{-1}}^{1-n^{-1}} \beta_2(x_1, \cdot)\right) \mathrm{d}x_1 \to \int_{\{1\}\times(-1,1)} u^\sharp\omega_{\mathbb{S}^1}.$$

Doing this for all faces,

$$\langle Ju, \alpha_n\rangle \to \int_{\partial\sigma} u^\sharp\omega_{\mathbb{S}^k}.$$

Since we are at a Lebesgue point for a generic cube (can you make this assertion precise and give a rigorous proof of it?), the conclusion follows. □

3.3 Towards more general targets

In this section, we give some ideas about how to extend the result from Theorem 3.4 to more general targets than $\mathbb{S}^k$. Looking into the proof, we see that the important point is that we are able to detect the homotopy class of a map *via* differential forms. We therefore focus on this specific aspect of the argument.

The starting point is that we do not need to restrict ourselves to the volume form of the target $\mathscr{N}$. We could as well define

$$\langle J_\omega u, \alpha\rangle = -\int_{\mathbb{B}^{k+1}} \mathrm{d}\alpha \wedge u^\sharp \omega,$$

with ω being a k-form on $\mathscr{N}$. In order for this object to vanish on limits of smooth maps, looking at the proof of Proposition 3.1, we should required that ω is a closed form, that is, $\mathrm{d}\omega = 0$.

For the degree, we had formula (3.2) that tells us that the degree is obtained as

$$\deg f = \frac{1}{|\mathbb{S}^k|}\int_{\mathbb{S}^k} f^\sharp \omega_{\mathbb{S}^k}.$$

If we can find a more general situation where *all* k-homotopy classes are detected by such formulas involving closed k-forms, we can essentially perform the same reasoning and characterize the closure of smooth maps *via* the objects J_ω.

Such a situation is provided for instance by the de Rham and Hurewicz theorems. Indeed, the de Rham theorem asserts that differential forms are dual to *cycles*, in the context of *homology*. That is, the condition that

$$\int_{\mathbb{S}^k} f^\sharp \omega = 0 \quad \text{for every closed } k\text{-form } \omega \text{ on } \mathscr{N} \tag{3.4}$$

says that $f^\sharp = 0$ *in homology*. However, it is known in algebraic topology that homology and homotopy may strongly differ. The link between both is provided by the Hurewicz theorem. For instance, if $\mathscr{N}$ is $(k-1)$-connected (that is, all its homotopy groups of order between 1 and $k-1$ are trivial) and if $\pi_k(\mathscr{N})$ has no torsion, then the homology and the homotopy of $\mathscr{N}$ in degree k can be identified. In this context, condition (3.4) implies that $f^\sharp = 0$ *in homotopy*, that is, f is homotopic to a constant.

The reasoning is at the core of the following result by F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein [BCDH91].

Theorem 3.5. *Assume that $\mathscr{N}$ is $(k-1)$-connected and that $\pi_k(\mathscr{N})$. Then, a map $u \in W^{1,k}(\mathbb{B}^{k+1}; \mathscr{N})$ can be approximated by smooth mappings if and only if*

$$J_\omega u = 0 \quad \textit{for every closed form } \omega \textit{ on } \mathscr{N}.$$

Just for the curiosity of the reader aware of algebraic topology, let us point out that the topological assumption on $\mathscr{N}$ is actually stronger than needed. Looking into the above reasoning, we see that the only thing that is required is that being zero in homology of degree k implies being zero in homotopy of degree k. The condition that we have imposed implies that homology and homotopy can be completely identified, which is more than needed. Actually, it suffices that the *Hurewicz morphism* from k-homotopy to k-homology is injective. This is implied by, but strictly weaker than, the previous assumption. For instance, the torus $\mathbb{T}^2$ is not simply connected, has trivial π_2 so in particular the Hurewicz morphism in degree 2 is injective, but has nontrivial H_2, so the Hurewicz morphism is not an isomorphism. We say that the 2-cycles in $\mathbb{T}^2$ are *non-spherical*, that is, they cannot be realized as homotopy classes of continuous maps. In particular, they cause no obstruction to the density of smooth maps, since they cannot arise as a topological singularity of a Sobolev mappings.

Extension of this to even more general target manifolds have been studied notably by R. Hardt and T. Rivière [HR03, HR08]. The key point is that, given any nontorsion homotopy class $a \in \pi_k(\mathscr{N})$, there is an algorithm involving tree-graphs, due to S. P. Novikov, that allows one to associate to any map $f\colon \mathbb{S}^k \to \mathscr{N}$ a differential form f^a on $\mathbb{S}^k$ such that

$$\int_{\mathbb{S}^k} f^a$$

computes exactly the topological degree of f with respect to a. This provides a criterion to detect which Sobolev mappings can be approximated by smooth maps when the k-order homotopy group of the target has no torsion. We shall not give further details about this, since it would bring us too far away from the scope of these notes and would require the introduction of additional notions from *rational homotopy theory*.

We also mention the work by M. Giaquinta, G. Modica, and J. Souček, culminating in the monumental monograph [GMS98a, GMS98b], and subsequent contributions by various authors, including also D. Mucci, aiming at detecting topological singularities of a Sobolev mapping as the "holes" in its graph, defined in the sense of distributions, using the language of geometric measure theory.

3.4 Extensions to $0 < s < 1$

In this section, we briefly explain some efforts that have been performed to extend the results explained in this lecture to the range $0 < s < 1$. The first difficulty that arises is how to even *define* the object Ju, and we focus on this issue. Indeed, for $u \in W^{1,k}$, the definition of Ju involves integrating the L^1 function $u^\sharp\omega_{\mathbb{S}^k}$ against a test function. However, when $0 < s < 1$, the object $u^\sharp\omega_{\mathbb{S}^k}$ may no longer make sense as an L^1 function, since the pullback operation by u involves one full derivative of u.

Let us show one formal computation that illustrates the philosophy to overcome this issue. The starting point is that a Sobolev map $u \in W^{s,p}(\mathbb{B}^{k+1}; \mathscr{N})$ has an extension

$U \in C^\infty(\mathbb{B}^{k+1} \times (0,1); \mathbb{R}^{k+1})$, which is essentially obtained by letting $U(x,t) = \rho_t * u(x)$, where ρ is a convolution kernel. Now the issue is that U need not take its values into $\mathbb{S}^k$, but luckily, the expression used to define ω still makes sense over $\mathbb{R}^{k+1}$. We define $\tilde{\alpha}(x,t) = \alpha(x)$, so that $\tilde{\alpha}$ extends α to $\mathbb{B}^{k+1} \times (0,1)$. We now compute formally

$$\begin{aligned}\langle Ju, \alpha \rangle = -\int_{\mathbb{B}^{k+1}} \mathrm{d}\alpha \wedge u^\sharp \omega_{\mathbb{S}^k} &= -\int_{\mathbb{B}^{k+1}} \mathrm{d}\tilde{\alpha} \wedge U^\sharp \omega_{\mathbb{S}^k} \\ &= \int_{\mathbb{B}^{k+1}\times(0,1)} \mathrm{d}(\mathrm{d}\tilde{\alpha} \wedge U^\sharp \omega_{\mathbb{S}^k}) = -\int_{\mathbb{B}^{k+1}\times(0,1)} \mathrm{d}\tilde{\alpha} \wedge U^\sharp (\mathrm{d}\omega_{\mathbb{S}^k}). \end{aligned} \tag{3.5}$$

We note that $\mathrm{d}\omega$ does not vanish here, since ω is now considered as a form over $\mathbb{R}^{k+1}$. As we explained, we thus have

$$\mathrm{d}\omega_{\mathbb{S}^k} = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{k+1}.$$

The twist here is that, since U is smooth, the right-hand-side of (3.5) makes sense even for $u \in W^{s,p}$ with $0 < s < 1$, even if the left-hand-side may not make sense.

The strategy to make this reasoning rigorous is in two key steps: (i) prove rigorously equation (3.5) when $u \in W^{1,k}(\mathbb{B}^{k+1}; \mathbb{S}^k)$; and (ii) prove that the right-hand-side of (3.5) is well-defined and continuous on $W^{s,p}(\mathbb{B}^{k+1}; \mathbb{S}^k)$ with $0 < s < 1$ and $sp = k$. This requires a suitable choice of the extension U, and relies on the *inverse theory of traces in weighted Sobolev spaces*.

We refer the reader to notably to the papers by J. Bourgain, H. Brezis, and P. Mironescu [BBM05], P. Bousquet and P. Mironescu [BM14], and D. Mucci [Muc22] for results in this direction.

3.5 Higher dimensional domains

We conclude by giving a short hint about what happens to what we explained in this lecture when the dimension of the domain is taken to be $m \geq k+1$. The model case is the study of the space $W^{1,k}(\mathbb{B}^m; \mathbb{S}^k)$ In this case, we have seen that the singular set of an almost smooth map in the suitable dense class, that is, the class $\mathscr{R}_{m-k-1}$, is of dimension $m-k-1$. The definition of the Jacobian given in equation (3.3) still makes sense, but one then has to take α to be a smooth compactly supported $(m-k-1)$-form, in order for the integrand to be an m-form. In the case $m = k+1$, we work with 0-forms, which are exactly functions.

We comment on the generalization of Proposition 3.3. As we explained, it asserts that in the limiting case of a general map $u \in W^{1,k}(\mathbb{B}^{k+1}; \mathbb{S}^k)$, the singular set consists of a countable union of points, while for mappings in the class $\mathscr{R}_0$, it is a *finite* union of points. In the general case where the singular set of a map of the class $\mathscr{R}_{m-k-1}$ is a finite union of $(m-k-1)$-affine spaces, one may wonder what should be the corresponding limiting singular set, if any. A beautiful result by G. Alberti, S. Baldo, and G. Orlandi [ABO03] asserts that the Jacobian can in this case be represented as the integration on a $(m-k-1)$-dimensional *rectifiable current*. For readers unfamiliar with the language of geometric measure theory, one may simply think of the singular set being a countable union of Lipschitz surfaces. Even more beautiful is the fact that *any* such current which is a boundary is the singular set of some Sobolev mapping.

Similarly, all the references from Sections 3.3 and 3.4 deal with the general case of an m-dimensional domain.

Lecture 4

The weak density problem: connections and dipoles

In this last session, we will briefly study what happens when strong convergence is replaced by weak convergence. We will focus on a famous technique to obtain positive results of weak approximation, namely eliminating the topological singularities of a Sobolev mapping along connections between them.

4.1 An introduction to weak density

In Lectures 1 and 2, we discovered that, in a striking contrast with the classical setting of *real-valued* Sobolev functions, smooth mappings need not be dense in the Sobolev space of mappings. Moreover, we provided a complete classification of the cases where strong density holds or fails, as well as a class of *almost smooth* mappings. In Lecture 3, we gave some hints about how, in a non strong density case, one can determine which specific Sobolev mappings are nevertheless approachable by smooth maps.

There is nevertheless one additional attempt to overcome the lack of strong density: weaken the notion of convergence under consideration. Indeed, what we were requiring in the three first lectures is convergence in the full $W^{1,p}$ norm, which is demanding a lot. What if we try to replace this by a weakest form of convergence? In this lecture, we restrict to the case of $W^{1,p}$, that is, $s = 1$.

Definition 4.1. *We say that a sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{1,p}(\mathbb{B}^m; \mathscr{N})$ weakly converges to u, and we write $u_n \rightharpoonup u$, whenever*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{B}^m} |\mathrm{D}u_n|^p < +\infty,$$

and $u_n \to u$ in measure.

In other words, weak convergence amounts to boundedness in $W^{1,p}$ along with a very weak convergence for the functions themselves. We could also have required convergence in L^1 or almost everywhere and this would have given rise to the same notion, possibly up to extraction of a subsequence (can you prove it?).

An important remark concerns the relation between this notion and the usual notion of linear weak convergence, inherited from the dual space of L^p. When $1 < p < +\infty$, both these notions coincide thanks to the reflexivity of L^p, *via* the Banach–Alaoglu theorem. (Can you provide the complete argument?) When $p = 1$ however, weak convergence is somehow ill-behaved due to the lack of reflexivity of L^1, and both notions cease to coincide. Actually, the linear notion is stronger than the one from Definition 4.1, and is equivalent to it under an equi-integrability requirement. From now on, we shall work with weak convergence as defined in Definition 4.1 without any further comments.

It is straightforward from the definition that strong convergence implies weak convergence. Additionally, weak convergence is somehow a natural notion to work with in the context of problems of partial differential equations or calculus of variations, since it is in general much more easy to obtain weak convergence for sequences of approximate solutions or approximated minimizers than strong convergence. It can even happen that weak convergence holds while strong convergence fails.

The natural question to ask is whether smooth mappings are always weakly dense in $W^{1,p}(\mathbb{B}^m;\mathscr{N})$, and if not, if it is possible to characterize the cases where weak density happens. Let us importantly mention that here, by *weak density*, we implicitly mean *weak sequential density*. Indeed, since weak convergence is not metrizable, density and sequential density do not coincide. It has been proved by F. Bethuel [Bet91] that the closure of the space of smooth mappings with respect to the topology induced by weak convergence is the whole Sobolev space of mappings. This comes from the fact that the mappings in the class $\mathscr{R}$ are always weakly approximable, as will become clear later on, combined with the strong density of the class $\mathscr{R}$. On the other hand, as we will see, if we look for weak approximability, things become more involved. For the sake of conciseness, we shall make the abuse of language of speaking about *weak density* when denoting weak sequential density. To use a notation similar as for strong density, we write $H^{1,p}_{\mathrm{W}}(\Omega;\mathscr{N})$ for the set of all $u \in W^{1,p}(\Omega;\mathscr{N})$ such that there exists a sequence of smooth $\mathscr{N}$-valued mappings on Ω weakly converging to u.

We start by the fact that, when p is not an integer, then weak density faces exactly the same obstruction as strong density. In this opportunity, we meet again our dear old friend the hedgehog.

Example 4.2. We define once again $u_0 \in W^{1,p}(\mathbb{B}^3;\mathbb{S}^2)$, with $1 \leq p < 3$ by

$$u_0(x) = \frac{x}{|x|}.$$

We claim that u_0 cannot be weakly approximated by smooth mappings in $W^{1,p}$ when $p > 2$. Indeed, assume by contradiction that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{B}^3;\mathbb{S}^2)$

such that $u_n \rightharpoonup u$. By the polar integration formula and Fatou's lemma, we find

$$\int_0^1 \left(\liminf_{n\to+\infty} \int_{\partial B_r} |\mathrm{D}u_n|^p \right) \mathrm{d}r \leq \liminf_{n\to+\infty} \int_0^1 \left(\int_{\partial B_r} |\mathrm{D}u_n|^p \right) \mathrm{d}r = \liminf_{n\to+\infty} \int_{\mathbb{B}^3} |\mathrm{D}u_n|^p < +\infty.$$

Therefore, for every $r \in (0,1)$, up to extraction of a subsequence (here possibly depending on r), we have

$$\sup_{n\in\mathbb{N}} \int_{\partial B_r} |\mathrm{D}u_n|^p < +\infty.$$

Another easy Fubini–Tonelli argument provides convergence in measure, whence we deduce weak convergence on ∂B_r. By the Rellich–Kondrashov theorem, this implies the uniform convergence of $u_{n|\partial B_r}$ to $u_{|\partial B_r}$, which allows to derive the required contradiction by a homotopy argument as for strong density. □

Exactly as for strong density, this can be extended to the general setting of an arbitrary target with nontrivial $\pi_{\lfloor p \rfloor}$, leading to the following theorem. It is due to F. Bethuel [Bet91].

Theorem 4.3. *Assume that $p < m$. Then, $H^{1,p}_{\mathrm{W}}(\Omega;\mathscr{N}) = W^{1,p}(\Omega;\mathscr{N})$ implies that $\pi_{\lfloor p \rfloor}(\mathscr{N}) \simeq \{0\}$.*

In Example 1.2, our argument could not be extended to p integer due to the failure of the embedding of $W^{1,p}$ into L^∞ in the limiting case $p = m$. However, this can be overcome by the limiting embedding of $W^{1,p}$ into VMO. As far as weak density is concerned, this failure is more dramatic, since the argument relies crucially on the *compactness* of the embedding, which is lost in the limiting case even when using VMO as a target space for the embedding.

Therefore, the interesting cases in which to study weak density are when $p \in \mathbb{N}_*$ and $\pi_p(\mathscr{N}) \not\simeq \{0\}$, so that strong density fails. We show by an example that there indeed *are* instances where strong density fails while weak density holds when $p \in \mathbb{N}$.

Example 4.4. Let u_0 again denote the hedgehog map. We claim that u_0 can be weakly approximated by smooth maps (although we already know that strong approximation is not feasible).

For this purpose, given any $\eta > 0$, we define a map $\varphi_\eta \colon \mathbb{S}^2 \to \mathbb{S}^2$ such that $\varphi_\eta = \mathrm{id}$ outside of a ball of radius η around the north pole, φ_η has degree zero, and $\sup_{\eta>0} \|\varphi_\eta\|_{W^{1,2}} < +\infty$. For instance, we can rely on the map $\psi \colon \mathbb{B}^2 \to \mathbb{S}^2$ defined by

$$\psi(x) = \left(\sin(\pi|x|) \cdot \frac{x}{|x|}, \cos(\pi(1-|x|)) \right),$$

which has degree -1 and maps $\overline{\mathbb{B}^2}$ on the north pole. It then suffices to scale ψ to a ball of radius η, and then to proceed to a suitable truncation and rescaling to match it with the identity.

We then define $u_\eta \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$ as the homogeneous extension of φ_η:

$$u_\eta(x) = \varphi_\eta\Big(\frac{x}{|x|}\Big).$$

By the properties of the homogeneous extension, these maps enjoys the following features: (i) $\sup_{\eta>0} \|u_\eta\|_{W^{1,2}} < +\infty$; (ii) $u_\eta \to u_0$ in measure; (iii) $u_{\eta|\partial B_r}$ has degree 0 for every η and every $r \in (0,1)$. By this third property, we may apply the shrinking procedure to approximate each u_η *strongly* in $W^{1,2}$ by smooth mappings into $\mathbb{S}^2$. A diagonal argument allows us to conclude. □

4.2 The dipole construction

The key feature in Example 4.4 was the possibility to insert, in an arbitrarily thin neighborhood of a line connecting the central singularity to the boundary, a degree -1 map that canceled the degree 1 singularity of the hedgehog at the origin. We use the fact that the integrability exponent here is 2 to have the proper scaling to ensure that the construction scales appropriately when the region where the surgery is performed thins out to a line.

In this section, we explain a more general insertion of singularities technique, called the *dipole construction*. For this purpose, we need to define a notion that we call a *pencil-shape neighborhood*. Let A and $B \in \overline{\mathbb{B}^{k+1}}$. We assume that the coordinates $x = (x_1, x')$ in $\mathbb{R}^{k+1}$ have been chosen so that x_1 is the coordinate in the direction of the segment from A to B, and x' is the coordinate in a hyperplane orthogonal to this segment. Given $\delta > 0$ and $\gamma > 0$, we define $g_{\delta,\lambda}(x_1) = \min\{\delta, \gamma \operatorname{dist}(x_1, \{A, B\})\}$. The set $U_{\delta,\gamma}$ is defined as

$$U_{\delta,\gamma} = \{x \in \overline{\mathbb{B}^{k+1}} \colon x_1 \in (A, B), |x'| < g_{\delta,\gamma}(x_1)\}.$$

Here, $x_1 \in (A, B)$ is an abuse of notation for the fact that the orthogonal projection of x on the line passing through A and B actually lies between A and B. This construction is taken from the work of G. Alberti, S. Baldi, and G. Orlandi [ABO03], with roots in the pioneering work of H. Brezis, J.-M. Coron, and E. Lieb [BCL86] who devised the dipole construction. We refer to [ABO03, Figure 2] for a very nice depiction of the construction.

Proposition 4.5. *Let $A, B \in \overline{\mathbb{B}^{k+1}}$, let $a \in \mathscr{N}$, and let $\alpha \in \pi_k(\mathscr{N})$. For every $\delta > 0$ and $\gamma > 0$, there exists a map $u_{\delta,\gamma} \in W^{1,k}(\mathbb{B}^{k+1}; \mathscr{N})$ which satisfies the following properties:*

1. *$u_{\delta,\gamma} = a$ outside of $U_{\delta,\gamma}$;*

2. $u_{\delta,\gamma} \in C(\overline{\mathbb{B}^{k+1}} \setminus \{A, B\}; \mathscr{N})$;

3. $[u_{\delta,\gamma}]_A = -\alpha$ *and* $[u_{\delta,\gamma}]_B = \alpha$;

where $[u_{\delta,\gamma}]_P$ *denotes the homotopy class of the restriction of* $u_{\delta,\gamma}$ *on a small sphere centered on* P, *which does not depend on the choice of the radius provided that it is taken to be sufficiently small. In addition, for any* $\eta > 0$, *there exists a choice of* $\delta > 0$ *and* $\gamma > 0$ *sufficiently small so that* $|U_{\delta,\gamma}| \leq \eta$ *and*

$$\int_{\mathbb{B}^{k+1}} |\mathrm{D}u_{\delta,\gamma}|^k \leq C_\alpha |B - A|,$$

where $C_\alpha >$ *is a constant depending only on* α.

Proof. The proof is in the spirit of Example 4.4, and consists in inserting a copy of a map in the class α on each slice of the pencil-shaped neighborhood $U_{\delta,\gamma}$. More specifically, if $f_\alpha \colon \mathbb{B}^k \to \mathscr{N}$ is a representative of the homotopy class α with basepoint a, we let

$$u_{\delta,\gamma}(x) = \begin{cases} f_\alpha\left(\frac{x'}{g_{\delta,\gamma}(x_1)}\right) & \text{if } x \in U_{\delta,\gamma}, \\ a & \text{otherwise.} \end{cases}$$

It is readily checked that this map satisfies properties 1, 2, and 3.

The proof of the last statement follows from careful estimates of $\mathrm{D}u_{\delta,\gamma}$. Instead of providing the details, we give the following intuitive explanation behind this fact. Due to the scaling of the k energy of the gradient in dimension k, the integral of $|\mathrm{D}u_{\delta,\gamma}|^k$ on each disk orthogonal to (A, B) inside $U_{\delta,\gamma}$ is essentially the same regardless of the choice of the parameters, that is, regardless of the scaling. Therefore, we end up integrating a constant (depending on the homotopy class through the map chosen to realize it) on the segment (A, B). □

The dipole construction allows one to insert two opposite singularities in a Sobolev map, while modifiying it only on a small set along a line connecting the inserted singularities, and with controlled energy. We note importantly that one of the points may be chosen to lie on the boundary of the domain, in case this amounts to inserting only *one* singularity, that is connected to a fictitious one on the boundary.

In order to derive weak density results from this construction, since the idea is to use the dipole procedure to cancel the existing singularities of a Sobolev mapping, we need a result which would give us a control on the minimal length required to pair all singularities of an arbitrary map in the class $\mathscr{R}$. This is the purpose of the next section, where we explore the notion of *connection*.

4.3 Connections of sphere-valued maps

In this section, we explain how to connect all the singularities of a function of the class $\mathscr{R}_0$ with the length of the connection being controlled by the energy of the map. The main result of the section is the following.

Proposition 4.6. *Let $u \in \mathscr{R}_0(\mathbb{B}^{k+1}; \mathbb{S}^k)$. There exists $y \in \mathbb{S}^k$ a regular point for u such that the set $u^{-1}(\{y\})$ is a smooth 1-dimensional manifold whose length satisfies*

$$\mathscr{H}^1(u^{-1}(\{y\})) \lesssim \int_{\mathbb{B}^{k+1}} |\mathrm{D}u|^k.$$

Before we prove this proposition, let us explain why it provides the desired connection. First of all, we observe that the curves constituting $u^{-1}(\{y\})$ may be given a canonical orientation induced by u. Indeed, at each point of these curves, the pullback by u of a basis of the tangent plane of the sphere at the point y orients a plane transversal to the curve at this point, and it suffices to take an orientation on the curve compatible with this plane and the orientation on the ambient space.

Then, by the formula for the degree given by the sum of the signs of the Jacobian at all points in the preimage of a regular point, we know that any sufficiently small sphere centered around a singularity a of u should cross $u^{-1}(\{y\})$ a number of times equal to the degree of u on this sphere, counted with a sign accounting for the orientation. Otherwise stated, in the language of geometric measure theory,

$$\partial(u^{-1}(\{y\})) = S_u \quad \text{in } \mathbb{B}^{k+1}.$$

We shall not attempt to give the formal definitions required to explain rigorously this identity, but it tells us exactly that $u^{-1}(\{y\})$ is a connection for the singular set of u, taking the multiplicities of the singularities into account.

Proof. Since u is smooth in $\mathbb{B}^{k+1} \setminus \mathscr{S}_u$, it follows from the Morse–Sard theorem that almost every point $y \in \mathbb{S}^k$ is a regular point for u. For such points, $u^{-1}(\{y\})$ is indeed a smooth 1-dimensional manifold by virtue of the submersion theorem.

We now invoke Federer's co-area formula:

$$\int_{\mathbb{S}^k} \mathscr{H}^1(u^{-1}(\{y\}))\,\mathrm{d}y = \int_{\mathbb{B}^{k+1}} Ju,$$

where the Jacobian is computed as the determinant of the differential from $\mathbb{R}^{k+1}$ to the tangent bundle to $\mathbb{S}^k$. Since the Jacobian is the square root of the product of the singular values of $\mathrm{D}u$, while the norm of the differential is the square root of the sum of those singular values, the arithmetico-geometric inequality implies that

$$Ju \leq \frac{1}{k^{k/2}} |\mathrm{D}u|^k.$$

Hence,

$$\int_{\mathbb{S}^k} \mathscr{H}^1(u^{-1}(\{y\}))\,\mathrm{d}y \lesssim \int_{\mathbb{B}^{k+1}} |\mathrm{D}u|^k.$$

This ensures the existence of a point y, regular image of u, satisfying

$$\mathscr{H}^1(u^{-1}(\{y\}))\,\mathrm{d}y \lesssim \int_{\mathbb{B}^{k+1}} |\mathrm{D}u|^k. \qquad \square$$

This proposition and the dipole construction are the key tools to prove the following weak density theorem, due to F. Bethuel [Bet90].

Theorem 4.7. *For every $k \in \mathbb{N}_*$,*

$$H^{1,k}_{\mathrm{W}}(\mathbb{B}^{k+1}; \mathbb{S}^k) = W^{1,k}(\mathbb{B}^{k+1}; \mathbb{S}^k).$$

The general strategy is exactly as in the example of the hedgehog: starting from a map in $\mathscr{R}_0$, construct a connection with controlled length by the means of Proposition 4.6, and then use the dipole construction as in Proposition 4.5 to approximate it weakly, with linear bound on the energy of the approximating sequence in terms of the energy of the approximated mapping, by maps in the class $\mathscr{R}_0$ all whose topological singularities are trivial. By a further *strong* approximation by smooth maps, this shows that any map in $\mathscr{R}_0$ can be weakly approximated by smooth maps, and that they can be chosen so that their Sobolev energy is controlled linearly by the Sobolev energy of the limiting map. The conclusion follows by the strong density of the class $\mathscr{R}_0$. Here it is important to have a control on the necessary energy to perform a weak approximation. Indeed, by the dipole insertion procedure, it is actually *always* possible to weakly approximate mappings in the class $\mathscr{R}$, connecting carelessly all their singularities to the boundary of the domain. However, without a control on the energy necessary for this approximation, one could not conclude from the strong density of the class $\mathscr{R}$ *via* a diagonal argument. Indeed, it could very well happen that for some function $u \in W^{1,k}(\mathbb{B}^m; \mathscr{N})$, for any sequence $(u_n)_{n\in\mathbb{N}}$ in the class $\mathscr{R}$ strongly converging to u, the Sobolev energy required to weakly approximate the mappings u_n blows up when $n \to +\infty$, preventing the use of a diagonal argument.

We only mention that, to obtain a complete proof of Theorem 4.7, some extra technicalities are required. Indeed, our dipole construction only permits to insert a dipole along a segment, and inside a region where the map is constant. There are two main paths to overcome this issue. The first one is to modify the dipole construction to make it work along an arbitrary curve. One then only needs to slightly perturb the map to be approximated in order to make it constant on a neighborhood of the connecting curves. The other one is to work instead along a *minimal connection*. Indeed, since we have a control on the length of *some* connection, in particular, it gives a control on the length of the *best* connections, which can be proven to be lines. The advantage is that this allows to insert dipoles along lines, but then the map on which to work is no longer constant on these lines. A more thorough adaptation of the dipole insertion technique is therefore required. An intermediate idea is to use a polyhedral approximation theorem from geometric measure theory to approximate the curves $u^{-1}(\{y\})$ by a piecewise linear curve.

4.4 The current state of the art concerning weak density

We conclude this lecture by a short (possibly non-exhaustive) review of the current state of the art concerning weak density, with a brief mention of the key ideas behind each result.

- In [Haj94], P. Hajłasz has generalized Theorem 4.7 by proving that

  $$H^{1,p}_{\mathrm{W}}(\Omega;\mathscr{N}) = W^{1,p}(\Omega;\mathscr{N})$$

  for every $p \in \mathbb{N}_*$, whenever $\mathscr{N}$ is $(p-1)$-connected. This relies on the *method of almost projection*: under this topological assumption, one may prove that there exists a map from the ambient space to $\mathscr{N}$, which is not a retraction (can you prove that this is impossible?), but which coincides with the identity on $\mathscr{N}$ *except on a set of small measure*. One then applies the standard strong density theorem, and projects back into $\mathscr{N}$ using this almost projection, which yields a weakly converging sequence of smooth maps. We mention that, by a standard lifting argument, this result can also be applied if $\pi_1(\mathscr{N})$ is nontrivial but finite, using a lifting to move to the universal covering and kill the π_1.

- In [PR03], M. R. Pakzad and T. Rivière have proved weak density in $W^{1,2}$ for more general target manifolds. Their result covers in particular Hajłasz's one, and also applies to some targets to which Hajłasz's theorem does not apply. Their approach relies on the dipole insertion technique, as we explained in this lecture. They key idea is that, under the assumptions they make, the manifold $\mathscr{N}$ can be transformed into a bouquet of spheres. The co-area argument and the dipole insertion technique can then be applied for each sphere separately. In [Pak03], M. R. Pakzad has adapted this technique to give an alternative proof of the weak density in $W^{1,1}$.

- In [HR15], R. Hardt and T. Rivière have proved that $H^{2,2}_{\mathrm{W}}(\mathbb{B}^5;\mathbb{S}^3) = W^{2,2}(\mathbb{B}^5;\mathbb{S}^3)$. Although their proof is restricted to $W^{2,2}$ for technical reasons, the key ingredient in their argument is the finiteness of the relevant homotopy group in this situation: $\pi_4(\mathbb{S}^3) = \mathbb{Z}/2\mathbb{Z}$.

- In [Bet20], F. Bethuel has constructed the first (and only, up to date) counterexample to the weak density property in $W^{1,p}$ when p is integer. Namely, he proved that

  $$H^{1,3}_{\mathrm{W}}(\Omega;\mathbb{S}^2) \neq W^{1,3}(\Omega;\mathbb{S}^2) \quad \text{whenever } \Omega \subset \mathbb{R}^m \text{ with } m \geq 4.$$

  Here, the topological obstructions to the strong density property arise from the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$, which generate $\pi_3(\mathbb{S}^2) \simeq \mathbb{Z}$.

  A major difference between $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$ (in which we proved that weak density holds) and $W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$ (which is the basic situation in which Bethuel's counterexample applies) lies in the rate of growth of the energy with respect to the relevant degree. Indeed, it can be shown that the minimum energy required to produce a $W^{1,2}$ map of degree d from $\mathbb{S}^2$ to itself is of order $|d|$, that is, the energy grows linearly with respect to the degree. On the other hand, it was shown by T. Rivière [Riv98] that the minimum $W^{1,3}$ energy of a map $\mathbb{S}^3 \to \mathbb{S}^2$ having Hopf degree d grows like $|d|^{4/3}$, that is, the growth is *superlinear*. The lower bound on the energy follows from a Sobolev inequality applied to the integral formula for the Hopf degree, while the optimality follows by constructing explicitly a map achieving this bound.

  Starting from this observation, and using the theory of *scans* developed by R. Hardt and T. Rivière [HR03, HR08], one reaches the following crucial difference with the case $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$. In the latter, if one wishes to connect together two singularities of degree d and $-d$, one can transport a topological charge of order d along a segment of length L connecting them, and the cost will be $|d| \cdot L$. But for $W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$, if one wishes to connect two Hopf singularities of degree d and $-d$, the cost of transporting a Hopf charge of order d along a segment of length L between them will be of order $|d|^{\frac{3}{4}} \cdot L$. The appropriate language to denote this phenomenon is that we are in presence of a *branched transportation problem* with exponent $\alpha = \frac{3}{4}$; see [BCM09] for a review of the field of branched optimal transportation.

  Here is where comes the crucial difference between both situations: the exponent $\alpha = \frac{3}{4} = 1-\frac{1}{4}$ is the critical exponent for branched optimal transportation in dimension 4 below (and starting from) which the Lebesgue measure cannot be irrigated. Using this feature, F. Bethuel was able to construct a sequence of mappings $u_n \colon \mathbb{B}^4 \to \mathbb{S}^2$

(which even belong to the class $\mathscr{R}_0$!) whose Sobolev energy do not grow faster than n^3, but whose weak approximation energy grows at least like $n^3 \ln n$. The conclusion then follows from the application of a nonlinear uniform boundedness principle (see [HL03b] for a principle specific to weak approximation, and [MVS19] for a general version), which implies that the maps u_n can be suitably scaled and patched together to construct a map $u \in W^{1,3}(\mathbb{B}^4; \mathbb{S}^2)$ which cannot be weakly approximated by a sequence of smooth maps. This principle is nothing more than a nonlinear counterpart of the classical Banach–Steinhaus theorem in functional analysis.

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