Quantitative suboptimal Sobolev embeddings

Antoine Detaille Petru Mironescu

September 18, 2025

Dedicated to the memory of Haïm Brezis, with immense admiration and gratitude

Abstract

We present two new families of integral inequalities involving Sobolev seminorms associated with compact Sobolev embeddings. These inequalities quantify the fact that, on "many" small balls of a given domain, quantitative Sobolev embeddings are "much better" than predicted by scaling arguments.

Une version quantitative des inégalités de Sobolev sous-optimales

Résumé

On présente deux nouvelles familles d'inégalités intégrales impliquant des seminormes de Sobolev, associées aux injections de Sobolev compactes. Ces inégalités quantifient le fait que, sur « un grand nombre » de boules contenues dans un domaine fixé, les inégalités de Sobolev quantitatives se comportent « bien mieux » que suggéré par un argument de mise à l'échelle.

1 Introduction and notation

Let $\Omega \subset \mathbb{R}^N$ be a domain, $0 \leq s < \infty$, and $1 \leq p \leq \infty$. We define the following seminorm on the space $W^{s,p}(\Omega)$:

$$|u|^p_{W^{s,p}(\Omega)} = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \int_{\Omega_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+sp}} \, \mathrm{d}z \, \mathrm{d}h, & \text{if } s \notin \mathbb{N} \\ \displaystyle\int_\Omega |D^s u|^p, & \text{if } s \in \mathbb{N} \end{cases}, \tag{1.1}$$

AD: Universite Claude Bernard Lyon 1, CNRS, Centrale Lyon, INSA Lyon, Université Jean Monnet, ICJ UMR5208, 69622 Villeurbanne, France.
ETH Zürich, Department of Mathematics, Rämistrasse 101, Zürich 8092, Switzerland. antoine.detaille@math.ethz.ch
PM: Universite Claude Bernard Lyon 1, CNRS, Centrale Lyon, INSA Lyon, Université Jean Monnet, ICJ UMR5208, 69622 Villeurbanne, France. mironescu@math.univ-lyon1.fr
Keywords: Sobolev embeddings, Weak L^p-estimates
Mots-clés : Injections de Sobolev, Estimées L^p faibles
MSC 2020 classification: 46E35

(with the obvious modifications when $p = \infty$). In the above: (i) $\Delta_h^m u$ denotes the m-th order difference operator of step h applied to u; (ii) $m = m(s) := \lceil s \rceil$ is the smallest integer $> s$; (iii) $\Omega_h = \Omega_{h,m}$ is the set of all $x \in \Omega$ such that $[x, x + mh] \subset \Omega$. (We recall that $\Delta_h^k u$ is defined by induction on k as follows: $\Delta_h^0 u = u$ and $\Delta_h^{k+1} u(z) = \Delta_h^k u(z + h) - \Delta_h^k u(z)$.) We note that the above definition includes the value $s = 0$, in which case $W^{0,p}$ is identified with L^p, and $|u|_{W^{0,p}} = \|u\|_p$.

We mention importantly that, when $s > 1$ is not an integer and Ω is sufficiently smooth (Lipschitz bounded suffices), the above seminorm is equivalent to the usual one involving the first order differences of $D^{\lceil s \rceil} u$. For a proof, we refer the reader, e.g., to R.A. DeVore and R.C. Sharpley [3, Section 6], but the result was known before. (To be precise, [3] only proves the equivalence between the full Sobolev norms, obtained by adding the L^p norm to the seminorms. Combining this with the Poincaré inequality

$$\inf_{P \in \mathscr{P}_{\lfloor s \rfloor}} \|u - P\|_p \leq C|u|_{W^{s,p}}, \forall\, u \colon \Omega \to \mathbb{R} \tag{1.2}$$

(were $\mathscr{P}_{\lfloor s \rfloor}$ denotes the set of all polynomial of degree at most $\lfloor s \rfloor$, see below), one obtains the equivalence of the seminorms. In turn, once the equivalence between the full norms is established, (1.2) is obtained *via* a straightforward compactness argument.)

In what follows, we consider parameters satisfying

$$0 \leq \alpha < s < \infty, 1 \leq p < \infty, 1 \leq q \leq \infty, \alpha - \frac{N}{q} < s - \frac{N}{p}. \tag{1.3}$$

For such parameters, and if $\omega \subset \mathbb{R}^N$ is a"sufficiently smooth" domain (Lipschitz bounded suffices), we have the well-known Sobolev embedding $W^{s,p}(\omega) \hookrightarrow W^{\alpha,q}(\omega)$. Moreover, we have Sobolev-type inequalities involving seminorms. In order to give examples of such inequalities, let ℓ be the largest integer $< s$, i.e.,

$$\ell = \ell(s) := \begin{cases} \lfloor s \rfloor, & \text{if } s \text{ is not an integer} \\ s - 1, & \text{if } s \text{ is an integer} \end{cases}. \tag{1.4}$$

Denote by $\mathscr{P}_\ell$ the set of all polynomials in $\mathbb{R}[X_1, \ldots, X_N]$ of degree at most ℓ (that we identify with the set of corresponding polynomial functions on ω). If, for example, $\omega = B_r(x)$ is a ball, then we have the following Sobolev inequality:

$$\inf_{P \in \mathscr{P}_\ell} |u - P|_{W^{\alpha,q}(B_r(x))} \leq C r^\beta |u|_{W^{s,p}(B_r(x))} \tag{1.5}$$

for some finite constant C depending on the parameters in (1.3), but not on r or x. Here,

$$\beta := s - \alpha - N\left(\frac{1}{p} - \frac{1}{q}\right). \tag{1.6}$$

When $r = 1$ and $x = 0$, the inequality (1.5) is deduced from the usual Sobolev inequality for the full $W^{s,p}$ norm on $\mathbb{B}^N$ — which can be obtained, e.g., by combining H. Triebel [7, Section 3.3] with the identification of Sobolev spaces as special types of Triebel–Lizorkin spaces [7, Sections 2.2.2 and 2.3.5] and the use of an extension operator [7, Section 3.3.4] — combined with the Poincaré inequality

$$\inf_{P\in\mathscr{P}_\ell} \|u-P\|_p \le C|u|_{W^{s,p}},\ \forall\, u\colon \mathbb{B}^N \to \mathbb{R}.$$

(We recall that the quantity $\|u\|_p + |u|_{W^{s,p}}$ is equivalent to the usual Sobolev norm on $\mathbb{B}^N$.) The case of a general r and x follows by a translation and scaling argument.

We consider next a general domain Ω and the collection of balls contained in Ω, described by the set

$$U := \{(x,r)\colon x\in\Omega,\ r>0,\ B_r(x)\subset\Omega\}.$$

As a consequence of (1.5), we have

$$\inf_{P\in\mathscr{P}_\ell} |u-P|_{W^{\alpha,q}(B_r(x))} \le C r^\beta |u|_{W^{s,p}(\Omega)},\ \forall\,(x,r)\in U. \tag{1.7}$$

Once we *fix* $(x,r)\in U$, the estimate (1.7) cannot be improved. Indeed, it suffices to start from a non-trivial bump function φ and to test (1.7) with $u := \varphi((\cdot - x)/r)$.

The purpose of this note is to exhibit a number of (in general, integral) inequalities showing that, for "many" couples $(x,r)\in U$, estimate (1.7) can be dramatically improved.

The next section provides a prototypical example of such inequalities, inspired by an inequality of J. Van Schaftingen presented in [5, equation (1.6)].

2 A model inequality

We consider parameters as in (1.3) and (1.6) and we let $\ell = \ell(s)$ as in (1.4). Consider some additional parameter t such that

$$p \le t \le \infty. \tag{2.1}$$

Proposition 2.1. *Assume that $s>0$ is not an integer. Then there exists a finite constant $C = C(\alpha,s,p,q,N)$ such that*

$$\left(\int_U \frac{\inf_{P\in\mathscr{P}_\ell} |u-P|^t_{W^{\alpha,q}(B_r(x))}}{r^{\beta t}} \frac{1}{r^{N+1}}\,\mathrm{d}x\,\mathrm{d}r\right)^{1/t} \le C|u|_{W^{s,p}(\Omega)} \tag{2.2}$$

(with the obvious modification when $t=\infty$).

Remark 2.2. In Proposition 2.1 and its counterparts stated below, the implicit assumption on $u\colon\Omega\to\mathbb{R}$ is that $|u|_{W^{s,p}(\Omega)}<\infty$. This ensures that

$$U \ni (x,r) \mapsto \inf_{P\in\mathscr{P}_\ell} |u-P|_{W^{\alpha,q}(B_r(x))}$$

is continuous. (This assertion can be easily proved by using the continuity of dilations and translations in Sobolev spaces and the fact that $\mathscr{P}_\ell$ is a finite dimensional normed space.)

Therefore, in what follows we do not discuss any measurability issue. □

Remark 2.3. Estimate (2.2) with $t = p$ implies that, for almost every $x \in \Omega$ and every positive constant K, the set

$$\left\{r\colon\ r > 0,\, B_r(x) \subset \Omega,\, \inf_{P\in\mathscr{P}_\ell} |u - P|_{W^{\alpha,q}(B_r(x))} > K r^{\beta+N/p} |u|_{W^{s,p}(\Omega)}\right\}$$

has zero density at $r = 0$, and thus, for "most" of small $r > 0$ it holds that

$$\inf_{P\in\mathscr{P}_\ell} |u - P|_{W^{\alpha,q}(B_r(x))} \le K r^{\beta+N/p} |u|_{W^{s,p}(\Omega)},$$

which is a significant improvement of (1.7). □

Proof of Proposition 2.1. We start by noting that it suffices to prove (2.2) at the endpoints $t = p$ and $t = \infty$. The intermediate case $p < t < \infty$ then follows (with a constant C independent of t) *via* the Hölder inequality.

When $t = \infty$, (2.2) is nothing but (1.7). Let next $t = p$. Consider some $\widetilde{s}$ sufficiently close to s such that

$$\max(\ell, \alpha) < \widetilde{s} < s \tag{2.3}$$

and

$$\widetilde{\beta} \coloneqq \widetilde{s} - \alpha - N\left(\frac{1}{p} - \frac{1}{q}\right) > 0. \tag{2.4}$$

For such $\widetilde{s}$ we have, thanks to the Sobolev inequality (1.5),

$$\inf_{P\in\mathscr{P}_\ell} |u - P|_{W^{\alpha,q}(B_r(x))} \lesssim r^{\widetilde{\beta}} |u|_{W^{\widetilde{s},p}(B_r(x))},\ \forall\,(x,r) \in U. \tag{2.5}$$

Let $\delta \coloneqq \beta - \widetilde{\beta} = s - \widetilde{s} > 0$. Using (2.5), the fact that $\beta = \delta + \widetilde{\beta}$, and the definition (1.1), we obtain

$$\begin{aligned}
&\int_U \frac{\inf_{P\in\mathscr{P}_\ell} |u - P|^p_{W^{\alpha,q}(B_r(x))}}{r^{\beta p}} \frac{1}{r^{N+1}}\,\mathrm{d}x\,\mathrm{d}r \\
&\quad\lesssim \int_U \frac{1}{r^{\delta p+N+1}} \int_{\mathbb{R}^N} \int_{(B_r(x))_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p}}\,\mathrm{d}z\,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}r \eqqcolon J.
\end{aligned} \tag{2.6}$$

We next note that, if $(x,r) \in U$, $h \in \mathbb{R}^N$, and $z \in (B_r(x))_h$, then $z \in \Omega_h$, $x \in B_r(z)$, and $r > |h|/2$. Therefore, Tonelli's theorem implies that

$$\begin{aligned}
J &\le \int_{\mathbb{R}^N} \int_{\Omega_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p}} \int_{|h|/2}^{\infty} \int_{B_r(z)} \mathrm{d}x \frac{1}{r^{\delta p+N+1}}\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}h \\
&\lesssim \int_{\mathbb{R}^N} \int_{\Omega_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p}} \int_{|h|/2}^{\infty} \frac{1}{r^{\delta p+1}}\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}h \lesssim \int_{\mathbb{R}^N} \int_{\Omega_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p+\delta p}}\,\mathrm{d}z\,\mathrm{d}h.
\end{aligned} \tag{2.7}$$

Recalling (1.1) and the fact that $s = \delta + \widetilde{s}$, we obtain (2.2) with $t = p$ from (2.6) and (2.7). $\square$

In the next section, we investigate the counterpart of estimate (2.2) when s is an integer. As we will see, new phenomena occur in this case.

3 The case where s is an integer

We consider again parameters as in (1.3), (1.6), and (2.1), with s integer (and thus $\ell = s-1$). We investigate whether (2.2) holds in this setting. This is indeed the case at the endpoint $t = \infty$ (see (1.7)). However, (2.2) *fails* at the other endpoint, $t = p$. This is a consequence of the following result.

Lemma 3.1. *Let $0 \le \ell < s$ be integers. Let $0 \le \alpha \le s$ and $1 \le q \le \infty$. Let $u(x) := x_1^s$, $\forall\, x = (x_1, \dots, x_N) \in \mathbb{R}^N$. Then there exists a finite constant $C = C(\ell, s, \alpha, q, N)$ such that*

$$\inf_{P \in \mathscr{P}_\ell} |u - P|_{W^{\alpha,q}(B_r(x))} \ge C r^{N/q+s-\alpha} = C r^{\beta - N/p}, \forall\, r > 0, \forall\, x \in \mathbb{R}^N. \tag{3.1}$$

The same holds for any non-zero polynomial $u \in \mathbb{R}[X_1, \dots, X_N]$, homogeneous of degree s.

Corollary 3.2. *Let $\Omega \subset \mathbb{R}^N$ be any non-empty bounded domain. Under the assumptions (1.3) and with β as in (1.6), the above u belongs to $W^{s,p}(\Omega)$, but satisfies*

$$\int_{\{r>0 : B_x(r) \subset \Omega\}} \frac{\inf_{P \in \mathscr{P}_\ell} |u - P|^p_{W^{\alpha,q}(B_r(x))}}{r^{\beta p}} \frac{1}{r^{N+1}}\, \mathrm{d}r = \infty, \forall\, x \in \Omega.$$

In particular, we have

$$\int_U \frac{\inf_{P \in \mathscr{P}_\ell} |u - P|^p_{W^{\alpha,q}(B_r(x))}}{r^{\beta p}} \frac{1}{r^{N+1}}\, \mathrm{d}x\, \mathrm{d}r = \infty.$$

Proof of Lemma 3.1. We first investigate the case where $\alpha < s$. Consider the finite dimensional vector space $E := \mathbb{R}u + \mathscr{P}_\ell$. Then $g \mapsto |g|_{W^{\alpha,q}(\mathbb{B}^N)}$ is a norm on E (since $\alpha < s$). Since $u \notin \mathscr{P}_\ell$, there exists some constant $C > 0$ such that

$$\inf_{P \in \mathscr{P}_\ell} |u - P|_{W^{\alpha,q}(\mathbb{B}^N)} \ge C. \tag{3.2}$$

Estimate (3.1) (with C the constant in (3.2)) follows from (3.2) *via* a suitable affine homothety.

The case where $\alpha = s$ is similar: (3.2) follows from the identity

$$|u - P|_{W^{\alpha,q}(\mathbb{B}^N)} = C > 0, \forall\, P \in \mathscr{P}_\ell. \qquad \square$$

As we will see below, this limiting case is the only pathological one when $p > 1$, while the case where $p = 1$ is more involved.

We present below two approaches allowing to settle the case where $p > 1$. The first one relies on a weak L^p-estimate and interpolation, requires no smoothness of Ω, and fails for

$p = 1$. The second one relies on Sobolev embeddings (whence a smoothness assumption on Ω); it also works when $p = 1$, *but only if* $N \geq 2$. The case where $p = 1$ and $N = 1$ requires a separate argument; in this situation, we actually show that the counterpart of (2.2) *fails*, in the whole range $p \leq t < \infty$ — only the case $t = \infty$ holds, as a direct consequence of (1.7).

In order to understand the statement of the *weak* L^p-estimate valid when $t = p$, it is convenient to rephrase the inequality provided by Proposition 2.1. Consider, on U, the measure $\mu := \dfrac{1}{r^{N+1}}\, dx\, dr$. Then Proposition 2.1 asserts that the map

$$U \ni (x, r) \mapsto G_u(x, r) := \frac{\inf_{P \in \mathscr{P}_\ell} |u - P|_{W^{\alpha,q}(B_r(x))}}{r^\beta} \tag{3.3}$$

satisfies

$$\|G_u\|_{L^t(U,\mu)} \leq C|u|_{W^{s,p}(\Omega)}. \tag{3.4}$$

The next result provides a weak version of (3.4) when $t = p > 1$.

Proposition 3.3. *Assume that s is an integer and $p > 1$. Then there exists a finite constant $C = C(\alpha, s, p, q, N)$ such that*

$$\|G_u\|_{L^{p,w}(U,\mu)} \leq C|u|_{W^{s,p}(\Omega)}.$$

Equivalently, Proposition 3.3 asserts that the set

$$E_{u,\lambda} := \left\{(x, r) \in U \colon \frac{\inf_{P \in \mathscr{P}_\ell} |u - P|_{W^{\alpha,q}(B_r(x))}}{r^\beta} > \lambda\right\} \tag{3.5}$$

satisfies

$$\mu(E_{u,\lambda}) \leq C^p \frac{|u|^p_{W^{s,p}(\Omega)}}{\lambda^p}.$$

Let us note that, in the scale of Lorentz spaces, the conclusion of Proposition 3.3 is optimal. Indeed, the map u considered in Lemma 3.1 satisfies, for each $0 < \sigma < \infty$, $G_u \notin L^{p,\sigma}(U, \mu)$.

Proof of Proposition 3.3. The proof relies on the following obvious inequality. If $f : \Omega \to \mathbb{R}$ is measurable, $(x, r) \in U$, and $0 < \widetilde{p} < \infty$, then

$$\|f\|_{L^{\widetilde{p}}(B_r(x))} \leq C r^{N/\widetilde{p}} [\mathscr{M}(|f|^{\widetilde{p}})(x)]^{1/\widetilde{p}}, \tag{3.6}$$

where $\mathscr{M}$ denotes the (centered or uncentered) maximal operator and the finite constant C depends only on $\widetilde{p}$ and N.

Let $1 \leq \widetilde{p} < p$ be sufficiently close to p in order to have

$$\widetilde{\beta} := s - \alpha - N\left(\frac{1}{\widetilde{p}} - \frac{1}{q}\right) > 0, \tag{3.7}$$

and thus $W^{s,\widetilde{p}}(B_r(x)) \hookrightarrow W^{\alpha,q}(B_r(x))$. For such $\widetilde{p}$ and $(x,r)\in U$ we have, *via* (1.5) and (3.6),

$$\inf_{P\in\mathscr{P}_\ell} |u-P|_{W^{\alpha,q}(B_r(x))} \lesssim r^{\widetilde{\beta}}|u|_{W^{s,\widetilde{p}}(B_r(x))} \lesssim r^{\widetilde{\beta}+N/\widetilde{p}}[\mathscr{M}(|D^s u|^{\widetilde{p}})(x)]^{1/\widetilde{p}}. \tag{3.8}$$

Recalling (3.3), (1.6), and (3.7), we find from (3.8) that

$$G_u(x,r) \lesssim r^{N/p}[\mathscr{M}(|D^s u|^{\widetilde{p}})(x)]^{1/\widetilde{p}}, \forall\, (x,r)\in U. \tag{3.9}$$

In turn, estimate (3.9) implies that, for every $x\in\Omega$, the section

$$E_{u,\lambda,x} := \{r>0\colon (x,r)\in E_{u,\lambda}\} \tag{3.10}$$

of $E_{u,\lambda}$ satisfies

$$r^N \gtrsim \frac{\lambda^p}{[\mathscr{M}(|D^s u|^{\widetilde{p}})(x)]^{p/\widetilde{p}}}, \ \forall\, r\in E_{u,\lambda,x},$$

and therefore

$$\int_{E_{u,\lambda,x}} \frac{1}{r^{N+1}}\,\mathrm{d}r \lesssim \inf_{r\in E_{u,\lambda,x}} \frac{1}{r^N} \lesssim \frac{1}{\lambda^p}[\mathscr{M}(|D^s u|^{\widetilde{p}})(x)]^{p/\widetilde{p}}, \forall\, x\in\Omega. \tag{3.11}$$

Integrating (3.11) with respect to x, we find that

$$\mu(E_{u,\lambda}) \lesssim \frac{1}{\lambda^p}\int_\Omega [\mathscr{M}(|D^s u|^{\widetilde{p}})(x)]^{p/\widetilde{p}} \lesssim \frac{1}{\lambda^p}\||D^s u|\|^p_{L^p(\Omega)} = \frac{1}{\lambda^p}|u|^p_{W^{s,p}(\Omega)},$$

where the second inequality follows from the maximal function theorem. □

Remark 3.4. An inspection of the above proof shows that we have proved more than stated in Proposition 3.3. More specifically, (3.11) combined with the maximal function theorem yields the following. With G_u as in (3.3), there exists a bounded sublinear operator $T\colon W^{s,p}(\Omega)\to L^p(\Omega;[0,\infty])$ such that

$$\int_{\{r>0: G_u(x,r)>\lambda\}} \frac{1}{r^{N+1}}\,\mathrm{d}r \le \frac{[Tu(x)]^p}{\lambda^p}, \forall\, u\in W^{s,p}(\Omega), \forall\, x\in\Omega, \forall\,\lambda>0.$$

(Indeed, $Tu(x) := C[\mathscr{M}(|D^s u|^{\widetilde{p}}(x)]^{1/\widetilde{p}}$ has, for sufficiently large C, the above properties.) □

By interpolation, we deduce from Proposition 3.3 the following result.

Proposition 3.5. *Assume that s is an integer. Let $p<t\le\infty$, and assume: either (i) $p>1$, or (ii) $t=\infty$. Then there exists a finite constant $C=C(\alpha,s,p,q,N,t)$ such that*

$$\left(\int_U \frac{\inf_{P\in\mathscr{P}_\ell}|u-P|^t_{W^{\alpha,q}(B_r(x))}}{r^{\beta t}}\frac{1}{r^{N+1}}\,\mathrm{d}x\,\mathrm{d}r\right)^{1/t} \le C|u|_{W^{s,p}(\Omega)} \tag{3.12}$$

(with the obvious modification when $t=\infty$).

Let us note that, by contrast with Proposition 2.1, the constant C in (3.12) *must depend on*

t. (This follows from Corollary 3.2.) The case $t = \infty$ can be obtained without the assumption $p > 1$, as it does not rely on Proposition 3.3 but only on the Sobolev inequality (1.7).

Proof. The case $t = \infty$ amounts to (1.7). Let $p < t < \infty$. By (a very easy part of) the Marcinkiewicz interpolation theorem — see, e.g., L. Grafakos [4, Proposition 1.1.14] for the exact statement that we use — we have

$$\|G_u\|_{L^t(U,\mu)} \le C(p,t)\|G_u\|^{p/t}_{L^{p,w}(U,\mu)}\|G_u\|^{1-p/t}_{L^\infty(U,\mu)} \lesssim |u|_{W^{s,p}(\Omega)},$$

where for the last inequality we have used Proposition 3.3 and (1.7). □

We next turn to the case where s is an integer and $p = 1$. We first prove that, *when* $N \ge 2$, the conclusion of Proposition 3.5 still holds in this setting.

Proposition 3.6. *Let Ω be a Lipschitz bounded domain. Assume that s is an integer and $N \ge 2$. Let $1 < t \le \infty$. Then there exists a finite constant $C = C(\alpha, s, q, \Omega, t)$ such that*

$$\left(\int_U \frac{\inf_{P\in\mathscr{P}_\ell}|u-P|^t_{W^{\alpha,q}(B_r(x))}}{r^{\beta t}}\frac{1}{r^{N+1}}\,dx\,dr\right)^{1/t} \le C|u|_{W^{s,1}(\Omega)} \tag{3.13}$$

(with the obvious modification when $t = \infty$). Similarly if $D^s u$ is a finite Borel measure.

Proof. When $t = \infty$, (3.13) amounts to (1.7) and thus holds true. It therefore suffices to find some $\varepsilon = \varepsilon(\alpha, q, N) > 0$ such that (3.13) holds when $1 < t < 1 + \varepsilon$, and finish the proof *via* the Hölder inequality (as in the proof of Proposition 2.1). For $1 < t < \dfrac{N}{N-1}$, define $\widetilde{s} > s - 1$ through the equation

$$\widetilde{s} - \frac{N}{t} = s - \frac{N}{1}. \tag{3.14}$$

Since $N \ge 2$, we have the embedding $W^{s,1}(\mathbb{B}^N) \hookrightarrow W^{\widetilde{s},t}(\mathbb{B}^N)$. (See V.A. Solonnikov [6] when $s = 1$. The case where $s \ge 2$ is a straightforward consequence of the case where $s = 1$.) In view of (3.14) and of the fact that $s > \alpha$, for sufficiently small $\varepsilon > 0$, if $1 < t < 1 + \varepsilon$ then $\widetilde{s} > \alpha$. For such t, Proposition 2.1 implies that

$$\left(\int_U \frac{\inf_{P\in\mathscr{P}_\ell}|u-P|^t_{W^{\alpha,q}(B_r(x))}}{r^{\beta t}}\frac{1}{r^{N+1}}\,dx\,dr\right)^{1/t} \lesssim |u|_{W^{\widetilde{s},t}(\Omega)}. \tag{3.15}$$

(We note that, by (1.3), (1.7), and (3.14), the constant β in (3.15) is the same as the one in (1.6).)

On the other hand, Ω being Lipschitz and bounded, we have the embedding $W^{s,1}(\Omega) \hookrightarrow W^{\widetilde{s},t}(\Omega)$. Since, moreover, Ω is connected and $\widetilde{s} > s - 1$, we have the Sobolev inequality

$$|u|_{W^{\widetilde{s},t}(\Omega)} \lesssim |u|_{W^{s,1}(\Omega)} \tag{3.16}$$

(which is an avatar of (1.5)).

We obtain (3.13) from (3.15) and (3.16).

Finally, once we know that (3.13) holds when $D^s u \in L^1(\Omega)$, the validity of its analogue when $D^s u$ is a finite Borel measure is established *via* a standard procedure, based on smoothing and the Fatou lemma. □

Remark 3.7. In the statement of Proposition 3.6, the assumption that Ω is Lipschitz and bounded can be relaxed to the requirement that the estimate (3.16) holds for t sufficiently close to 1.
On the other hand, the proof of Proposition 3.6 provides an alternative proof of Proposition 3.5 under the (superfluous) extra assumption that Ω is Lipschitz and bounded. □

Question 3.8. Is it true that the conclusion of Proposition 3.6 holds in any domain Ω (without any smoothness assumption)?

Question 3.9. Unlike the proof of Proposition 3.5, which relies on the limiting weak-type estimate provided by Proposition 3.3 when $t = p$, the proof of Proposition 3.6 provides *directly* a strong-type estimate when $t > p$, but says nothing about $t = p$. Is there a weak-type estimate in the limiting case $t = p = 1$, that is, does the counterpart of Proposition 3.3 hold when $p = 1$?

When $N = 1$, the proof of Proposition 3.6 breaks down due to the non-embedding of $W^{1,1}$ in $W^{1/p,p}$ when $p > 1$. Actually, the situation is more dramatic: the conclusion of Proposition 3.6 *fails* when $N = 1$, as Corollary 3.11 below shows — except if $t = \infty$, which is covered by Proposition 3.5.

Lemma 3.10. *Let $N = 1$, $s \geq 1$ be an integer, $p = 1$, $\Omega = (-1, 1)$, and $1 \leq t < \infty$. Let u be a map in Ω such that $u(x) = 0$ if $x \leq 0$ and $u(x) = x^{s-1}$ if $0 < x \leq 1/2$. Then, with G_u as in (3.3), we have $\|G_u\|_{L^{t,\infty}(U,\mu)} = \infty$.*

Corollary 3.11. *Let $N = 1$, $s \geq 1$ an integer, $p = 1$, and $1 \leq t < \infty$. Then there exists some $u \in W^{s,1}(\Omega)$ such that $\|G_u\|_{L^{t,\infty}(U,\mu)} = \infty$.*

Proof of Lemma 3.10. If $r < 1/4$ and $|x| \leq r/2$, then $B_r(x) \supset B_{r/2}(0)$, and therefore, for some $C > 0$, we have

$$\inf_{P \in \mathscr{P}_\ell} |u - P|_{W^{\alpha,q}(B_r(x))} \geq \inf_{P \in \mathscr{P}_\ell} |u - P|_{W^{\alpha,q}(B_{r/2}(0))} = C r^{1/q+s-1-\alpha} = C r^\beta, \tag{3.17}$$

where we have used: (i) a scaling argument for the first equality; (ii) the fact that u is not a polynomial near the origin to justify the fact that $C > 0$; (iii) (1.6).

We find that

$$G_u(x, r) \geq \lambda \quad \text{for every } 0 < \lambda < C, 0 < r < 1/4, |x| < r/2.$$

Therefore, if $\lambda < C$ and $E_{u,\lambda}$ is as in (3.5), then we have

$$\mu(E_{u,\lambda}) \gtrsim \int_0^{1/4} \int_{\{|x|<r/2\}} \frac{1}{r^2}\, dx\, dr = \int_0^{1/4} \frac{1}{r}\, dr = \infty. \qquad \square$$

Sketch of proof of Corollary 3.11. Consider a function u as in Lemma 3.10, with the additional properties that u is compactly supported and $D^s u$ is a finite Borel measure. By smoothing and the Fatou lemma, there exists a sequence $(v_j) \subset C_c^\infty((-1,1))$ such that $\|v_j^{(s)}\|_{L^1} \le 1$ and $\|G_{v_j}\|_{L^{t,w}(U,\mu)} \to \infty$. By rescaling and translating these maps, there exist mutually disjoint intervals $I_k \subset \Omega$ and maps $u_k \in C_c^\infty(I_k)$ such that $\|u_k^{(s)}\|_{L^1} \le 2^{-k}$ and $\|G_{u_k}\|_{L^{t,w}(U_k,\mu)} \to \infty$ (where U_k is the set U adapted to I_k). Then $u := \sum_k u_k$ satisfies the requirements of the corollary. $\square$

4 Two-norms families of inequalities

The inequalities presented in Sections 2 and 3 involve the L^t norm on U with respect to the measure $\mu = \dfrac{1}{r^{N+1}}\,\mathrm{d}x\,\mathrm{d}r$. One may wonder whether there exists a more general family of inequalities that would involve: (i) the L^{t_1} norm with respect to some weighted measure in the variable r; (ii) the L^{t_2} norm with respect to the measure $\mathrm{d}x$ in the variable x, that would specialize to the results in Section 2 when $t_1 = t = t_2$. This is the main topic of this section. In this perspective, the relevant substitutes of U are its sections

$$U_x := \{r > 0 \colon B_r(x) \subset \Omega\},\ x \in \Omega,\ \text{and } U^r := \{x \in \Omega \colon B_r(x) \subset \Omega\},\ r > 0.$$

Case 1. $t_1 \le t_2$. As in the previous sections, we start with the fractional regularity case. We keep the same notation as above.

Proposition 4.1. *Assume that $s > 0$ is not an integer.*

(1) *Let $p \le t_1 \le t_2 < \infty$. Then there exists a finite constant $C = C(\alpha, s, p, q, N, t_1)$ such that*

$$\left(\int_\Omega \left(\int_{U_x} \frac{\inf_{P \in \mathscr{P}_\ell(B_r(x))} |u - P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}} \frac{1}{r^{N t_1/t_2 + 1}}\,\mathrm{d}r\right)^{t_2/t_1} \mathrm{d}x\right)^{1/t_2} \le C|u|_{W^{s,p}(\Omega)}. \tag{4.1}$$

(2) *Let $p \le t_1 < \infty$. Then there exists a finite constant $C = C(\alpha, s, p, q, N, t_1)$ such that*

$$\left(\int_{U_x} \frac{\inf_{P \in \mathscr{P}_\ell(B_r(x))} |u - P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}} \frac{1}{r}\,\mathrm{d}r\right)^{1/t_1} \le C|u|_{W^{s,p}(\Omega)},\ \forall\, x \in \Omega. \tag{4.2}$$

Remark 4.2. Estimate (4.2) is, formally, the limiting case $t_2 = \infty$ of (4.1). Similar remarks for Propositions 4.3, 4.4, 4.7, 4.8 below. $\square$

Proof. (1) For the comfort of notation, here and in all the proofs of this section, we denote by I the left-hand side of the inequality to be shown.

Let $\widetilde{s}$ be such that (2.3) and (2.4) hold. Then, thanks to (1.5) and (2.5), we have

$$\frac{\inf_{P \in \mathscr{P}_\ell(B_r(x))} |u - P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}} \lesssim \frac{|u|^p_{W^{\widetilde{s},p}(B_r(x))}}{r^{\delta p}} |u|^{t_1 - p}_{W^{s,p}(\Omega)}, \tag{4.3}$$

where we recall that $\delta = \beta - \widetilde{\beta} = s - \widetilde{s} > 0$. Therefore,

$$I \lesssim |u|_{W^{s,p}(\Omega)}^{(t_1-p)/t_1} \left(\int_\Omega \left(\int_{U_x} \int_{\mathbb{R}^N} \int_{(B_r(x))_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p}} \, \mathrm{d}z \, \mathrm{d}h \frac{1}{r^{\delta p + N t_1/t_2 + 1}} \, \mathrm{d}r \right)^{t_2/t_1} \mathrm{d}x \right)^{1/t_2}. \tag{4.4}$$

We let F be the set of all $(x, r, h, z) \in \Omega \times (0, \infty) \times \mathbb{R}^N \times \Omega$ such that $B_r(x) \subset \Omega$ and $z \in (B_r(x))_h$, and we define

$$f(x, r, h, z) := \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p}} \chi_F(x, r, h, z),$$

so that (4.4) rewrites as

$$I^{t_1} \lesssim |u|_{W^{s,p}(\Omega)}^{t_1-p} \left(\int_\Omega \left(\int_{(0,\infty)\times\mathbb{R}^N\times\Omega} f \, \mathrm{d}z \, \mathrm{d}h \frac{1}{r^{\delta p + N t_1/t_2 + 1}} \, \mathrm{d}r \right)^{t_2/t_1} \mathrm{d}x \right)^{t_1/t_2}.$$

Since $t_2 \geq t_1$, we may invoke the Minkowski integral inequality to deduce that

$$\begin{aligned}
I^{t_1} &\lesssim |u|_{W^{s,p}(\Omega)}^{t_1-p} \int_{(0,\infty)\times\mathbb{R}^N\times\Omega} \left(\int_\Omega f^{t_2/t_1} \, \mathrm{d}x \right)^{t_1/t_2} \mathrm{d}z \, \mathrm{d}h \frac{1}{r^{\delta p + N t_1/t_2 + 1}} \, \mathrm{d}r \\
&\lesssim |u|_{W^{s,p}(\Omega)}^{t_1-p} \int_{\mathbb{R}^N} \int_{\Omega_h} \int_{|h|/2}^\infty \left(\int_{B_r(z)} \frac{|\Delta_h^m u(z)|^{t_2 p/t_1}}{|h|^{t_2(N+\widetilde{s}p)/t_1}} \, \mathrm{d}x \right)^{t_1/t_2} \frac{1}{r^{\delta p + N t_1/t_2 + 1}} \, \mathrm{d}r \, \mathrm{d}z \, \mathrm{d}h \\
&\lesssim |u|_{W^{s,p}(\Omega)}^{t_1-p} \int_{\mathbb{R}^N} \int_{\Omega_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p}} \int_{|h|/2}^\infty \frac{r^{N t_1/t_2}}{r^{N t_1/t_2 + 1 + \delta p}} \, \mathrm{d}r \, \mathrm{d}z \, \mathrm{d}h \\
&\lesssim |u|_{W^{s,p}(\Omega)}^{t_1-p} \int_{\mathbb{R}^N} \int_{\Omega_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+sp}} \, \mathrm{d}z \, \mathrm{d}h \lesssim |u|_{W^{s,p}(\Omega)}^{t_1}.
\end{aligned}$$

This concludes the proof when $t_2 < \infty$.

(2) Using (4.3) and arguing as above, we find that

$$\begin{aligned}
I^{t_1} &\lesssim |u|_{W^{s,p}(\Omega)}^{t_1-p} \int_{U_x} \frac{|u|^p_{W^{\widetilde{s},p}(B_r(x))}}{r^{\delta p}} \frac{1}{r} \, \mathrm{d}r \\
&\lesssim |u|_{W^{s,p}(\Omega)}^{t_1-p} \int_{\mathbb{R}^N} \int_{\Omega_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p}} \int_{|h|/2}^\infty \frac{1}{r^{\delta p + 1}} \, \mathrm{d}r \, \mathrm{d}z \, \mathrm{d}h \\
&\lesssim |u|_{W^{s,p}(\Omega)}^{t_1-p} \int_{\mathbb{R}^N} \int_{\Omega_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+sp}} \, \mathrm{d}z \, \mathrm{d}h = |u|_{W^{s,p}(\Omega)}^{t_1}.
\end{aligned}$$

□

We now state our result for integer order regularity. The case $t_1 = t_2$ being already contained in Propositions 3.3 and 3.5, we focus on the case $p \leq t_1 < t_2$. As in Section 3, we rely on two different approaches: one based on a weak-type estimate along with interpolation, that requires no smoothness on Ω but fails if $p = 1$, and another one, based on Sobolev embeddings, that requires some regularity on Ω but carries on to $p = 1$ *if* $N \geq 2$. Afterwards, we will show that there is no estimate when $N = 1$ and $p = 1$.

We start with the first approach, valid when $p > 1$ and $t_2 < \infty$.

Proposition 4.3. *Assume that s is an integer and $p > 1$.*

(1) *Let $p \leq t_1 < t_2 < \infty$. Then there exists a finite constant $C = C(\alpha, s, p, q, N, t_1, t_2)$ such that*

$$\left(\int_{\Omega}\left(\int_{U_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}} \frac{1}{r^{Nt_1/t_2+1}}\,\mathrm{d}r\right)^{t_2/t_1}\mathrm{d}x\right)^{1/t_2} \leq C|u|_{W^{s,p}(\Omega)}. \tag{4.5}$$

(2) *Let $p \leq t_1 < \infty$. Then there exists a finite constant $C = C(\alpha, s, p, q, N, t_1)$ such that*

$$\left(\int_{U_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}} \frac{1}{r}\,\mathrm{d}r\right)^{1/t_1} \leq C|u|_{W^{s,p}(\Omega)}, \forall\, x \in \Omega. \tag{4.6}$$

Proof. (1) Let $E_{u,\lambda}$, respectively $E_{u,\lambda,x}$, be as in (3.5), respectively (3.10). Set $\nu := \dfrac{1}{r^{Nt_1/t_2+1}}\,\mathrm{d}r$. On the one hand, thanks to the Sobolev inequality (1.5), we have, with C the constant in (1.5),

$$E_{u,\lambda,x} = \varnothing \text{ for } \lambda \geq C|u|_{W^{s,p}(\Omega)}. \tag{4.7}$$

On the other hand, arguing exactly as in the proof of (3.11), we find that, for $1 \leq \widetilde{p} < p$ sufficiently close to p, we have

$$\nu(E_{u,\lambda,x}) \lesssim \frac{1}{\lambda^{pt_1/t_2}}[\mathscr{M}(|D^s u|^{\widetilde{p}})(x)]^{(pt_1)/(\widetilde{p}t_2)}, \forall\, x \in \Omega. \tag{4.8}$$

Combining (4.7) and (4.8), we find that

$$\begin{aligned}
&\int_{U_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}} \frac{1}{r^{Nt_1/t_2+1}}\,\mathrm{d}r = t_1\int_0^\infty \lambda^{t_1-1}\nu(E_{u,\lambda,x})\,\mathrm{d}\lambda\\
&\lesssim \int_0^{C|u|_{W^{s,p}(\Omega)}} \lambda^{t_1-1-pt_1/t_2}\Big(\mathscr{M}(|D^s u|^{\widetilde{p}})(x)\Big)^{(pt_1)/(\widetilde{p}t_2)}\,\mathrm{d}\lambda\\
&\lesssim |u|^{t_1-pt_1/t_2}_{W^{s,p}(\Omega)}\Big(\mathscr{M}(|D^s u|^{\widetilde{p}})(x)\Big)^{(pt_1)/(\widetilde{p}t_2)},
\end{aligned}$$

where the last inequality relies on the fact that $t_1 - 1 - pt_1/t_2 > -1$ (since $t_2 > t_1 \geq p$).

This implies that

$$I^{t_2} \lesssim |u|^{t_2-p}_{W^{s,p}(\Omega)}\int_\Omega \Big(\mathscr{M}(|D^s u|^{\widetilde{p}})(x)\Big)^{p/\widetilde{p}} \lesssim |u|^{t_2-p}_{W^{s,p}(\Omega)}|u|^p_{W^{s,p}(\Omega)},$$

owing to the maximal function theorem. This yields (4.5).

(2) We start with the case where $p < t_1$. When $\Omega \neq \mathbb{R}^N$, we argue as follows. Let $R := \operatorname{dist}(x, \Omega^c) < \infty$, so that $U_x = (0, R)$. Set $\omega := B_R(x)$. Let $\widetilde{s}$ and $\widetilde{p}$ be such that

$$\max(s-1, \alpha) < \widetilde{s} < s,\ \widetilde{s} - \frac{N}{\widetilde{p}} = s - \frac{N}{p}. \tag{4.9}$$

Using (4.9), (4.2), and the embedding $W^{s,p}(\omega) \hookrightarrow W^{\widetilde{s},\widetilde{p}}(\omega)$, we find that, with C' independent of R (by scaling),

$$\left(\int_{U_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}}\frac{1}{r}\,\mathrm{d}r\right)^{1/t_1} \begin{aligned}[t] &\le C|u|_{W^{\widetilde{s},\widetilde{p}}(\omega)} \le C'|u|_{W^{s,p}(\omega)} \\ &\le C'|u|_{W^{s,p}(\Omega)}, \forall\, x\in\Omega. \end{aligned} \tag{4.10}$$

Next, assume that $p < t_1$ and $\Omega = \mathbb{R}^N$. Then $\omega = \mathbb{R}^N$ and (4.10) still holds in this context. (For the analogue of (1.5) in the full space, see, e.g., [2, Appendix B].) Alternatively, one could start from (4.10) with $\Omega = B_R(x)$, then let $R \to \infty$.

Finally, we consider the case where $p = t_1$. In this case, we will rely on a variant of (3.8). More specifically, let, for $r \in U_x$,

$$A_r := \{y \in \mathbb{R}^N : r/2 < |y-x| < r\} \subset \Omega.$$

With $\widetilde{p}$ satisfying (3.7), we have, by the proof of (3.8),

$$\inf_{P\in\mathscr{P}_\ell} |u-P|_{W^{\alpha,q}(B_r(x))} \lesssim r^{\widetilde{\beta}}|u|_{W^{s,\widetilde{p}}(B_r(x))} \lesssim r^{\widetilde{\beta}+N/\widetilde{p}}[\mathscr{M}(|D^s u|^{\widetilde{p}})(y)]^{1/\widetilde{p}}, \forall\, y\in A_r, \tag{4.11}$$

where we have extended $D^s u$ with the value 0 outside Ω.

Taking the average of (4.11) in y, integrating the result in r, applying the Jensen inequality, and using (3.7) and the maximal function theorem, we find that

$$\begin{aligned} &\int_{U_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{p}_{W^{\alpha,q}(B_r(x))}}{r^{\beta p}}\frac{1}{r}\,\mathrm{d}r \\ &\lesssim \int_{U_x}\left(\fint_{A_r}[\mathscr{M}(|D^s u|^{\widetilde{p}})(y)]^{1/\widetilde{p}}\,\mathrm{d}y\right)^p r^N\frac{1}{r}\,\mathrm{d}r \\ &\lesssim \int_{U_x}\int_{A_r}[\mathscr{M}(|D^s u|^{\widetilde{p}})(y)]^{p/\widetilde{p}}\,\mathrm{d}y\frac{1}{r}\,\mathrm{d}r \\ &\le \int_{\Omega}[\mathscr{M}(|D^s u|^{\widetilde{p}})(y)]^{p/\widetilde{p}}\int_{|y-x|}^{2|y-x|}\frac{1}{r}\,\mathrm{d}r\,\mathrm{d}y \lesssim |u|^p_{W^{s,p}(\Omega)}, \end{aligned}$$

whence (4.6) with $t_1 = p$. □

We next turn to the limiting case where $p = 1$ and $N \ge 2$, that we are able to treat under an extra regularity assumption on Ω.

Proposition 4.4. *Let Ω be a bounded Lipschitz domain. Assume that s is an integer and $N \ge 2$.*

(1) *Let $1 < t_1 < t_2 < \infty$. Then there exists a finite constant $C = C(\alpha, s, q, \Omega, t_1, t_2)$ such that*

$$\left(\int_\Omega\left(\int_{U_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}}\frac{1}{r^{Nt_1/t_2+1}}\,\mathrm{d}r\right)^{t_2/t_1}\mathrm{d}x\right)^{1/t_2} \le C|u|_{W^{s,1}(\Omega)}. \tag{4.12}$$

(2) *Let $1 < t_1 < \infty$. Then there exists a finite constant $C = C(\alpha, s, q, \Omega, t_1)$ such that*

$$\left(\int_{U_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}}\frac{1}{r}\,\mathrm{d}r\right)^{1/t_1} \le C|u|_{W^{s,1}(\Omega)}, \forall\, x\in\Omega. \tag{4.13}$$

Proof. (1) We follow the proof of Proposition 3.6. It suffices to establish the conclusion when $1 < t_1 < t_2$, with t_1 sufficiently close to 1. For such t_1, defining $\tilde{s}$ *via* (3.14) and using estimate (4.1), we obtain

$$\left(\int_\Omega\left(\int_{U_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}}\frac{1}{r^{Nt_1/t_2+1}}\,\mathrm{d}r\right)^{t_2/t_1}\mathrm{d}x\right)^{1/t_2} \le C|u|_{W^{\tilde{s},t_1}(\Omega)}. \tag{4.14}$$

We conclude by combining (4.14) with the Sobolev inequality (3.16).

(2) The argument is essentially the same: we use (4.2) instead of (4.1). □

We call the attention of the reader to the great similarity between this section and the two previous ones. Indeed, both when working with one or two parameters, the approach is essentially the same: when $s > 0$ is not an integer, the core of the proof is an suitable use of either Tonelli's theorem or Minkowski's integral inequality, while the case where s is an integer relies on an interpolation of weak-type estimates, or on Solonnikov's inequality to obtain the limiting case $p = 1$ by appealing to the case where s is not an integer.

The latter argument breaks down when $N = 1$. As in Section 3, we are able to show that there is actually no estimate in this case; see Corollary 4.6 below.

Lemma 4.5. *Let $N = 1$, $s \ge 1$ be an integer, $p = 1$, and $\Omega = (-1, 1)$. Let u be as in Lemma 3.10.*

(1) *If $1 \le t_1 \le t_2 < \infty$, then*

$$\left(\int_\Omega\left(\int_{U_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}}\frac{1}{r^{t_1/t_2+1}}\,\mathrm{d}r\right)^{t_2/t_1}\mathrm{d}x\right)^{1/t_2} = \infty. \tag{4.15}$$

(2) *If $1 \le t_1 < \infty$, then*

$$\left(\int_{U_0} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(0))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(0))}}{r^{\beta t_1}}\frac{1}{r}\,\mathrm{d}r\right)^{1/t_1} = \infty. \tag{4.16}$$

Proof. Items (1) and (2) are straightforward consequences of (3.17). □

Corollary 4.6. *Let $N = 1$, $s \ge 1$ be an integer, and $p = 1$.*

(1) *If $1 \le t_1 \le t_2 < \infty$, then there exists some $u \in W^{s,1}(\Omega)$ such that*

$$\left(\int_\Omega\left(\int_{U_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}}\frac{1}{r^{t_1/t_2+1}}\,\mathrm{d}r\right)^{t_2/t_1}\mathrm{d}x\right)^{1/t_2} = \infty. \tag{4.17}$$

(2) *If $1 \le t_1 < \infty$, then there exists some $u \in W^{s,1}(\Omega)$ such that*

$$\sup_{x\in\Omega}\left(\int_{U_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(0))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(0))}}{r^{\beta t_1}}\frac{1}{r}\,\mathrm{d}r\right)^{1/t_1} = \infty. \tag{4.18}$$

Sketch of proof of Corollary 4.6. (1) As explained in the proof of Corollary 3.11, (4.15) implies that there exist mutually disjoint intervals $I_k \subset \Omega$ and maps $u_k \in C^\infty_c(I_k)$ such that $\|u_k^{(s)}\|_{L^1} \le 2^{-k}$ and

$$\left(\int_{I_k}\left(\int_{(U_k)_x} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_1}}\frac{1}{r^{t_1/t_2+1}}\,\mathrm{d}r\right)^{t_2/t_1}\mathrm{d}x\right)^{1/t_2} \to \infty, \tag{4.19}$$

where U_k is the set U adapted to I_k. We may then let $u := \sum_k u_k$.
(2) We perform a similar construction, starting this time from (4.18). Instead of (4.19), we require the existence of $x_k \in I_k$ such that

$$\left(\int_{(U_k)_{x_k}} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x_k))}|u-P|^{t_1}_{W^{\alpha,q}(B_r(x_k))}}{r^{\beta t_1}}\frac{1}{r}\,\mathrm{d}r\right)^{1/t_1} \to \infty. \qquad \square$$

Case 2. $t_1 \ge t_2$. Recall that, in this case, the relevant section is $U^r := \{x \in \Omega\colon B_r(x) \subset \Omega\}$. We start again with the fractional regularity case.

Proposition 4.7. *Assume that $s > 0$ is not an integer.*

(1) *Let $p \le t_2 \le t_1 < \infty$. Then there exists a finite constant $C = C(\alpha, s, p, q, N, t_1, t_2)$ such that*

$$\left(\int_0^\infty\left(\int_{U^r} \frac{\inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_2}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_2}}\,\mathrm{d}x\right)^{t_1/t_2}\frac{1}{r^{Nt_1/t_2+1}}\,\mathrm{d}r\right)^{1/t_1} \le C|u|_{W^{s,p}(\Omega)}. \tag{4.20}$$

(2) *Let $p \le t_2 < \infty$. Then there exists a finite constant $C = C(\alpha, s, p, q, N)$ such that*

$$\left(\int_{U^r} \inf_{P\in\mathscr{P}_\ell(B_r(x))}|u-P|^{t_2}_{W^{\alpha,q}(B_r(x))}\,\mathrm{d}x\right)^{1/t_2} \le C r^{\beta+N/t_2}|u|_{W^{s,p}(\Omega)},\ \forall\, r > 0. \tag{4.21}$$

Proof. (1) The proof is very similar to the one of Proposition 4.1 (1). We let $\widetilde{s}$ and δ be as there. Using: (i) (4.3) (with t_2 in place of t_1); (ii) $z \in (B_r(x))_h \Rightarrow x \in B_r(z)$; (iii) $z \in (B_r(x))_h \ne \emptyset \Rightarrow |h| < 2r$ (see the proof of (2.7)), we find that

$$\begin{aligned}
I^{t_1} &\lesssim |u|^{t_1(t_2-p)/t_2}_{W^{s,p}(\Omega)} \int_0^\infty\left(\int_{U^r}\int_{\mathbb{R}^N}\int_{(B_r(x))_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p}}\,\mathrm{d}z\,\mathrm{d}h\,\mathrm{d}x\right)^{t_1/t_2}\frac{1}{r^{t_1\delta p/t_2+Nt_1/t_2+1}}\,\mathrm{d}r \\
&\lesssim |u|^{t_1(t_2-p)/t_2}_{W^{s,p}(\Omega)} \int_0^\infty\left(\int_{B_{2r}(0)}\int_{\Omega_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p}}\,\mathrm{d}z\,\mathrm{d}h\right)^{t_1/t_2}\frac{1}{r^{t_1\delta p/t_2+1}}\,\mathrm{d}r.
\end{aligned}$$

We now invoke the Minkowski inequality, as in the proof of Proposition 4.1, to find

$$
\begin{aligned}
I^{t_2} &\lesssim |u|_{W^{s,p}(\Omega)}^{t_2-p} \int_{\mathbb{R}^N} \int_{\Omega_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p}} \,\mathrm{d}z\,\mathrm{d}h \left(\int_{|h|/2}^{\infty} \frac{1}{r^{\delta t_1 p/t_2+1}} \,\mathrm{d}r \right)^{t_2/t_1} \mathrm{d}z\,\mathrm{d}h \\
&\lesssim |u|_{W^{s,p}(\Omega)}^{t_2-p} \int_{\mathbb{R}^N} \int_{\Omega_h} \frac{|\Delta_h^m u(z)|^p}{|h|^{N+\widetilde{s}p+\delta p}} \,\mathrm{d}z\,\mathrm{d}h = |u|_{W^{s,p}(\Omega)}^{t_2},
\end{aligned}
$$

concluding the proof of item (1).

(2) The case where $t_2 = p$ is obtained by integrating (1.5) in $x \in U^r$ and using property (ii) in the proof of item (1). The case where $p < t_2 < \infty$ follows from (1.5), the case where $t_2 = p$, and the Hölder inequality. □

In the case where s is an integer, we do not know whether or not it is possible to obtain the counterpart of Proposition 4.7 *via* interpolation of a weak-type estimate, even when $p > 1$, due to the reversed order of integration. However, it is still possible to obtain the desired estimate by relying on Sobolev embeddings, assuming some extra regularity on the domain, and that $N \geq 2$ in the limiting case where $p = 1$, but *only* when $t_1 < \infty$. Indeed, in this particular case, the estimate when $t_1 = \infty$ holds *without any extra assumption* on Ω, p, or N.

Proposition 4.8. *Assume that s is an integer.*

(1) *Let Ω be a Lipschitz bounded domain, $p < t_2 < t_1 < \infty$, and assume: either (i) $p > 1$, or (ii) $p = 1$ and $N \geq 2$. Then there exists a finite constant $C = C(\alpha, s, p, q, \Omega, t_1, t_2)$ such that*

$$
\left(\int_0^{\infty} \left(\int_{U^r} \frac{\inf_{P \in \mathscr{P}_\ell(B_r(x))} |u - P|_{W^{\alpha,q}(B_r(x))}^{t_2}}{r^{\beta t_2}} \,\mathrm{d}x \right)^{t_1/t_2} \frac{1}{r^{N t_1/t_2+1}} \,\mathrm{d}r \right)^{1/t_1} \leq C|u|_{W^{s,p}(\Omega)}. \tag{4.22}
$$

(2) *Let $p \leq t_2 < \infty$. Then there exists a finite constant $C = C(\alpha, s, p, q, N)$ such that*

$$
\left(\int_{U^r} \inf_{P \in \mathscr{P}_\ell(B_r(x))} |u - P|_{W^{\alpha,q}(B_r(x))}^{t_2} \,\mathrm{d}x \right)^{1/t_2} \leq C r^{\beta + N/t_2} |u|_{W^{s,p}(\Omega)}, \;\forall\, r > 0. \tag{4.23}
$$

Proof. (1) The proof is the same as the proof of Propositions 3.6 and 4.4 (1), and we therefore omit the details. The key idea is to rely on the Sobolev embedding $W^{s,p} \hookrightarrow W^{\widetilde{s},t}$, where $p < t < t_2$ is chosen sufficiently close to p and satisfying

$$
\widetilde{s} - \frac{N}{t} = s - \frac{N}{p}.
$$

Such a Sobolev inequality always holds on bounded Lipschitz domains — and in particular on balls (see, e.g., [1, Theorem B] and the references therein) — except in the case where $N = 1$ and $p = 1$ that we already mentioned. We then conclude by the means of Proposition 4.7 (1).

(2) We repeat the proof of (4.21). We observe that here, the assumption that either (i) $p > 1$, or (ii) $p = 1$ and $N \geq 2$, is not required, as we only rely on the suboptimal Sobolev embedding (1.5), which is always valid. □

To conclude, we state the by now familiar non-inequality in the limiting case where $N = 1$, $s \geq 1$ integer, and $p = 1$.

Lemma 4.9. *Let $N = 1$, $s \geq 1$ be an integer, $p = 1$, $\Omega = (-1, 1)$, and $1 \leq t_2 \leq t_1 < \infty$. Let u be as in Lemma 3.10. Then*

$$\left(\int_0^\infty \left(\int_{U^r} \frac{\inf_{P \in \mathscr{P}_\ell(B_r(x))} |u - P|^{t_2}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_2}} \, dx \right)^{t_1/t_2} \frac{1}{r^{t_1/t_2+1}} \, dr \right)^{1/t_1} = \infty. \tag{4.24}$$

Corollary 4.10. *Let $N = 1$, $s \geq 1$ be an integer, $p = 1$, and $1 \leq t_2 \leq t_1 < \infty$. Then there exists some $u \in W^{s,1}(\Omega)$ such that*

$$\left(\int_0^\infty \left(\int_{U^r} \frac{\inf_{P \in \mathscr{P}_\ell(B_r(x))} |u - P|^{t_2}_{W^{\alpha,q}(B_r(x))}}{r^{\beta t_2}} \, dx \right)^{t_1/t_2} \frac{1}{r^{t_1/t_2+1}} \, dr \right)^{1/t_1} = \infty. \tag{4.25}$$

The proofs of Lemma 4.9 and Corollary 4.10 are essentially the same as the ones of Lemma 4.5 and Corollary 4.6, and are omitted.

We end by collecting here the limiting cases that are not covered by our analysis of the case where $t_1 \geq t_2$.

Question 4.11. (1) Is there a counterpart to Propositions 4.4 and 4.8 (1) for an arbitrary domain?

(2) What happens with Propositions 4.4 (1) and 4.8 (1) in the limiting case $t_1 = 1$, respectively $t_2 = p$?

References

[1] H. Brezis and P. Mironescu. Where Sobolev interacts with Gagliardo-Nirenberg. *J. Funct. Anal.*, 277(8):2839–2864, 2019. 16

[2] T. Bullion-Gauthier. Higher-order affine Sobolev inequalities, 2025. hal-05067637. 13

[3] R.A. DeVore and R.C. Sharpley. Besov spaces on domains in $\mathbb{R}^d$. *Trans. Amer. Math. Soc.*, 335(2):843–864, 1993. 2

[4] L. Grafakos. *Classical Fourier analysis*. Number 249 in Grad. Texts in Math. Springer, 2008. 8

[5] P. Mironescu and J. Van Schaftingen. Lifting in compact covering spaces for fractional Sobolev mappings. *Anal. PDE*, 14(6):1851–1871, 2021. 3

[6] V. A. Solonnikov. Certain inequalities for functions from the classes $\vec{W}_p(R^n)$. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 27:194–210, 1972. Boundary value problems of mathematical physics and related questions in the theory of functions. 6. 8

[7] H. Triebel. *Theory of function spaces*. Number 78 in Monogr. Math. Birkhäuser, 1983. 2, 3