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Problem 1

For $T(n)$ to be a member of the set $O(f(n))$ there must exist a constant c such that $T(n) \leq c f(n)$ for all $n \geq n_0$ where n_0 is a constant.

$T(n) = \log_2(n) \qquad f(n) = \log_{10}(n)$

$(\exists c)(\forall n)(T(n) \in O(f(n)) \rightarrow T(n) \leq c f(n))$

$T(n) \in O(f(n)) \rightarrow T(n) \leq 4 f(n)$

$\neg(T(n) \in O(f(n))) \vee T(n) \leq 4 f(n)$

Because $\neg(T(n) \in O(f(n)))$ is false for all functions $T(n)$ that are members of $O(f(n))$ we will prove that $T(n) \leq 4 f(n)$ is true for $T(n) = \log_2(n)$ and $f(n) = \log_{10}(n)$ thus proving our initial conjecture.

Base case $n_0 = 1$

$\log_2(1) \leq 4(\log_{10}(1))$

$0 \leq 4(0)$

$0 \leq 0$

True

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Inductive Step $n = n+1$

$\log_2(n+1) \leq 4 \log_{10}(n+1)$

$\log_2(n+1) \leq 4\left(\frac{\log_2(n+1)}{\log_2(10)}\right)$ {Prop. of logs}

$\frac{\log_2(n+1)}{\log_2(n+1)} \leq 4 \frac{1}{\log_2(10)}$

$1 \leq 4\left(\frac{1}{\log_2(10)}\right)$

$1 \leq 4(.30103)$

$1 \leq 1.2$

True

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Problem 2

$$T(n) = \begin{cases} 1 & n \le 1 \\ 2T\left(\frac{n}{3}\right) + \left(\frac{n}{3}\right)^2 \end{cases}$$

$$\Rightarrow \quad T(n) = 2T\left(\frac{n}{3}\right) + \left(\frac{n}{3}\right)^2$$

$$= 2\left(2T\left(\frac{n}{9}\right) + \left(\frac{n}{9}\right)^2\right) + \left(\frac{n}{3}\right)^2$$

$$= 4T\left(\frac{n}{9}\right) + \frac{11n^2}{81}$$

$$= 4\left(2T\left(\frac{n}{27}\right) + \left(\frac{n}{27}\right)^2\right) + \frac{11n^2}{81}$$

$\ldots$

$$= 2^i T\left(\frac{n}{3^i}\right) + \left(\frac{n}{3^i}\right)^2$$

for $i = \lg n$

$$= n\,T\left(\frac{n}{n}\right) + \left(\frac{n}{n}\right)^2$$

$$= n\,T(1) + 1 = n \cdot 1 + 1 = n + 1 \in \Theta(n)$$

$$\boxed{\therefore \; T(n) \in \Theta(n)}$$

Problem 3

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[1, 7, 1, 5, 3, -1, 9]

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[-1, 1, 1, 3, 5, 7, 9]

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Problem 4

The runtime complexity for our mergesort function is:

$$T(n) = \begin{cases} 1 & n \leq 1 \\ n & \text{List is sorted} \\ \frac{n^5-n^4}{8^i} - \frac{2n^4-2n^3}{4^i} + \frac{n^3-n^2}{2^i} & n > 1 \end{cases}$$

$i = \log n$

$$\frac{n^5-n^4}{n} - \frac{2n^4-2n^3}{n} + \frac{n^3-n^2}{n}$$

$$n^4-n^3-(2n^3-2n^2)+n^2-n$$

$$n^4-3n^3+3n^2-n$$

$$n^4-3n^3+3n^2-n \in \Theta(n^4)$$

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Problem 5

From "Discrete Mathematics and its Applications" by Kenneth H. Rosen the probability of an event occuring is $P|E| = \frac{|E|}{|S|}$ where $|E|$ is the event and $|S|$ is the sample space of equally likely outcomes.

$$|E| = (n/2 - 1)(n/2)\,3!$$

The event is choosing a value between $n/4$ and $3n/4$ with 3! possible pivot combinations. (low, good, or high)

$$|S| = \frac{n!}{(n-3)!} = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = n(n-1)(n-2)$$

The sample space of equally likely outcomes is equal to the total possible samples divided by the remaining possible samples after removing 3 elements.

Variable added to find probablistic ratio.

$$P|E| = \frac{(n/2-1)(n/2)3!}{n(n-1)(n-2)} \cdot n = \frac{(n-2)(n)3}{2(n-1)(n-2)n} \cdot n = \frac{3}{2n-1} \cdot n$$

which gives us a probalistic ratio of 1.5 when n goes to infinity.

Thus since 1 pivot quicksort has a 50% chance of finding a good pivot, the best possible input size for a 3 pivot quicksort, baring lists of size 0, 1, 2 and 3 will have a 75% chance of choosing a good pivot. From the best case the probability of choosing a good pivot using median of 3 will slowly

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decrease to 50% at ∞. Since we are not sorting infinitly long lists quicksort utilizing a median of 3 pivot system will have a higher chance of finding a good pivot than a single pivot implimentation of quicksort.